Please use \LaTeX to type up your solutions!

**Problem 1.** Let $G$ be a $d$-regular graph with vertex set $V(G) = [n]$. Let the eigenvalues of $G$ be $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$ and set $\lambda := \max\{|\lambda_2|, |\lambda_n|\}$. We know that $d$ is an eigenvalue of $G$. Moreover, we have used repeatedly that $d$ is the largest eigenvalue of $G$, but we have not proved this in class.

(a) Show that $\lambda_1 = d$ and, in fact, $\lambda \leq d$.

(b) Show that $\lambda_2 = d$ if and only if the graph $G$ is disconnected.

(c) Show that $\lambda_n = -d$ if and only if $G$ has a bipartite connected component.

(d) Conclude that $\lambda < d$ if and only if $G$ is connected and not bipartite.

**Problem 2.** Let $G$ be a $d$-regular graph with vertex set $V(G) = [n]$. Let the eigenvalues of $G$ be $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$ and set $\lambda := \max\{|\lambda_2|, |\lambda_n|\}$. Consider the random walk $v_0, v_1, \ldots$ on $G$ with starting distribution $w_0$ and let $w_k$ be the distribution of $v_k$, i.e., $w_k(i) := P[v_k = i]$. We have shown in class that if $\lambda < d$, then $\lim_{n \to \infty} w_k = 1/n$, the uniform distribution. Prove the converse of this statement: If for every starting distribution $w_0$, we have $\lim_{k \to \infty} w_k = 1/n$, then $\lambda < d$.

**Problem 3** (Non-bipartite expanders). In class we gave a probabilistic proof that there exist arbitrarily large graphs with maximum degree 3 whose Cheeger constant is separated from zero. Our proof, however, had a drawback: the graphs we obtained were bipartite. In this problem you will give a probabilistic construction of non-bipartite graphs with Cheeger constant at least $\varepsilon$, for some fixed $\varepsilon > 0$.

Fix $n \in \mathbb{N}^+$ (you may assume that $n$ is sufficiently large). Build a random graph $G$ with vertex set $[n]$ as follows: Pick two permutations $\pi_1, \pi_2$ of $[n]$ uniformly at random and put an edge between distinct vertices $i$ and $j$ if and only if at least one of the following statements holds:

$$\pi_1(i) = j, \quad \pi_1(j) = i, \quad \pi_2(i) = j, \quad \text{or} \quad \pi_2(j) = i.$$  

By construction, the maximum degree of $G$ is at most 4. Show that there is some constant $\varepsilon > 0$, independent of $n$, such that, with positive probability, $G$ is not bipartite and $\chi(G) \geq \varepsilon$.

**Problem 4** (Bipartite expanders). Sometimes it is more convenient to work with “expander-like” bipartite graphs in which the two parts have different sizes. A **bipartite $(n, m, d, \alpha, c)$-expander** is a bipartite graph $G$ with fixed bipartition $V(G) = L \cup R$ such that:

- $|L| = n$ and $|R| = m$;
- every vertex in $L$ has degree at most $d$;
- for every $S \subseteq L$ of size at most $n/2$, we have $|N_G(S)| \geq |S|$;
- for every $S \subseteq L$ of size at most $\alpha n$, we have $|N_G(S)| \geq c|S|$.

(a) Show that there exist $d_0, n_0 \in \mathbb{N}$ such that for all $d \geq d_0$, $n \geq n_0$, and $m \geq 3n/4$, there exists a bipartite $(n, m, d, 1/(10d), 5d/8)$-expander.

*Hint:* Use a probabilistic construction.

(b) Let $G$ be a bipartite $(n, m, d, \alpha, c)$-expander with $c > d/2$. Show that for every nonempty set $S \subseteq L$ of size at most $\alpha n$, there is a vertex $y \in R$ with precisely one neighbor in $S$.

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