Please use \LaTeX to type up your solutions!

**Problem 1.** Show that the dimension of the space of all polynomials in \( n \) variables \( x_1, \ldots, x_n \) of degree at most \( d \) is \( \binom{n+d}{d} - d \). (For simplicity, you may work with polynomials over \( \mathbb{R} \); but your proof will most likely generalize to polynomials over an arbitrary field.)

**Problem 2** (Oddtown). Fix \( n \in \mathbb{N}^+ \). An **oddtown** is a family \( \mathcal{F} \) of subsets of \([n]\) such that:

- for all \( A \in \mathcal{F} \), \(|A|\) is odd;
- for all distinct \( A, B \in \mathcal{F} \), \(|A \cap B|\) is even.

(a) Construct an oddtown \( \mathcal{F} \) of size \( n \).
(b) Show that \(|\mathcal{F}| \leq n\) for every oddtown \( \mathcal{F} \).

*Hint:* Proceed as in the proof of Fisher’s theorem, but over \( \mathbb{Z}_2 \).

**Problem 3** (Eventown). Fix \( n \in \mathbb{N}^+ \). An **eventown** is a family \( \mathcal{F} \) of subsets of \([n]\) such that:

- for all \( A \in \mathcal{F} \), \(|A|\) is even;
- for all distinct \( A, B \in \mathcal{F} \), \(|A \cap B|\) is even.

(a) Construct an eventown \( \mathcal{F} \) of size \( 2^{\lfloor n/2 \rfloor} \).

Suppose that \( \mathcal{F} \) is an eventown and let \( M \) be the matrix over \( \mathbb{Z}_2 \) whose columns are the characteristic vectors of the members of \( \mathcal{F} \).

(b) Show that \( M^T M = 0 \), where \( M^T \) denotes the transpose of \( M \).

(c) Use the rank–nullity theorem to deduce that the rank of \( M \) is at most \(|n/2|\).

(d) Conclude that \(|\mathcal{F}| \leq 2^{\lfloor n/2 \rfloor}\).

**Problem 4.** Prove the mod \( p \) version of the Frankl–Wilson theorem:

**Theorem 4.1** (Deza–Frankl–Singhi). Let \( p \) be a prime number and let \( L \subseteq \mathbb{Z}_p \) be a set of size \( k \).

Suppose that \( \mathcal{F} \) is a family of subsets of \([n]\) such that:

- for all \( A \in \mathcal{F} \), \(|A| \pmod{p} \neq L");
- for all distinct \( A, B \in \mathcal{F} \), \(|A \cap B| \pmod{p} \in L").

Then \(|\mathcal{F}| \leq \binom{n}{p} + \binom{n}{p} + \cdots + \binom{n}{p} = O(n^k)\).

**Problem 5.** Fix \( s \in \mathbb{N}^+ \) and let \( G \) be the graph whose vertices are the 3-element subsets of \([s]\) and two distinct vertices \( A, B \) are adjacent if and only if \(|A \cap B| = 1\). Show that \( G \) is \( s \)-Ramsey (i.e., \( G \) contains neither a clique nor an independent set of size strictly greater than \( s \)).

*Remark.* Note that \(|V(G)| = \binom{s}{3}\) is cubic in \( s \).

**Problem 6.** Fix positive integers \( k \) and \( s \) such that \( k < s \). By Frankl–Wilson, if \( \mathcal{F} \) is a family of subsets of \([n]\) such that \(|A \cap B| < k\) for all distinct \( A, B \in \mathcal{F} \), then \(|\mathcal{F}| \leq \binom{n}{k} + \binom{n}{k} + \cdots + \binom{n}{k} = O(n^k)\).

We also know that this upper bound is tight and is attained by the family of all subsets of \( \mathcal{F} \) of size less than \( k \). But what if we require every set in \( \mathcal{F} \) to have size exactly \( s \)?

Let \( p \geq s \) be a prime. Show that if \( n = sp \), then there is a family \( \mathcal{F} \) of subsets of \([n]\) such that:

- for all \( A \in \mathcal{F} \), \(|A| = s\);
- for all distinct \( A, B \in \mathcal{F} \), \(|A \cap B| < k\);
- \(|\mathcal{F}| \geq (n/s)^k = \Omega(n^k)\).

*Hint:* Use that two distinct polynomials of degree less than \( k \) can only agree on fewer than \( k \) inputs.

---

*Date: November 5, 2019.*