Name: Anton Bernshteyn

<table>
<thead>
<tr>
<th></th>
<th>/20</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>/20</td>
</tr>
<tr>
<td>2</td>
<td>/20</td>
</tr>
<tr>
<td>3</td>
<td>/20</td>
</tr>
<tr>
<td>4</td>
<td>/20</td>
</tr>
<tr>
<td>5</td>
<td>/20</td>
</tr>
<tr>
<td>Total</td>
<td>/100</td>
</tr>
</tbody>
</table>

No calculators, books, notes, &tc. are allowed. Please justify all your answers.
Problem 1. (20 pts.) Consider the following set of four matrices in $M_{2 \times 2}(\mathbb{F}_2)$:
\[
\left\{ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \right\}.
\]
Show that this set, equipped with the matrix addition and multiplication operations, is a field.

For brevity, let us write
\[
0 := \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad I := \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad A := \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}, \quad B := \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix},
\]
and let $F := \{0, I, A, B\}$. To begin with, we need to show that $F$ is closed under addition and multiplication. This is indeed the case, as the addition and multiplication tables for $F$ are:

\[
\begin{array}{c|cccc}
+ & 0 & I & A & B \\
0 & 0 & I & A & B \\
I & I & 0 & B & A \\
A & A & B & 0 & I \\
B & B & 0 & A & I \\
\end{array}
\quad
\begin{array}{c|cccc}
\cdot & 0 & I & A & B \\
0 & 0 & 0 & 0 & 0 \\
I & 0 & I & A & B \\
A & 0 & A & B & I \\
B & 0 & B & I & A \\
\end{array}
\]

The set $F$ has an additive identity (namely 0) and a multiplicative identity (namely $I$). Notice that for every matrix $X \in M_{2 \times 2}(\mathbb{F}_2)$, we have $X + X = 0$, i.e., $-X = X$, and thus $F$ is automatically closed under taking additive inverses. The properties such as associativity of $+$ and $\cdot$, commutativity of $\cdot$, and distributivity of $\cdot$ over $+$ hold in $M_{2 \times 2}(\mathbb{F}_2)$, and hence they also hold in $F$. This means that $F$ is a ring. Furthermore, since the multiplication table of $F$ is symmetric with respect to the main diagonal, $F$ is a commutative ring. Since $0 \neq I$, it remains to observe that $F$ has a multiplicative inverse for every nonzero element, which is true as $I^{-1} = I$, $A^{-1} = B$, and $B^{-1} = A$. 
Problem 2. (10+10 pts.) Let $\mathbb{R}^+$ denote the set of all positive real numbers.

(a) (10 pts.) For $\alpha, \beta \in \mathbb{R}^+$ and $r \in \mathbb{R}$, define

$$\alpha \oplus \beta := \alpha \beta \quad \text{and} \quad r \odot \alpha := \alpha^r.$$ 

Does this definition make $\mathbb{R}^+$ into an $\mathbb{R}$-vector space?

(b) (10 pts.) For $\alpha, \beta \in \mathbb{R}^+$ and $r \in \mathbb{R}$, define

$$\alpha \boxplus \beta := \alpha \beta \quad \text{and} \quad r \boxdot \alpha := r^\alpha.$$ 

Does this definition make $\mathbb{R}^+$ into an $\mathbb{R}$-vector space?

(a) Yes. It is not hard to verify all the requirements directly, but here’s a more conceptual argument. Consider the function $f: \mathbb{R}^+ \to \mathbb{R}$ given by

$$f(x) := \log(x).$$ 

Then $f$ is a bijection, and for $\alpha, \beta \in \mathbb{R}^+$ and $r \in \mathbb{R}$, we have

$$f(\alpha \oplus \beta) = \log(\alpha \beta) = \log(\alpha) + \log(\beta) = f(\alpha) + f(\beta),$$ 

$$f(r \odot \alpha) = \log(\alpha^r) = r \cdot \log(\alpha) = r \cdot f(\alpha).$$

In other words, $f$ is an isomorphism between the structure $\mathbb{R}^+$, with operations $\oplus$ and $\odot$, and $\mathbb{R}$, viewed as an $\mathbb{R}$-vector space.

(b) No. For instance, $1 \boxdot \alpha = 1^\alpha = 1 \neq \alpha$, for all $\alpha \neq 1$. (Also, what is $(-1) \boxdot (1/2)$?)
Problem 3. (20 pts.) For what values of $t \in \mathbb{R}$ does there exist a $\mathbb{Q}$-linear map $f : \mathbb{R} \to \mathbb{R}$ such that $f(1) = 1$ and $f(t) = -1$?

**Answer:** $t \in (\mathbb{R} \setminus \mathbb{Q}) \cup \{-1\}$.

If $t$ is irrational, then the set $\{1, t\}$ is independent over $\mathbb{Q}$. Therefore, there is a basis $B$ for $\mathbb{R}$ as a $\mathbb{Q}$-vector space such that $\{1, t\} \subseteq B$. Thus, any assignment of values to 1 and $t$ (in particular, the assignment $1 \mapsto 1$, $t \mapsto -1$) can be extended to a $\mathbb{Q}$-linear function $\mathbb{R} \to \mathbb{R}$.

On the other hand, if $t$ is rational, then, since $f$ is $\mathbb{Q}$-linear, we must have

$$-1 = f(t) = f(t \cdot 1) = t \cdot f(1) = t \cdot 1 = t,$$

and the linear function $f(x) := x$ satisfies $f(1) = 1$, $f(-1) = -1$. 
Problem 4. (5+5+5+5 pts.) Let $V$ be a vector space over a field $F$ and let $X, Y \subseteq V$ be subspaces. Define the sum $X + Y$ of $X$ and $Y$ by

$$X + Y := \{ x + y : x \in X, y \in Y \}. $$

(a) (5 pts.) Show that $X + Y$ is a subspace of $V$. 

(b) (5 pts.) Show that $X + Y = \text{Span}(X \cup Y)$. 

(c) (5 pts.) Show that if the spaces $X$ and $Y$ are finite-dimensional, then so is $X + Y$. 

(d) (5 pts.) Assuming that the spaces $X$ and $Y$ are finite-dimensional, prove that

$$\dim(X + Y) = \dim X + \dim Y - \dim(X \cap Y).$$

*Hint:* First prove the inequality $\leq$.

(a) Since $X, Y \neq \emptyset$, $X + Y$ is nonempty as well. To show that $X + Y$ is closed under addition, note that for any $x_1, x_2 \in X$ and $y_1, y_2 \in Y$, we have

$$(x_1 + y_1) + (x_2 + y_2) = (x_1 + x_2) + (y_1 + y_2) \in X + Y. $$

Similarly, for all $x \in X, y \in Y$, and $a \in F$,

$$a \cdot (x + y) = \sum_{x \in X} a \cdot x + \sum_{y \in Y} a \cdot y \in X + Y,$$

so $X + Y$ is closed under scaling. Hence, $X + Y$ is a subspace.

(b) Since $X, Y \subseteq X + Y$ and $X + Y$ is a subspace, we have $X + Y \supseteq \text{Span}(X \cup Y)$. Conversely, since $\text{Span}(X \cup Y)$ contains $X$ and $Y$ and is closed under addition, we have $X + Y \subseteq \text{Span}(X \cup Y)$.

(c) Let $B_X \subseteq X$ and $B_Y \subseteq Y$ be finite bases for $X$ and $Y$ respectively. Then $B_X \cup B_Y$ is a finite set that spans $X + Y$, and hence $\dim(X + Y) \leq |B_X \cup B_Y|$ is finite.

(d) Let $B \subseteq X \cap Y$ be an arbitrary basis for $X \cap Y$. Since $B$ is an independent set, there exist bases $B_X \subseteq X$ and $B_Y \subseteq Y$ for $X$ and $Y$ respectively such that $B \subseteq B_X, B_Y$. Then

$$\dim(X + Y) \leq |B_X \cup B_Y| = |B_X| + |B_Y| - |B_X \cap B_Y| = |B_X| + |B_Y| - |B|$$

$$= \dim X + \dim Y - \dim(X \cap Y).$$

We claim that, in fact, $B_X \cup B_Y$ is a basis for $X + Y$, and thus

$$\dim(X \cup Y) = |B_X \cup B_Y| = \dim X + \dim Y - \dim(X \cap Y),$$

as desired. We only need to show that the set $B_X \cup B_Y$ is independent. Consider any linear combination of elements of $B_X \cup B_Y$ that evaluates to zero: $\sum_{v \in B_X \cup B_Y} c(v) \cdot v = 0$. Then we have

$$w := \sum_{x \in B_X} c(x) \cdot x = \sum_{y \in B_Y \backslash B_X} c(y) \cdot y.$$ 

Thus, $w \in \text{Span}(B_X) = X$ and $w \in \text{Span}(B_Y \backslash B_X) \subseteq Y$, hence $w \in X \cap Y$. But $B_Y$ is an independent set, and so $\text{Span}(B_Y \backslash B_X) \cap \text{Span}(B_Y \cap B_X) = \{0\}$, which means that $w = 0$. Since the sets $B_X$ and $B_Y \backslash B_X$ are independent, this implies that $c(v) = 0$ for all $v \in B_X \cup B_Y$, as desired.
**Problem 5.** (20 pts.) Let $V$ be a vector space and let $I, J \subseteq V$ be finite independent sets. Suppose that $|J| > |I|$. Show that there is an element $y \in J \setminus I$ such that the set $I \cup \{y\}$ is independent.

Suppose, towards a contradiction, that for all $y \in J \setminus I$, the set $I \cup \{y\}$ is not independent. This means that for all $y \in J \setminus I$, we have $y \in \text{Span}(I)$; in other words, $J \setminus I \subseteq \text{Span}(I)$. Also, $J \cap I \subseteq I \subseteq \text{Span}(I)$, and hence $J \subseteq \text{Span}(I)$. But $I$ is a basis for $\text{Span}(I)$, hence $\dim \text{Span}(I) = |I| < |J|$, and $\text{Span}(I)$ cannot contain an independent set of size $|J|$; a contradiction.
This page is intentionally left blank for use as scrap paper. If you want your work on this page to be graded, please indicate so clearly (including the problem number).
This page is intentionally left blank for use as scrap paper. If you want your work on this page to be graded, please indicate so clearly (including the problem number).
This page is intentionally left blank for use as scrap paper. If you want your work on this page to be graded, please indicate so clearly (including the problem number).
This page is intentionally left blank for use as scrap paper. If you want your work on this page to be graded, please indicate so clearly (including the problem number).