Please use \LaTeX{} to type up your solutions!

**Problem 1** (Kővári–Sós–Turán theorem). Recall that, given a graph $H$, we write $\text{ex}(n, H)$ for the extremal number of $H$, i.e., the maximum number of edges in an $n$-vertex $H$-free graph, where a graph $G$ is said to be $H$-free if it has no subgraph isomorphic to $H$. In class we proved Reiman’s theorem stating that

$$\text{ex}(n, C_4) \leq \frac{1}{4} n \left(1 + \sqrt{4n - 3}\right) \sim \frac{n^{3/2}}{2}, \quad (1.1)$$

where $C_4$ denotes the 4-cycle. In this problem you will prove a generalization of this fact.

You may use the following without proof: For $k \in \mathbb{N}$ and $x_1, \ldots, x_n \geq 0$,

$$\left(\frac{x_1 + \cdots + x_n}{n}\right)^k \leq \frac{x_1^k + \cdots + x_n^k}{n}. \quad (1.2)$$

(a) (Warm-up; not for credit.) Draw the graph of the function $x \mapsto x^k$ and use it to convince yourself that (1.2) should be true when $n = 2$.

(b) (Warm-up; not for credit.) When $k = 2$, deduce (1.2) from the Cauchy–Schwarz inequality.

(c) (Bonus; not for credit.) Prove (1.2) by induction on $k$.

Remark. The inequality (1.2) is an instance of the so-called Jensen’s inequality (look it up!).

A graph $H$ is bipartite if the vertex set of $H$ can be partitioned as $V(H) = A \sqcup B$ so that all the edges of $H$ go between $A$ and $B$ (i.e., the sets $A$ and $B$ are independent in $H$). The complete bipartite graph $K_{s,t}$ with part sizes $s$ and $t$ is the bipartite graph with vertex set $A \sqcup B$ and edge set $\{ab : a \in A, b \in B\}$, where $|A| = s$ and $|B| = t$. For instance, $K_{2,2}$ is the 4-cycle (see Fig. 1).

![Figure 1. Examples of complete bipartite graphs.](image)

(d) Let $s \geq 2$ and let $G$ be an $n$-vertex $K_{s,s}$-free graph. Show that

$$\sum_{x \in V(G)} \left(\frac{\deg_G(x)}{s}\right) \leq (s - 1) \binom{n}{s}. \quad (1.3)$$

(e) Prove the Kővári–Sós–Turán theorem: For every $s \geq 2$, there is $c > 0$ such that

$$\text{ex}(n, K_{s,s}) \leq cn^{2 - 1/s} \quad \text{for all } n \in \mathbb{N}^+. \quad (1.3)$$

Remark. For $s = 2$, (1.3) is a consequence of Reiman’s bound (1.1).

(f) Show that a finite graph $H$ is bipartite if and only if there are $c, \varepsilon > 0$ such that

$$\text{ex}(n, H) \leq cn^{2 - \varepsilon} \quad \text{for all } n \in \mathbb{N}^+.$$
Problem 2 (Moon–Moser theorem). Suppose that $G$ is a graph with $n$ vertices and $m$ edges. Let $t(G)$ denote the number of triangles in $G$ (i.e., the number of triples $\{x, y, z\} \subseteq V(G)$ of pairwise adjacent vertices). Recall that, by Turán’s theorem, $\text{ex}(n, K_3) \leq n^2/4$. This means that if $m > n^2/4$, then $G$ contains a triangle, i.e., $t(G) \geq 1$. In this problem you will show that if $m > n^2/4$, then $G$ in fact must contain many triangles.

(a) For each edge $e = xy \in E(G)$, let $t(e)$ denote the number of triangles in $G$ that contain $e$. Show that $\deg_G(x) + \deg_G(y) - t(e) \leq n$.

(b) Deduce the following inequality due to Moon and Moser:

$$t(G) \geq \frac{4m}{3n} \left( m - \frac{n^2}{4} \right).$$

**Hint:** At some point you will need to use Cauchy–Schwarz.

(c) Conclude that if $m \geq n^2/4 + 1$, then $G$ contains at least $n/3$ triangles.

Problem 3 (Sidon sets). In class we discussed an algebraic construction that shows that Reiman’s bound (1.1) is asymptotically tight; namely, we proved that if $p$ is a prime number, then

$$\text{ex}(p(p-1), C_4) \geq \frac{p(p-1)(p-2)}{2}.$$  

In this problem you will obtain an alternative (but also algebraic) construction.

Let $n$, $k \geq 2$ be integers and consider the set $\mathbb{Z}_n^k$ of all $k$-tuples of residues mod $n$. We equip $\mathbb{Z}_n^k$ with the operation of coordinate-wise addition:

$$(x_1, \ldots, x_k) + (y_1, \ldots, y_k) := (x_1 + y_1, \ldots, x_k + y_k),$$

where the sums $x_1 + y_1, \ldots, x_k + y_k$ are interpreted modulo $n$. A **Sidon set** is a subset $A \subseteq \mathbb{Z}_n^k$ such that for all $a, b, c, d \in A$, we have the following implication:

$$a + b = c + d \implies \{a, b\} = \{c, d\};$$

in other words, the equation $a + b = c + d$ with $a, b, c, d \in A$ has only “trivial” solutions.

(a) Show that if $A \subseteq \mathbb{Z}_n^k$ is a Sidon set, then $|A|||A| - 1| \leq n^k - 1$.

**Hint:** Consider the values $a - b$ for distinct $a, b \in A$.

(b) Let $p \geq 3$ be a prime number. Show that $\{(a, a^2) : a \in \mathbb{Z}_p\}$ is a Sidon set in $\mathbb{Z}_p^2$ of size $p$.

(c) Let $A \subseteq \mathbb{Z}_n^k$ be a Sidon set. Let $G_A$ be the graph with vertex set $\mathbb{Z}_n^k$ in which two distinct vertices $x, y \in \mathbb{Z}_n^k$ are adjacent if and only if $x + y \in A$. Show that $G_A$ contains no 4-cycles.

(d) Conclude that for prime $p \geq 3$,

$$\text{ex}(p^2, C_4) \geq \frac{p^2(p-1)}{2}. \quad (3.1)$$

Note that if we set $n := p^2$, then the right-hand side of (3.1) is $\sim n^{3/2}/2$, which is asymptotically best possible by Reiman’s theorem.

Problem 4.

(a) Recall that a **derangement** of $[n]$ is a permutation $\pi : [n] \to [n]$ without fixed points, i.e., such that $\{x \in [n] : \pi(x) = x\} = \emptyset$. Show that if $n$ is even, then the number of derangements of $[n]$ is odd.

(b) Let $n$ be even and suppose that $A$ is an $n$-by-$n$ matrix all of whose diagonal entries are 0 and all of whose non-diagonal entries are in $\{1, -1\}$. Use part (a) to conclude that $\det A \neq 0$.

Problem 5. Use the matrix–tree theorem to compute the number of spanning trees in the complete bipartite graph $K_{s,t}$.