Please use \LaTeX{} to type up your solutions!

Recall that the zero locus of a multivariate polynomial $p \in F[x_1, \ldots, x_n]$ over a field $F$ is the set

$$Z_F(p) := \{(c_1, \ldots, c_n) \in F^n : p(c_1, \ldots, c_n) = 0\}.$$ 

For the next problem, we need to define resultants of multivariate polynomials. Let $F$ be a field and let $p, q \in F[x_1, \ldots, x_n]$ be a pair of nonzero polynomials over $F$ in $n$ variables. For each $1 \leq i \leq n$, we may consider $p$ and $q$ as polynomials in $x_i$ whose coefficients are, in turn, polynomials in the remaining $n-1$ variables. This allows us to compute the resultant of $p$ and $q$ with respect to the variable $x_i$, denoted by $\text{res}_{x_i}(p, q)$. For instance,

$$\text{res}_{x_i}(x^2 + y^2 + z^2, xyz) = \det \begin{bmatrix} y^2 + z^2 & 0 & 0 \\ 0 & yz & 0 \\ 1 & 0 & yz \end{bmatrix} = y^4z^2 + y^2z^4.$$ 

Note that $\text{res}_{x_i}(p, q)$ is itself a polynomial over $F$ in the remaining $n-1$ variables.

**Problem 1.** In this problem we work over $\mathbb{C}$. Given a pair of polynomials $f, g \in \mathbb{C}[t]$ of degree at least 1, we consider the set $\mathcal{S}_{f,g} \subseteq \mathbb{C}^2$ parameterized by $f$ and $g$:

$$\mathcal{S}_{f,g} := \{(x, y) \in \mathbb{C}^2 : x = f(t), y = g(t) \text{ for some } t \in \mathbb{C}\}.$$ 

(a) Consider the polynomials $x - f(t)$ and $y - g(t)$ in the three variables $t, x, y$. Let $p \in \mathbb{C}[x, y]$ be the polynomial given by the formula

$$p := \text{res}_t(x - f(t), y - g(t)).$$

Show that $\mathcal{S}_{f,g}$ is the zero locus of $p$.

(b) Find a polynomial $p \in \mathbb{C}[x, y]$ whose zero locus is parameterized by the polynomials

$$f(t) = t^2 + t \quad \text{and} \quad g(t) = t^3 + t.$$ 

(You may use a computer algebra system to aid you in calculating the determinant.)

**Problem 2** (Kakeya problem over a finite field). For this problem, $F$ is a finite field of size $q$. A subset $E \subseteq F^n$ is called a Kakeya set if it “contains a line in every direction,” i.e., if for each nonzero vector $v \in F^n$, there is some $a \in E$ such that

$$\{a + tv : t \in F\} \subseteq E.$$ 

The goal of this problem is to establish the following lower bound on the size of a Kakeya set:

**Theorem 2.1** (Dvir). If $E \subseteq F^n$ is a Kakeya set, then

$$|E| \geq \binom{q + n - 1}{n}.$$ 

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$^1$Named after the Japanese mathematician Sōichi Kakeya.
Notice that
\[
\binom{q + n - 1}{n} \geq \frac{q^n}{n!} = (1/n!)|F^n|,
\]
and hence it follows from Theorem 2.1 that a Kakeya set must occupy at least a \((1/n!)\) proportion of the entire space \(F^n\), regardless of the size of the finite field \(F\).

Suppose, towards a contradiction, that \(E \subseteq F^n\) is a Kakeya set such that \(|E| < \binom{q + n - 1}{n}\).

(a) Show that there is a nonzero polynomial \(p \in F[x_1, \ldots, x_n]\) of degree \(d < q\) such that \(E \subseteq \mathbb{Z}_F(p)\).

*Hint:* Compare the dimension of the space of all polynomials in \(n\) variables of degree less than \(q\) with that of the space \(F^E\) of all functions from \(E\) to \(F\).

Let \(p\) be the polynomial obtained in part (a) and let \(d := \deg p\). Write
\[
p = p_0 + p_1 + \cdots + p_d,
\]
where \(p_i\) is the \(i\)-th *homogeneous component* of \(p\), i.e., the polynomial obtained from \(p\) by only retaining the monomials of degree \(i\). By definition, \(p_d \neq 0\) and \(d \geq 1\).

Take any nonzero vector \(v = (v_1, \ldots, v_n) \in F^n\) and let \(a = (a_1, \ldots, a_n) \in E\) be such that
\[
\{a + tv : t \in F\} \subseteq E.
\]
(Such \(a\) exists because \(E\) is a Kakeya set.) Define \(f_{v,a} \in F[t]\) to be the polynomial given by
\[
f_{v,a}(t) := p(a + tv) = p(a_1 + tv_1, \ldots, a_n + tv_n).
\]
(b) Show that \(f_{v,a}\) is the zero polynomial.

(c) Show that \([t^d]f_{v,a} = p_d(v)\), and hence \(p_d(v) = 0\) for all \(v \in F^n\).

(d) Finish the proof of Theorem 2.1. *Hint:* Use Schwartz–Zippel.

**Problem 3.** Let \(F\) be a field and let \(V\) be an \(F\)-vector space. Let \(\varphi, \psi: V \to V\) be linear functions. Suppose that \(\varphi\) and \(\psi\) commute, i.e., that \(\varphi \circ \psi = \psi \circ \varphi\). Let \(\lambda \in \text{Spec}(\varphi)\) and let \(W \subseteq V\) be the corresponding eigenspace.

(a) Show that the space \(W\) is \(\psi\)-invariant.

(b) Conclude that if the field \(F\) is algebraically closed and the space \(V\) is finite-dimensional, then \(W\) contains an eigenvector of \(\psi\); in particular, \(\varphi\) and \(\psi\) have a common eigenvector.

**Problem 4.** In this problem we prove the following result of Sylvester:

**Theorem 4.1 (Sylvester).** Let \(F\) be an algebraically closed field and let \(A \in M_{n \times n}(F), B \in M_{m \times m}(F)\). If \(A\) and \(B\) have no common eigenvalues, then for each \(C \in M_{n \times m}(F)\), the equation
\[
AX - XB = C
\]
has a unique solution \(X \in M_{n \times m}(F)\).

Let \(V := M_{n \times m}(F)\) and define linear functions \(\varphi_A, \varphi_B: V \to V\) by
\[
\varphi_A(X) := AX \quad \text{and} \quad \varphi_B(X) := XB.
\]

(a) Show that the functions \(\varphi_A\) and \(\varphi_B\) commute with each other.

(b) Let \(\lambda \in \text{Spec}(\varphi_A - \varphi_B)\). Show that \(\lambda = \mu - \nu\) for some \(\mu \in \text{Spec}(\varphi_A)\) and \(\nu \in \text{Spec}(\varphi_B)\).

(c) Show that \(\text{Spec}(\varphi_A) = \text{Spec}(A)\) and \(\text{Spec}(\varphi_B) = \text{Spec}(B)\).

(d) Conclude that all eigenvalues of \(\varphi_A - \varphi_B\) are nonzero and finish the proof of Theorem 4.1.