Please use \LaTeX{} to type up your solutions!

In what follows, the graph of a function \( f: X \to Y \) is the set
\[
\Gamma_f := \{(x, y) \in X \times Y : f(x) = y\}.
\]

**Problem 1.** For a real number \( x \in \mathbb{R} \), let \( x \mod 1 \) denote the fractional part of \( x \), i.e., the unique number \( \alpha \in [0; 1) \) such that \( x - \alpha \) is an integer. For \( \alpha, \beta \in [0; 1) \) and \( r \in \mathbb{R} \), define
\[
\alpha \oplus \beta := (\alpha + \beta) \pmod{1} \quad \text{and} \quad r \odot \alpha := (r\alpha) \pmod{1}.
\]

Does this definition make \([0; 1)\) into an \( \mathbb{R} \)-vector space?

**Problem 2** (First isomorphism theorem). Let \( V \) and \( W \) be vector spaces over a field \( F \) and let \( \varphi: V \to W \) be a linear function. Show that the space \( \text{im} \varphi \) is isomorphic to \( V / \ker(\varphi) \).

**Problem 3** (Direct sums). Fix a field \( F \). The direct sum of two \( F \)-vector spaces \( V \) and \( W \) is the \( F \)-vector space \( V \oplus W \) defined as follows. As a set, \( V \oplus W \) is equal to \( V \times W \), and addition and scalar multiplication on \( V \oplus W \) are defined component-wise:
\[
(v_1, w_1) + (v_2, w_2) := (v_1 + v_2, w_1 + w_2) \quad \text{and} \quad a \cdot (v, w) := (a \cdot v, a \cdot w).
\]

Prove that a function \( f: V \to W \) is linear if and only if its graph is a subspace of \( V \oplus W \).

**Problem 4.** Let \( V \) be a vector space over a field \( F \) and let \( W \subseteq V \) be a subspace of \( V \).

(a) Show that there is a subspace \( W' \subseteq V \) such that every vector \( v \in V \) can be uniquely expressed as a sum \( v = w + w' \) with \( w \in W \) and \( w' \in W' \).

(b) Show that every subspace \( W' \subseteq V \) as in (a) is isomorphic to \( V/W \).

(c) Conclude that \( V \) is isomorphic to \( W \oplus (V/W) \).

**Problem 5** (\( \mathbb{Q} \)-linear functions are weird). Let \( f: \mathbb{R} \to \mathbb{R} \) be a function that is \( \mathbb{Q} \)-linear but not \( \mathbb{R} \)-linear. Show that the graph of \( f \) is dense in \( \mathbb{R}^2 \).

**Remark.** A set \( S \subseteq \mathbb{R}^2 \) is dense in \( \mathbb{R}^2 \) if for every point \( p \in \mathbb{R}^2 \) and for every positive real number \( \varepsilon \), there is a point \( q \in S \) such that the distance between \( p \) and \( q \) is less than \( \varepsilon \). In other words, \( S \) is dense in \( \mathbb{R}^2 \) if \( S \) intersects every disc \( D \subset \mathbb{R}^2 \) of positive radius:

\[
\text{There is a point of } S \text{ somewhere in here}
\]
If we were to draw a picture of a dense subset of the plane giving each point an arbitrarily small positive thickness, it would look like this:

\[ \text{Hint. What is Span}_R(\Gamma_f)? \]

**Problem 6.** Consider the \( \mathbb{R} \)-vector space \( \mathbb{R}^\mathbb{N} \) of all infinite sequences of reals. For each \( \alpha \in \mathbb{R} \), let
\[ e_\alpha := (1, \alpha, \alpha^2, \alpha^3, \ldots). \]
Show that the set \( \{ e_\alpha : \alpha \in \mathbb{R} \} \) is independent.

*Remark.* This means that you can find as many independent vectors in \( \mathbb{R}^\mathbb{N} \) as there are real numbers!