Supersaturation Problem for Color-Critical Graphs

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Abstract

The *Turán function* $\mathrm{ex}(n, F)$ of a graph F is the maximum number of edges in an F-free graph with n vertices. The classical results of Turán and Rademacher from 1941 led to the study of supersaturated graphs where the key question is to determine $h_F(n, q)$, the minimum number of copies of F that a graph with n vertices and $\mathrm{ex}(n, F) + q$ edges can have.

We determine $h_F(n, q)$ asymptotically when F is *color-critical* (that is, F contains an edge whose deletion reduces its chromatic number) and $q = o(n^2)$.

Determining the exact value of $h_F(n, q)$ seems rather difficult. For example, let c_1 be the limit superior of q/n for which the extremal structures are obtained by adding some q edges to a maximal F-free graph. The problem of determining c_1 for cliques was a well-known question of Erdős that was solved only decades later by Lovász and Simonovits. Here we prove that $c_1 > 0$ for every color-critical F. Our approach also allows us to determine c_1 for a number of graphs, including odd cycles, cliques with one edge removed, and complete bipartite graphs plus an edge.

1 Introduction

The *Turán function* $\mathrm{ex}(n, F)$ of a graph F is the maximum number of edges in an F-free graph with n vertices. In 1907, Mantel [12] proved that $\mathrm{ex}(n, K_3) = \lfloor n^2/4 \rfloor$, where K_r denotes the complete graph on r vertices. The fundamental paper of Turán [20] solved this extremal problem for cliques: the Turán graph $T_r(n)$, the complete r-partite graph of order n with parts of size $\lfloor n/r \rfloor$ or $\lceil n/r \rceil$, is the unique maximum K_{r+1}-free graph of order n. Thus we have $\mathrm{ex}(n, K_{r+1}) = t_r(n)$, where $t_r(n) = |E(T_r(n))|$.

Stated in the contrapositive, this implies that a graph with $t_r(n) + 1$ edges (where, by default, n denotes the number of vertices) contains at least one copy of K_{r+1}. Rademacher (1941, unpublished) showed that a graph on $\lfloor n^2/4 \rfloor + 1$ edges contains not just one but at least $\lfloor n/2 \rfloor$ copies of a triangle. This is perhaps the first result in the so-called "theory of supersaturated graphs" that

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focuses on the function

$$h_F(n,q) = \min\{\#F(G) : |V(G)| = n,\ |E(G)| = \mathrm{ex}(n,F) + q\},$$

the mininum number of F-subgraphs in a graph G with n vertices and $\mathrm{ex}(n,F)+q$ edges. One possible construction is to add some q edges to a maximum F-free graph; let $t_F(n,q)$ be the smallest number of F-subgraphs that can be achieved this way. Clearly, $h_F(n,q) \le t_F(n,q)$.

Erdős [3] extended Rademacher's result by showing that $h_{K_3}(n,q) = t_{K_3}(n,q) = q\lfloor n/2\rfloor$ for $q \le 3$. Later, he [4, 5] showed that there exists some small constant $\epsilon_r > 0$ such that $h_{K_r}(n,q) = t_{K_r}(n,q)$ for all $q \le \epsilon_r n$. Lovász and Simonovits [10, 11] found the best possible value of ϵ_r, settling a long-standing conjecture of Erdős [3]. If fact, the second paper [11] completely solved the $h_{K_r}(n,q)$-problem when $q = o(n^2)$. The case $q = \Omega(n^2)$ of the supersaturation problem for cliques has been actively studied and proved notoriously difficult. Only recently was an asymptotic solution found: Razborov [15] (for K_3), Nikiforov [14] (for K_4), and Reiher [16] (for general K_r).

If F is bipartite, then there is a beautiful (and still open) conjecture of Erdős–Simonovits [19] and Sidorenko [17] whose positive solution would determine $h_F(n,q)$ asymptotically for $q = \Omega(n^2)$. We refer the reader to two recent papers on the topic, by Conlon, Fox, and Sudakov [2] and by Hatami [8], that contain many references.

Obviously, if we do not know $\mathrm{ex}(n,F)$, then it is difficult to say much about the supersaturation problem for small q. A large and important class of graphs for which the Turán function is well understood is formed by *color-critical* graphs, that is, graphs whose chromatic number can be decreased by removing an edge:

Definition 1.1. *A graph F is* r-critical *if $\chi(F) = r+1$ but F contains an edge e such that $\chi(F-e) = r$.*

Simonovits [18] proved that for an r-critical graph F we have $\mathrm{ex}(n,F) = t_r(n)$ for all large enough $n \ge n_0(F)$; furthermore, $T_r(n)$ is the unique maximum F-free graph. The supersaturation problem for a color-critical graph that is not a clique was first considered probably by Erdős [7] who proved that $h_{C_5}(2m,1) = 2m(2m-1)(2m-2)$, where C_k denotes the cycle of length k.

Very recently, Mubayi [13] embarked on a systematic study of this problem for color-critical graphs:

Definition 1.2. *Fix $r \ge 2$ and let F be an r-critical graph. Let $c(n,F)$ be the minimum number of copies of F in the graph obtained from $T_r(n)$ by adding one edge.*

Observe that if n is large enough, then $c(n,F) = t_F(n,1)$.

Theorem 1.3 (Mubayi [13])**.** *For every r-critical graph F, there exists a constant $c_0 = c_0(F) > 0$ such that for all sufficiently large n and $1 \le q < c_0 n$ we have*

$$h_F(n,q) \ge qc(n,F). \tag{1}$$

As it is pointed out in [13], the bound in (1) is asymptotically best possible. Also, (1) is sharp for some graphs F, including odd cycles and $K_4 - e$, the graph obtained from K_4 by deleting an edge.

In this paper, we study the function $h_F(n,q)$. We show that in order to determine $h_F(n,q)$ asymptotically for $q = o(n^2)$, it is enough to consider graphs constructed as follows: $V(H) = X \cup V_1 \cup \ldots \cup V_r$ where $V_1 \cup \ldots \cup V_r$ form a Turán graph, $|X| = O(q/n)$, and V_1 contains some extra edges spread uniformly. Determining the asymptotic behavior of $h_F(n,q)$ then reduces to optimizing a function of $|X|$, the neighborhoods of $x \in X$, and the number of extra edges in V_1. We solve this problem when $q/n \to \infty$ (see Theorem 3.4).

Let $\mathcal{T}_r^q(n)$ be the set of graphs obtained from the Turán graph $T_r(n)$ by adding q edges. These graphs are natural candidates, particularly if q is small, for membership in $\mathcal{H}_F(n,q)$, the set of graphs on n vertices and $\mathrm{ex}(n,F)+q$ edges which contain the fewest number of copies of F. Of particular interest is identifying a threshold for when graphs in $\mathcal{T}_r^q(n)$ are optimal or asymptotically optimal. It is not hard to show, when n is large and $q = o(n^2)$, that

$$t_F(n,q) = (1+o(1))qc(n,F).$$

Formally, we define

$$c_2(F) = \sup\left\{c : \forall \epsilon > 0\ \exists n_0\ \forall n \geq n_0\ \forall q \leq cn\ \ (H \in \mathcal{H}_F(n,q)) \Rightarrow \left(\frac{\#F(H)}{qc(n,F)} \geq 1-\epsilon\right)\right\},$$

to be the threshold for the asymptotic optimality of $\mathcal{T}_r^q(n)$.

Our Theorem 3.5 determines this parameter for every color-critical F. Its statement requires some technical definitions so we postpone it until Section 3. Informally speaking, Theorem 3.5 states that c_2 is the limit inferior of q/n when the following construction starts beating the bound $(1-o(1))qc(n,F)$: add a new vertex x of degree $t_r(n)+q-t_r(n-1)$ to $T_r(n-1)$ so that the number of the created F-subgraphs is minimized. For some instances of F and values of q, this construction indeed wins. On the other hand, there are also examples of F with $c_2(F) = \infty$; in the latter case we prove the stronger claim that $h_F(n,q) = (1+o(1))qc(n,F)$ for all $q = o(n^2)$ (not just for $q = O(n)$).

We then focus on the optimality of $\mathcal{T}_r^q(n)$ and our result qualitatively extends Theorem 1.3 as follows:

Theorem 1.4. *For every r-critical graph F, there exist $c_1 > 0$ and n_0 such that for all $n > n_0$ and $q < c_1 n$, we have $h_F(n,q) = t_F(n,q)$ (in fact, more strongly, we have $\mathcal{H}_F(n,q) \subseteq \mathcal{T}_r^q(n)$).*

A natural question arises here, namely, how large $c_1 = c_1(F)$ in Theorem 1.4 can be. So we define

$$\begin{aligned} q_F(n) &= \max\{q : h_F(n,q') = t_F(n,q') \text{ for all } q' \leq q\}, \\ c_1(F) &= \liminf_{n\to\infty}\left\{\frac{q_F(n)}{n}\right\}. \end{aligned}$$

In 1955 Erdős [3] conjectured that $q_{K_3}(n) \geq \lfloor n/2 \rfloor - 1$ and observed that, if true, this inequality would be sharp for even n. This conjecture (and even its weaker version if $c_1(K_3) \geq 1/2$) remained open for decades until it was finally proved by Lovász and Simonovits [10, 11] whose more general results imply that $c_1(K_{r+1}) = 1/r$ for every r.

Our approach allows us to determine the value of $c_1(F)$ for a number of other graphs. Here are some examples.

Theorem 1.5. *Let F be an odd cycle. Then $c_1(F) = 1/2$.*

Theorem 1.6. *Let $r \geq 2$ and $F = K_{r+2} - e$ be obtained from K_{r+2} by removing an edge. Then $c_1(F) = (r-1)/r^2$.*

Also, we can determine $c_1(F)$ if F is obtained from a complete bipartite graph by adding an edge (see Theorem 4.6) and for a whole class of what we call *pair-free* graphs. Unfortunately, these results are rather technical to state so instead we refer the reader to Section 4.

In all these examples (as well as for $F = K_{r+1}$), if $q < (c_1(F) - \epsilon)n$ and $n \geq n_0(\epsilon, F)$, then not only $h_F(n,q) = t_F(n,q)$ but $\mathcal{H}_F(n,q) \subseteq \mathcal{T}_r^q(n)$, that is, every extremal graph is obtained by adding edges to the Turán graph.

In Theorems 1.5 and 1.6, the c_1-threshold coincides with the moment when the number of copies of F may be strictly decreased by using a non-equitable partition. For example, if $F = C_3 = K_3$ and $n = 2\ell$ is even, then instead of adding $q = \ell$ edges to the Turán graph $T_2(n) = K_{\ell,\ell}$, one can add $q+1$ edges to the larger part of $K_{\ell+1,\ell-1}$ and get fewer triangles. However, some other and more complicated phenomena can occur at the c_1-threshold. In Section 4 we give an example of a graph F such that if we start with $K_{\ell+1,\ell-1}$ (resp. $K_{\ell,\ell}$), then all extra edges have to go into the larger part (resp. have to be divided equally between the parts) and this does affect the value of $c_1(F)$. This indicates that a general formula for $c_1(F)$ may be difficult to obtain.

Interestingly, the congruence class of n modulo r may also affect the value of $q_F(n)$. For example, if $n = 2\ell+1$ and we start with $K_{\ell+2,\ell-1}$ instead of $T_2(n) = K_{\ell+1,\ell}$, then we need to add extra $q+2$ edges (not $q+1$ as it is for even n); so, in fact, $q_{K_3}(2\ell+1)$ is about twice as large as $q_{K_3}(2\ell)$. Hence, we also define the following r constants

$$c_{1,i}(F) = \liminf_{\substack{n\to\infty \\ n\equiv i \bmod r}} \left\{ \frac{q_F(n)}{n} \right\}, \quad 0 \leq i \leq r-1.$$

Clearly, we have $c_1(F) = \min\{c_{1,i}(F) : 0 \leq i \leq r-1\}$. In some cases, we are able to determine the constants $c_{1,i}(F)$ as well.

The rest of the paper is organized as follows. In the next section we introduce the functions and parameters with which we work. Our asymptotic results on the case $q = o(n^2)$, including the value of $c_2(F)$, as well as some general lower bounds on $c_1(F)$ are proved in Section 3. We use the last section to determine $c_1(F)$ for some special graphs.

2 Parameters

In the arguments and definitions to follow, F will be an r-critical graph and we let $f = |V(F)|$ be the number of vertices of F. We identify graphs with their edge set, e.g. $|F| = |E(F)|$. Typically, the order of a graph under consideration will be denoted by n and viewed as tending to infinity. We will use the asymptotic terminology (such as, for example, the expression $O(1)$) to hide constants independent of n. We write $x = y \pm z$ to mean $|x - y| \leq z$.

Let us begin with an expression for $c(n, F)$.

Lemma 2.1. *Let F be an r-critical graph on f vertices. There is a positive constant α_F such that*

$$c(n,F) = \alpha_F n^{f-2} + O(n^{f-3}).$$

This is proved by Mubayi [13] by providing an explicit formula for $c(n,F)$. If F is an r-critical graph, we call an edge e (resp., a vertex v) a *critical edge* (resp., a *critical vertex*) if $\chi(F-e)=r$ (resp., $\chi(F-v)=r$).

Given disjoint sets $V_1,\dots,V_r$, let $K(V_1,\dots,V_r)$ be formed by connecting all vertices $v_i\in V_i, v_j\in V_j$ with $i\neq j$, i.e., $K(V_1,\dots,V_r)$ is the complete r-partite graph on vertex classes $V_1,\dots,V_r$. Let H be obtained from $K(V_1,\dots,V_r)$ by adding one edge xy in the first part and let $c(n_1,\dots,n_r;F)$, where $n_i=|V_i|$, denote the number of copies of F contained in H. Let $uv\in F$ be a critical edge and let χ_{uv} be a proper r-coloring of $F-uv$ where $\chi_{uv}(u)=\chi_{uv}(v)=1$. Let x^i_{uv} be the number of vertices of F excluding u,v that receive color i. An edge preserving injection of F into H is obtained by picking a critical edge uv of F, mapping it to xy, then mapping the remaining vertices of F to H so that no two adjacent vertices get mapped to the same part of H. Such a mapping corresponds to some coloring χ_{uv}. So, with $\mathrm{Aut}(F)$ denoting the number of automorphisms of F, we obtain

$$c(n_1,\dots,n_r;F)=\frac{1}{\mathrm{Aut}(F)}\sum_{uv\ \text{critical}}\sum_{\chi_{uv}}2(n_1-2)_{x^1_{uv}}\prod_{i=2}^{r}(n_i)_{x^i_{uv}}, \tag{2}$$

where $(n)_k=n(n-1)\cdots(n-k+1)$ denotes the *falling factorial*. We obtain a formula for $c(n,F)$ by picking $H\in\mathcal{T}^1_r(n)$. If $r\,|\,n$, we get a polynomial expression in n of degree $f-2$ and α_F is the leading coefficient. Also, if $n_1\le n_2\le\cdots\le n_r$ and $n_r-n_1\le 1$, then

$$c(n,F)=\min\{c(n_1,\dots,n_r;F),c(n_r,\dots,n_1;F)\}. \tag{3}$$

A recurring argument in our proofs involves moving vertices or edges from one class to another, potentially changing the partition of n. To this end, we compare the values of $c(n_1,\dots,n_r;F)$. In [13], Mubayi proves that

$$c(n_1,\dots,n_r;F)\ge c(n,F)+O(an^{f-3})$$

for all partitions $n_1+\ldots+n_r=n$ where $\lfloor n/r\rfloor-a\le n_i\le\lceil n/r\rceil+a$ for every $i\in[r]$. We need the following, more precise estimate:

Lemma 2.2. *There exists a constant ζ_F such that the following holds for all $\delta>0$ and $n>n_0(\delta,F)$. Let $c(n,F)=c(n_1',\dots,n_r';F)$ as in (3). Let $n_1+\ldots+n_r=n$, $a_i=n_i-n_i'$ and $M=\max\{|a_i|\ :\ i\in[r]\}$. If $M<\delta n$, then*

$$c(n,F)-c(n_1,\dots,n_r;F)=\zeta_F a_1 n^{f-3}+O(M^2n^{f-4}).$$

Proof. We bound $c(n_1,\dots,n_r;F)$ using the Taylor expansion about $(n_1',\dots,n_r')$. We first note that $c(n_1,\dots,n_r;F)$ is symmetric in the variables $n_2,\dots,n_r$. Hence,

$$\frac{\partial c}{\partial n_i}(n/r,\dots,n/r)=\frac{\partial c}{\partial n_j}(n/r,\dots,n/r) \tag{4}$$

for all $2\le i,j\le r$. Furthermore, as $|n_i'-n/r|\le 1$ for all $1\le i\le r$,

$$\left|\frac{\partial c}{\partial n_i}(n_1',\dots,n_r')-\frac{\partial c}{\partial n_i}(n/r,\dots,n/r)\right|=O(n^{f-4}).$$

We have

$$c(n_1,\dots,n_r;F)-c(n_1',\dots,n_r';F)=\sum_{j=1}^{r}a_j\frac{\partial c}{\partial n_j}\left(\frac{n}{r},\dots,\frac{n}{r}\right)+O(M^2n^{f-4}).$$

As $\sum_{i=1}^{r} a_i = 0$, the lemma follows from (4) with ζ_F being the coefficient of n^{f-3} in $\frac{\partial c}{\partial n_2}(n/r,\ldots,n/r) - \frac{\partial c}{\partial n_1}(n/r,\ldots,n/r)$. ∎

Definition 2.3. *For an r-critical graph F, let* $\pi_F = \begin{cases} \frac{\alpha_F}{|\zeta_F|} & \text{if } \zeta_F \neq 0 \\ \infty & \text{if } \zeta_F = 0. \end{cases}$

To give a brief foretaste of the arguments to come, we compare the number of copies of a 2-critical graph F in some $H \in \mathcal{T}_2^q(n)$ and a graph H' with $K(V_1, V_2) \subseteq H'$ where $n = 2\ell$ is even, $|V_1| = \ell + 1$, and $|V_2| = \ell - 1$. While H contains q 'extra' edges, $(\ell+1)(\ell-1) = \ell^2 - 1$ implies that the number of 'extra' edges in H' is $q + 1$. Ignoring, for now, the copies of F that use more than one 'extra' edge, we compare the quantities $qc(n, F) \approx q\alpha_F n^{f-2}$ and $(q+1)(\alpha_F n^{f-2} - \zeta_F n^{f-3})$. It becomes clear that the ratio α_F/ζ_F will play a significant role in bounding $c_1(F)$.

Another phenomenon of interest is the existence of a vertex with large degree in each part. Let $\boldsymbol{d} = (d_1, \ldots, d_r)$ and let $\#F(n_1, \ldots, n_r; \boldsymbol{d})$ be the number of copies of F in the graph $H = K(V_1, \ldots, V_r) + z$ where $|V_i| = n_i$ and the extra vertex z has d_i neighbors in V_i. Let $\#F(n, \boldsymbol{d})$ correspond to the case when $n_1 + \ldots + n_r = n - 1$ are almost equal and $n_1 \geq \ldots \geq n_r$.

We have the following formula for $\#F(n_1, \ldots, n_r; \boldsymbol{d})$. An edge preserving injection from F to H is obtained by choosing a critical vertex u, mapping it to z, then mapping the remaining vertices of F to H so that neighbors of u get mapped to neighbors of z and no two adjacent vertices get mapped to the same part. Such a mapping is given by an r-coloring χ_u of $F - u$. Thus

$$\#F(n_1, \ldots, n_r; \boldsymbol{d}) = \frac{1}{\mathrm{Aut}(F)} \sum_{u \text{ critical}} \sum_{\chi_u} \prod_{i=1}^{r} (n_i - y_i)_{x_i} (d_i)_{y_i}$$

where y_i is the number of neighbors of u that receive color i and x_i is the number of non-neighbors that receive color i. We find it convenient to work instead with the following polynomial. For $\boldsymbol{\xi} = (\xi_1, \ldots, \xi_r) \in \mathbb{R}^r$, let

$$P_F(\boldsymbol{\xi}) = \frac{1}{\mathrm{Aut}(F)} \sum_{u \text{ critical}} \sum_{\chi_u} \prod_{i=1}^{r} \frac{1}{r^{x_i}} \xi_i^{y_i}.$$

As a first exercise, let us characterize all connected graphs for which $\deg(P_F) = r$ (we will later treat such graphs separately).

Lemma 2.4. *If F is a connected r-critical graph and* $\deg(P_F) = r$*, then $F = K_{r+1}$ or $r = 2$ and $F = C_{2k+1}$ is an odd cycle.*

Proof. The degree of P_F is determined by the largest degree of a critical vertex. Therefore, $\deg(u) \leq r$ for each critical vertex $u \in F$. However, any r-coloring χ_u of $F - u$ must assign all r colors to the neighbors of u. Thus, $\deg(u) = r$ and $y_i = 1$ for all $i \in [r]$. Therefore, every edge incident to u is a critical edge and, by extension, every neighbor of u is a critical vertex. As F is connected, it follows that every vertex is critical and has degree r. The lemma follows from Brooks' Theorem [1]. ∎

So, if F is r-critical and $\deg(P_F) = r$, then F contains a connected r-critical component isomorphic to K_{r+1} or C_{2k+1} and some (possibly none) connected components that are r-colorable.

We now state a few easy properties of the polynomial $P_F(\boldsymbol{\xi})$.

Lemma 2.5. *$P_F(\boldsymbol{\xi})$ is a symmetric polynomial with nonnegative coefficients.* ∎

Lemma 2.6. *For every $\epsilon > 0$, there exists $\delta > 0$ satisfying the following: if $n = \sum_{i=1}^{r} n_i > 1/\delta$ and if, for all $i \in [r]$, we have $0 \le d_i \le n_i$, $|n_i - n/r| \le \delta n$ and $|\xi_i - d_i/n| \le \delta$, then $|\#F(n_1, \dots, n_r; \boldsymbol{d}) - n^{f-1} P_F(\boldsymbol{\xi})| < \epsilon n^{f-1}$.* ∎

Let us now restrict the domain of P_F to those $\boldsymbol{\xi}$ which may arise as the density vector of some vertex. Note that if $\boldsymbol{d}$ corresponds to the degrees of a vertex and we let $\boldsymbol{\xi} = \boldsymbol{d}/n$, it would follow that $\xi_i \ge 0$ for all $i \in [r]$. Furthermore, as $\sum_{i=1}^{r} d_i \le n - 1$, we have $\sum_{i=1}^{r} \xi_i \le 1$. However, we mostly encounter equitable partitions and, therefore, use the more restrictive set

$$\mathcal{S} = \{\boldsymbol{\xi} \in \mathbb{R}^r : \forall i \in [r]\ 0 \le \xi_i \le 1/r\}.$$

Most of the arguments that follow involve minimizing P_F, usually over some subset of $\mathcal{S}$. One such subset is $\mathcal{S}_\rho = \{\boldsymbol{\xi} \in \mathcal{S} : \sum_{i=1}^{r} \xi_i = \rho\}$ where $\rho \in [0, 1]$. Let

$$p(\rho) = \min\{P_F(\boldsymbol{\xi}) : \boldsymbol{\xi} \in \mathcal{S}_\rho\}.$$

Definition 2.7. *If $\deg(P_F) \ge r+1$, let $\rho_F = \inf\left\{\rho \in (\frac{r-1}{r}, 1) : p(\rho) \le \alpha_F(\rho - \frac{r-1}{r})\right\}$. If $\deg(P_F) = r$ or $p(\rho) > \alpha_F(\rho - (r-1)/r)$ for all $\rho \in (\frac{r-1}{r}, 1)$, then $\rho_F = \infty$.*

Definition 2.8. *If $\deg(P_F) \ge r+1$, let $\hat{\rho}_F = \inf\left\{\rho \in (\frac{r-1}{r}, 1) : p(\rho) < \alpha_F(\rho - \frac{r-1}{r})\right\}$. If $\deg(P_F) = r$ or $p(\rho) \ge \alpha_F(\rho - (r-1)/r)$ for all $\rho \in (\frac{r-1}{r}, 1)$, then $\hat{\rho}_F = \infty$.*

We restrict ρ to the interval $(\frac{r-1}{r}, 1)$ in the definitions above as $p(\rho) = 0$ whenever $\rho \le \frac{r-1}{r}$ and it is easily checked that $p(1) > \alpha_F/r$ unless $\deg(P_F) = r$. Clearly, $\rho_F \le \hat{\rho}_F$. Let us show that ρ_F is strictly greater than $1 - 1/r$.

Lemma 2.9. $\rho_F - (1 - 1/r) > 0$.

Proof. Assume that $\deg(P_F) \ge r + 1$ for otherwise the stated inequality holds by the definition of ρ_F. Given F, choose sufficiently small positive δ and then $\epsilon \ll \delta$. Let us show that $\rho_F - \frac{r-1}{r} \ge \epsilon$.

Take any $\boldsymbol{\xi} \in \mathcal{S}_\rho$ with $0 < \rho - \frac{r-1}{r} < \epsilon$. As P_F is symmetric, assume without loss of generality that $\xi_1 \le \xi_i$ for all $i \in [r]$. Also, we may assume that $\xi_1 < \delta$ for otherwise, since P_F has non-negative coefficients, we are done:

$$P_F(\boldsymbol{\xi}) \ge P_F(\delta, \dots, \delta) > \epsilon\alpha_F > (\rho - (1 - 1/r))\alpha_F.$$

Fix some index i with $2 \le i \le r$. Since $\sum_{j=1}^{r} \xi_j = \rho > \frac{r-1}{r}$, we have that $0 \le 1/r - \xi_i \le \xi_1 < \delta$. Also, the i-th partial derivative of P_F at $\boldsymbol{x_0} = (0, \frac{1}{r}, \dots, \frac{1}{r})$ is 0. (This follows from the combinatorial fact that if the special vertex u from the definition of $F(n_1, \dots, n_r; \boldsymbol{d})$ has no neighbors in the first part, then the number of F-subgraphs at u is identically 0.) Thus, as $\frac{\partial}{\partial \xi_1} P_F(\boldsymbol{x_0}) = \alpha_F > 0$ and δ is small, we have that

$$\frac{\partial}{\partial \xi_1} P_F(\boldsymbol{\xi}) > \frac{\partial}{\partial \xi_i} P_F(\boldsymbol{\xi}) + \frac{\alpha_F}{2}.$$

If follows that, if we increase ξ_i to $1/r$ and decrease ξ_1 by the same amount, then the value of P_F does not go up.

Iteratively repeating the above perturbation for each $i \geq 2$, we obtain the required:

$$P_F(\boldsymbol{\xi}) \geq P_F\left(\rho - \frac{r-1}{r}, \frac{1}{r}, \ldots, \frac{1}{r}\right) > \left(\rho - \frac{r-1}{r}\right) \frac{\partial}{\partial \xi_1} P_F(\boldsymbol{x_0}) = (\rho - (1-1/r))\alpha_F.$$

Here the second inequality follows from the fact that at least one derivative $\frac{\partial^j}{\xi_1^j} P_F(\boldsymbol{x_0})$ with $j \geq 2$ is positive (in view of $\deg(P_F) \geq r+1$). ∎

To give a better picture of proceedings, let us recall some previous parameters. First, consider starting with the Turán graph and 'growing' the graph by adding extra edges. Loosely speaking, the number of copies of F grows 'linearly' with q with a slope of α_F. On the other hand, if we start with a slight perturbation of the partition sizes, we have a slope slightly smaller than α_F (but a higher intercept). The ratio π_F gives the intersection of these two curves. Alternatively, we may start with a Turán graph on one fewer vertices and grow the graph by introducing a vertex of appropriate degree. The number of copies then grows according to $p(\rho)$. In this scenario, ρ_F and $\hat{\rho}_F$ identify the first time when this curve, respectively, intersects and crosses the line of slope α_F. In a sense, the values ρ_F and $\hat{\rho}_F$ signify critical densities when comparing $H \in \mathcal{T}_r^q(n)$ with those graphs obtained by altering the neighborhood of a vertex.

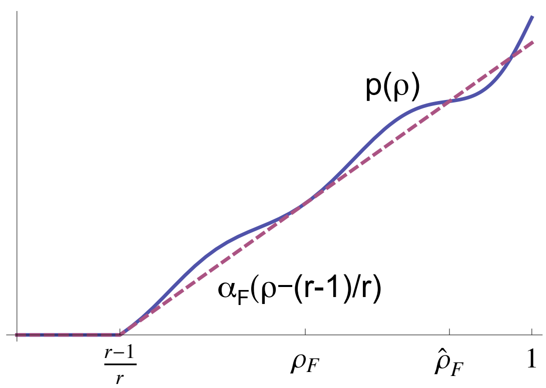

Figure 1: ρ_F and $\hat{\rho}_F$.

The values ρ_F and $\hat{\rho}_F$ in Figure 1 do not coincide. However, $\rho_F = \hat{\rho}_F$ for all graphs we have thus far encountered, and it may be possible that equality holds for all graphs. In some instances, this would imply that $c_1(F) = c_2(F)$.

3 Results

In this section we determine $h_F(n, q)$ asymptotically for $q = o(n^2)$. We then proceed to prove our results on the optimality and asymptotic optimality of graphs in $\mathcal{T}_r^q(n)$. A key step is a lemma on the structure of F-optimal graphs which we prove by building upon the method of Mubayi [13]. As was the case in Mubayi's result, the graph removal lemma (see [9, Theorem 2.9]) and the Erdős-Simonovits Stability Theorem are key components of our proof.

Theorem 3.1 (Graph Removal Lemma)**.** *Let F be a graph with f vertices. Suppose that an n-vertex graph H has at most $o(n^f)$ copies of F. Then there is a set of edges of H of size $o(n^2)$*

whose removal results in a graph with no copies of F.

Theorem 3.2 (Erdős [6] and Simonovits [18]). *Let $r \geq 2$ and F be a graph with chromatic number $r+1$. Let H be a graph with n vertices and $t_r(n) - o(n^2)$ edges that contains no copy of F. Then H can be obtained from $T_r(n)$ by adding and deleting $o(n^2)$ edges.*

Let us start by defining some constants. Given an r-critical graph F, pick constants satisfying the following hierarchy:

$$\delta_0 \gg \delta_1 \gg \delta_2 \gg \delta_3 \gg \delta_4 \gg \delta_5 \gg \delta_6 \gg \delta_7 \gg 1/n_0,$$

each being sufficiently small depending on the previous ones. Let $H \in \mathcal{H}_F(n,q)$ where $n \geq n_0$ and $1 \leq q < \delta_7 n^2$. We fix a max-cut r-partition $V = V_1 \cup \ldots \cup V_r$ of the vertex set $V = V(H)$. We call the edges of H that intersect two parts *good* and those that lie within one part *bad* and denote the sets of good and bad edges by G and B, respectively. Let $M = K(V_1, \ldots, V_r) \setminus G$ be the set of *missing* edges. Observe that $|B| \geq |M| + q$.

Note that every copy of F in H must contain at least one bad edge. To this end, we denote by $\#F(uv)$ the number of copies of F that contain the bad edge $uv \in B$ but no other bad edges. In addition, for $u \in V(H)$, the number of copies of F that use the vertex u is denoted by $\#F(u)$.

As an arbitrary graph in $\mathcal{T}_r^q(n)$ has at most $2\alpha_F \delta_7 n^f$ copies of F, it follows that $h_F(n,q) \leq t_F(n,q) \leq 2\alpha_F\delta_7 n^f$. Thus the Removal Lemma applies to H and gives an F-free subgraph $H' \subseteq H$ with at least $t_r(n) - \delta_6 n^2$ edges. We then apply the Erdős-Simonovits Stability Theorem and obtain an r-partite subgraph $H'' \subseteq H'$ with at least $t_r(n) - \delta_5 n^2$ edges. We observe here that $n_i = |V_i| = n/r \pm \delta_4 n$ and $|B| \leq 2\delta_5 n^2$.

Lemma 3.3. *If $M \neq \emptyset$, then there exists a vertex x with $d_B(x) \geq \delta_1 n$ and $d_H(x) \geq \left(\frac{r-1}{r} + \delta_2\right) n$.*

Proof of Lemma 3.3 Assume that $M \neq \emptyset$. We first show that $\Delta(B) \geq \delta_1 n$.

Assume, for contradiction, that $d_B(u) < \delta_1 n$ for all $u \in V$. Let $uv \in M$ and let

$$\#F'(uv) = \#F(H + uv) - \#F(H)$$

be the number of potential copies of F associated with uv, that is, the number of copies of F introduced by including the edge uv. As $d_B(u) + d_B(v) < 2\delta_1 n$, it follows that $\#F'(uv) < 2\delta_1 n^{f-2}$. However, note that

$$\#F(xy) \leq \#F'(uv) \tag{5}$$

for all $xy \in B$, as otherwise we may reduce the number of copies of F by removing xy and replacing it by uv. As

$$\#F(xy) \geq \alpha_F n^{f-2} - (d_M(x) + d_M(y))Cn^{f-3} - \delta_1 n^{f-2} \tag{6}$$

for some $C = C(F) > 0$, (5) implies that $d_M(x) + d_M(y) \geq (\alpha_F - 3\delta_1)n/C$. Now, counting in two ways the number of adjacent pairs (e, e') with $e \in B$ and $e' \in M$, we obtain

$$2\Delta(B)|M| \geq (\alpha_F - 3\delta_1)n|B|/C,$$

contradicting the fact that $|B| \geq |M|$.

Let x be a vertex with $d_B(x) = \Delta(B) \geq \delta_1 n$. As we picked a max-cut partition, $|N(v) \cap V_i| \geq \delta_1 n$ for all $i \in [r]$, it follows that $\#F(x) \geq P_F(\delta_1, \ldots, \delta_1)n^{f-1} - \delta_3 n^{f-1} \geq P_F(\delta_1, \ldots, \delta_1)n^{f-1}/2$. As

$P_F(\boldsymbol{\xi})$ is continuous and $P_F(1/r,\ldots,1/r,0)=0$, we have $P_F(1/r,\ldots,1/r,\delta_2)\le P_F(\delta_1,\ldots,\delta_1)/4$. Hence, unless $d_H(x)\ge(\frac{r-1}{r}+\delta_2)n$, we can strictly reduce the number of copies of F by connecting x to all vertices in other classes and to at most $\delta_2 n$ vertices in its own class. ∎

We now apply Lemma 3.3 to provide a lower bound on the value of $h_F(n,q)$. In so doing, we describe a class of graphs that allow us to determine $h_F(n,q)$ asymptotically. If the first case of Lemma 3.3 holds (i.e., $M=\emptyset$), then we have the inequality

$$\#F(H)\ge|B|c(n_1,\ldots,n_r;F)\ge(1-\delta_3)qc(n,F).$$

On the other hand, if $M\neq\emptyset$, we build a sequence (H_i,q_i,x_i) as follows. Initially, set $i=0$, $H_0=H$, and $q_0=q$. We iterate by picking a vertex x_i in H_i satisfying Lemma 3.3 and let $q_{i+1}=|H_i-x_i|-t_r(n-i)$. Finally, we pick $H_{i+1}\in\mathcal{H}_F(n-i-1,q_{i+1})$, i.e., H_{i+1} is a graph of the same order and size as H_i-x_i that minimizes the number of F-subgraphs. If $\#F(H_{i+1})\ge(1-\delta_3)q_{i+1}c(n,F)$, and, in particular, if $q_{i+1}\le 0$, we let $k=i+1$ and stop.

Note that, by construction,

$$\#F(H)\ge\#F(H_k)+\sum_{i=0}^{k-1}\#F(x_i). \tag{7}$$

As $d_{B_i}(x_i)\ge\delta_1(n-i)$, it follows that $\#F(x_i)\ge\delta_2(n-i)^{f-1}$ for all $i<k$. In particular, as $\#F(H)\le 2\alpha_F\delta_7 n^f$, we have that $k\le\delta_6 q/n$. Rewrite (7) as

$$\#F(H)\ge(1-\delta_3)q_k c(n,F)+n^{f-1}\sum_{i=0}^{k-1}p\left(\frac{d_{H_i}(x_i)}{n}\right)-k\,\delta_3 n^{f-1}. \tag{8}$$

On the other hand, given the vector $(d_{H_0}(x_0),\ldots,d_{H_{k-1}}(x_{k-1}),q_k)$, one can construct a graph that achieves (8) asymptotically. Specifically, we can start with the graph $T_r(n-k)$ and, for each $i\in[0,k-1]$, add a vertex u_i whose density vector is in $\mathcal{S}_{\rho_i}$ and minimizes $F(u_i)$, where $\rho_i=d_{H_i}(x_i)/n$. Next, we add q_k (plus up to k^2) edges uniformly into one part. The number of copies of F that use more than one vertex u_i may be bounded from above by $k^2 f!\,n^{f-2}$. Therefore, this new graph contains at most

$$(1+\delta_3)q_k c(n,F)+(n-k)^{f-1}\sum_{i=0}^{k-1}p\left(\frac{d_{H_i}(x_i)}{n}\right)+k^2c(n,F)+k^2f!\,n^{f-2} \tag{9}$$

copies of F. Note that the sum of the last two terms in (9) is dominated by the sum of the first two terms.

The above discussion provides us with asymptotically optimal graphs for $q=o(n^2)$ and, in some cases, allows us to determine $h_F(n,q)$ asymptotically. Let β_F be the infimum of the ratio $\frac{p(\rho)}{\rho-(1-1/r)}$ (observe that $\beta_F=\alpha_F$ if $\hat{\rho}_F=\infty$ and $\beta_F<\alpha_F$ otherwise).

Theorem 3.4. *If $q=o(n^2)$ and $q/n\to\infty$, then*

$$h_F(n,q)=(\beta_F+o(1))qn^{f-2}.$$

Proof of Theorem 3.4 It follows, by (8) and the definition of β_F, that $h_F(n,q)\ge(\beta_F-o(1))qn^{f-2}$. On the other hand, we may construct a graph as above where each vertex u_i has density vector $\boldsymbol{\xi}$ with $P_F(\boldsymbol{\xi})=\beta_F+o(1)$, thereby proving the upper bound. ∎

If $q = O(n)$, then $k = O(1)$ and determining $h_F(n,q)$ reduces to an optimization problem. To be more precise, for fixed $c > 0$ and $q = (c + o(1))n$, we have that $h_F(n,q) = (1 + o(1))\phi(c)n^{f-1}$ where

$$\phi(c) = \min_k \min_{\boldsymbol{s}} \left(s_k \alpha_F + \sum_{i=0}^{k-1} p(s_i) \right),$$

with the minimum taken over all $k \in \mathbb{Z}$ and all $\boldsymbol{s} \in \mathbb{R}^{k+1}$ satisfying $\sum_{i=0}^{k} s_i = c + k(r - 1/r)$, $s_k \geq 0$, and $s_i \geq (r-1)/r + \delta_2$ for $0 \leq i \leq k-1$. We can now compute $c_2(F)$ as the supremum of c for which $\phi(c) = \alpha_F c$.

Theorem 3.5. $c_2(F) = \hat{\rho}_F - (1-1/r)$. *Furthermore, if* $\hat{\rho}_F = \infty$, *then* $h_F(n,q) = (1+o(1))qc(n,F)$ *for all* $q = o(n^2)$.

Proof of Theorem 3.5 Let us first show that $c_2(F) \leq \hat{\rho}_F - (1 - 1/r)$. We may assume that $\hat{\rho}_F$ is finite, for otherwise the upper bound holds vacuously. Let $c > \hat{\rho}_F - (1 - 1/r)$ be arbitrary. Take $\boldsymbol{\xi} \in \mathcal{S}_\rho$ such that $\hat{\rho}_F < \rho < c + \frac{r-1}{r}$ and $\lambda > 0$, where $\lambda = \alpha_F(\rho - \frac{r-1}{r}) - P_F(\boldsymbol{\xi})$.

Let n be large. Let H be obtained from $T_r(n-1)$ by adding a new vertex u that has $(\xi_i + o(1))n$ neighbors in each part V_i. Thus H has $t_r(n) + q$ edges, where $q = (\rho - \frac{r-1}{r} + o(1))n$. Then

$$\#F(H) = (P_F(\boldsymbol{\xi}) + o(1))\, n^{f-1} < \left(\alpha_F \left(\rho - \frac{r-1}{r}\right) - \lambda/2\right) n^{f-1} < (1 - \lambda/3)qc(n,F),$$

This infinite sequence of graphs implies the stated upper bound on $c_2(F)$.

Now assume that $q \leq cn$ where $c < (\hat{\rho}_F - (1-1/r))$ or that $\hat{\rho}_F = \infty$ and $q = o(n^2)$. We apply Lemma 3.3 to some $H \in \mathcal{H}_F(n,q)$ and obtain the sequence (H_i, x_i, q_i) as in the discussion above. First, if $M = \emptyset$ (in other words, $k = 0$ and $q_k = q$), then $\#F(H) \geq (1-\delta_3)qc(n,F)$ and we are done. So, suppose now that $M \neq \emptyset$ and $k \geq 1$. If there exists some x_i with $d_{H_i}(x_i) \geq (\hat{\rho}_F - (1-1/r))(n-i)$, then, by monotonicity of $p(\rho)$, we have that

$$\#F(x_i) \geq (p(\hat{\rho}_F) - \delta_3)(n-i)^{f-1} > (1-\delta_2)qc(n,F).$$

Finally, if $d_{H_i}(x_i) < (\hat{\rho}_F - (1-1/r))(n-i)$ for all $i \leq k$, then

$$\#F(x_i) \geq \alpha_F(d_{H_i}(x_i) - (1-1/r)n)n^{f-2} - \delta_3 n^{f-1}.$$

We get the required inequality by summing this quantity over all vertices x_i as in (7). ∎

We now prove Theorem 1.4 in a stronger sense. Recall that the sign of $x \in \mathbb{R}$, denoted $\operatorname{sgn}(x)$, is 0 if x is 0 and $x/|x|$ otherwise. For notational convenience define

$$\theta_F = \rho_F - (1-1/r).$$

Theorem 3.6. *Let F be an r-critical graph. Then,* $c_1(F) > 0$. *Specifically,*

- *if* $\deg(P_F) = r$, *then* $c_1(F) \geq 1/r$,
- *if* $\deg(P_F) \geq r+1$, *then* $c_{1,i}(F) \geq c_1(F) \geq \min(\pi_F, \theta_F)$ *for all* $i \in [r]$.
  Furthermore, $c_{1,t}(F) \geq \min(2\pi_F, \theta_F)$ *for* $t = \operatorname{sgn}(\zeta_F)$.

Proof of Theorem 3.6 As stated in the Introduction, we prove the theorem by showing something stronger; namely, for n large, $c \in \mathbb{R}$ in the appropriate range, and $q < cn$, not only is $h_F(n,q) = t_F(n,q)$ but, in fact, $\mathcal{H}(n,q) \subseteq \mathcal{T}_r^q(n)$.

Let $c(n,F) = c(n'_1, \ldots, n'_r; F)$ as in (2). Pick some $c > 0$ and consider $H \in \mathcal{H}_F(n,q)$ where $q < cn$. Let $V(H) = V_1 \cup \ldots \cup V_r$ be a max-cut partition with $|V_1| \geq |V_2| \geq \ldots \geq |V_r|$, let G, B and M be the sets of good, bad and missing edges, respectively, and recall the sequence of constants δ_i. Consider the following cases:

Case 1: $M = \emptyset$.

Let $a = \max(|V_1| - \lceil n/r \rceil, \lfloor n/r \rfloor - |V_r|)$. We have

$$q + \frac{a^2 r}{2(r-1)} \leq |B| \leq q + \frac{a^2 r}{2}.$$

As $a \leq \delta_4 n$, Lemma 2.2 implies that

$$\begin{aligned} \#F(H) &\geq \left(q + \frac{a^2 r}{2(r-1)}\right)\left(c(n,F) - |\zeta_F| a n^{f-3} - \frac{a^2 n^{f-4}}{\delta_0}\right) \\ &\geq qc(n,F) + a n^{f-2}\left(\alpha_F \frac{ar}{2(r-1)} - c\,|\zeta_F| - \delta_1 a\right). \end{aligned}$$

So, if $a \geq 2(c|\zeta_F| + 1)/(\alpha_F - \delta_1)$, we have $\#F(H) > qc(n,F) + an^{f-2}$. On the other hand, $t_F(n,q) \leq qc(n,F) + O(n^{f-2})$, which is demonstrated by adding extra q edges to one part of $T_r(n)$ so that they form a graph of bounded maximum degree. Since H is optimal, we have $a = O(1)$.

We now refine the argument to show that if $c < \pi_F - \delta_0$, then all optimal graphs are contained in the set $\mathcal{T}_r^q(n)$. In other words, if $|H| = t_r(n) + q$ and H contains the complete r-partite graph on parts of size $n_1, \ldots, n_r$ where $n = n_1 + \ldots + n_r$ and $n_1 \geq n_r + 2$, then H is not optimal.

For $i \in [r]$, let $B_i = B[V_i]$ be the set of bad edges contained in V_i.

Claim 3.7. *If $|V_j| = |V_k| + s$, where $s > 1$, then*

$$(|B_j| - |B_k|)\zeta_F \geq (s-1)(1-\delta_3)\alpha_F n. \tag{10}$$

Proof of Claim. Assume otherwise. Consider H' obtained from H by moving one vertex from V_j to V_k. Namely, pick a vertex $v \in V_j$ with $d_B(v) \leq d_B(u)$ for all $u \in V_j$. We replace v with a vertex v' such that $uv' \in H'$ for all $u \in V \setminus V_k$. Next, we pick $d_B(v)$ vertices in V_k with low bad degrees as neighbors of v'. Then, as $(|V_j| - 1)(|V_k| + 1) = |V_j||V_k| + s - 1$, we remove $s - 1$ bad edges chosen arbitrarily.

The change in the cardinalities of V_j and V_k alters the number of copies of F for each bad edge. However, by Lemma 2.2, the difference between $\#F_H(e)$ and $\#F_{H'}(e)$ is $O(n^{f-4})$ unless $e \in B_j \cup B_k$ or e is one of the $s-1$ edges that were deleted. So,

$$\begin{aligned} \#F(H) - \#F(H') &= (s-1)c(n,F) + \sum_{e \in B_j \cup B_k} (\#F_H(e) - \#F_{H'}(e)) + O(n^{f-3}) \\ &= (s-1)\alpha_F n^{f-2} - (|B_j| - |B_k|)\zeta_F n^{f-3} + O(n^{f-3}), \end{aligned}$$

where the second identity follows, once again, from Lemma 2.2. As H is optimal, (10) must hold. ∎

Note that if $t = 0$, then $\zeta_F = 0$ and Claim 3.7 cannot be satisfied. Therefore, $|V_j| - |V_k| \leq 1$ for all $1 \leq j, k \leq r$, that is, $H \in \mathcal{T}_r^q(n)$.

Suppose that $t \in \{-1, 1\}$ and $H \notin \mathcal{T}_r^q(n)$. Then there exist j, k such that $|V_j| - |V_k| \geq 2$. Hence, $|B| \geq \max(|B_j|, |B_k|) \geq (1 - \delta_3)\pi_F n$, and $q \geq |B| - a^2 \geq (1 - \delta_0)\pi_F n$ as required. Next, if $n \equiv t \pmod{r}$, then there exist j, k, l with $|V_j| - |V_k| \geq 3$ or $j \neq k$ and $|V_j| - |V_l| = |V_k| - |V_l| = 2t$. In the first case, we apply Claim 3.7 directly to obtain $q \geq 2(1 - \delta_0)\pi_F$. On the other hand, if the second case holds with $t\zeta_F > 0$, we have, by applying Claim 3.7 twice, that $|B_j|, |B_k| \geq (1 - \delta_3)\pi_F$, again implying that $q \geq 2(1 - \delta_0)\pi_F$.

Case 2: $M \neq \emptyset$.

As $M \neq \emptyset$, it follows from Lemma 3.3 that $X \neq \emptyset$, where we define

$$X = \left\{ x \in V(H) : d_B(x) \geq \delta_1 n \text{ and } d_H(x) \geq \left(\frac{r-1}{r} + \delta_2 \right) n \right\}. \tag{11}$$

We will now handle the two cases $\deg(P_F) = r$ and $\deg(P_F) \geq r + 1$ separately. Let us first consider the case $\deg(P_F) \geq r + 1$.

Claim 3.8. *If $\deg(P_F) \geq r + 1$, then $d(x) \geq (\rho_F - \delta_2)n$ for all $x \in X$.*

Proof of Claim. Let, for example, $x \in X \cap V_1$ contradict the claim. By the definition of ρ_F, we have $p(\rho) > \alpha_F(\rho - \frac{r-1}{r})$ for $\rho \in (\frac{r-1}{r}, \rho_F)$. Since p is a continuous function, we can assume that $p\left(\frac{d(x)}{n}\right) - \alpha_F\left(\frac{d(x)}{n} - \frac{r-1}{r}\right) \geq 5\delta_3$.

Let us replace x with a vertex u whose neighborhood is $V(H) \setminus V_1$. Clearly, $\#F(u) \leq \delta_3 n^{f-1}$. Next, we distribute the remaining $d(x) - d(u)$ edges evenly among vertices in V_1 with bad degree at most $\delta_4 n$. This creates at most $(d(x) - \frac{r-1}{r}\, n)c(n, F) + 2\delta_3 n^{f-1}$ copies of F. Comparing with the old value $\#F(x) \geq p(d(x)/n) - \delta_3 n^{f-1}$, we see that the number of F-subgraphs decreases, a contradiction to the optimality of H. ▌

Note that, by Claim 3.8 and Definition 2.7, we have

$$\rho_F < 1 \tag{12}$$

and $\theta_F = \rho_F - (r-1)/r \in (0, 1/r)$. Furthermore, if $x \in X$, then

$$\#F(x) \geq n^{f-1}(p(\rho_F - \delta_2) - \delta_3) > n^{f-1}(\alpha_F\theta_F - \delta_1) > (\theta_F - \delta_0)nc(n, F) + \delta_1 n^{f-1}.$$

Thus, if $c \leq \theta_F - \delta_0$, the number of copies of F at some vertex $x \in X$ exceeds the bound of $qc(n, F) + O(n^{f-2})$, contradicting our assumption that $\#F(H) = h_F(n, q)$.

This completes the proof of Theorem 3.6 for r-critical graphs with $\deg(P_F) \geq r + 1$.

We now consider the case when $\deg(P_F) = r$. We will first show that $t_F(n, q) = qc(n, F)$ for $q \leq \lfloor n/r \rfloor - 1$. This value is obtained by the graph $H^*(n, q) \in \mathcal{T}_r^q(n)$ constructed as follows: $V(H^*) = U_1 \cup \ldots \cup U_r$ where $|U_i| = n_i$ is either $\lceil n/r \rceil$ or $\lfloor n/r \rfloor$, $c(n, F) = c(n_1, \ldots, n_r; F)$ and $E(H^*) = K(U_1, \ldots, U_r) \cup K(\{u^*\}, W)$, where $u^* \in U_1$, $W \subseteq U_1 \setminus \{u^*\}$ and $|W| = q$. That is, $H^*(n, q)$ is obtained from $T_r(n)$ by adding (the edges of) a star of size q in U_1. Observe that any copy of F in H^* must use the vertex u^*. Furthermore, u^* is contained in the r-critical component, which, in this case, is isomorphic to K_{r+1} or C_{2k+1}. So, each copy of F uses exactly one bad edge incident to u^* and $\#F(H^*(n, q)) = t_F(n, q) = qc(n, F)$.

Now let $H \in \mathcal{H}_F(n, q)$ where $q \leq (1/r - \delta_0)n$. Recall the set X defined in (11).

Claim 3.9. *Every $x \in X$ is incident to at most $(r-1)\delta_2 n$ missing edges.*

Proof of Claim. Let $x \in X \cap V_1$ with d_i neighbors in each part V_i. By the max-cut property we have $d_1 \leq d_i$ for all $i \in [r]$. In particular, each d_i is at least $\delta_1 n$.

If, for example, x has at least $\delta_2 n$ non-neighbors in V_2, then we can move $\delta_2 n/2$ edges at x from V_1 to V_2, decreasing the product $d_1 d_2$ by at least $3\delta_3 n^2$. Since $P_F(\boldsymbol{\xi}) = C_F \prod_{i=1}^r \xi_i$ for some constant $C_F > 0$, this would strictly decrease $\#F(x)$, a contradiction to the extremality of H. The claim follows. ▌

Claim 3.10. *Every missing edge intersects X.*

Proof of Claim. Assume there exists $uv \in M$ with $u, v \notin X$. As both endpoints have bad degree of at most $\delta_1 n$, it follows that $\#F'(uv) \leq 2\delta_1 n^{f-2}$. On the other hand, consider a vertex $x \in X$. There is a bad edge xw such that $d_M(w) < \delta_3 n$ (otherwise, $|B| \geq |M| > 2\delta_4 n^2$). By Claim 3.9, we have $\#F(xw) \geq \alpha_F n^{f-2} - \delta_1 n^{f-2} > \#F'(uv)$, resulting in a contradiction. ▌

As $\#F(x) \geq \delta_2 n^{f-1}$ for all $x \in X$ and $\#F(H) \leq 2\alpha_F \delta_7 n^f$, it follows that $|X| = o(n) < \delta_4 n$. Thus, by Claims 3.9 and 3.10 we have for every $u \in V(H)$ that

$$d_M(u) \leq \max\Big(|X|, (r-1)\delta_2 n\Big) = (r-1)\delta_2 n.$$

It follows that $\#F(u'v') \geq (1-\delta_1)c(n,F)$ for every bad edge $u'v' \in B$. That is,

$$|B| \leq \frac{q}{1-\delta_1} < (1/r - \delta_0/2)n. \tag{13}$$

Now pick some vertex u^* and consider a graph H' where all $|B|$ bad edges and $|M|$ missing pairs are incident to u^*. This procedure removes all copies of F using multiple bad edges. Furthermore, observe that if a missing pair uv and a bad edge $u'v'$ are disjoint, then we can increase the number of potential copies that contain both by making them adjacent. Therefore, by choosing u^* appropriately, we have $\#F_H(u'v') \geq \#F_{H'}(u^*w)$ for every $u'v' \in B(H)$ and $u^*w \in B(H')$. However, as $H \in \mathcal{H}_F(n,q)$, we have $\#F(H) = \#F(H')$ and $\#F_{H'}(u^*w) \geq (1-\delta_2)c(n,F)$. Now, by construction,

$$\#F'_{H'}(u^*v) \leq r\alpha_F n^{f-3} d_B(u^*) + \delta_2 n^{f-2}$$

for all $u^*v \in M(H')$. It follows that

$$|B(H)| = |B(H')| = d_B(u^*) > (1/r - \delta_2)n,$$

contradicting (13).

This completes the proof of Theorem 3.6. ▌

4 Special Graphs

In this section we obtain upper bounds on $c_{1,i}(F)$ for a class of graphs and compute the exact value for some special instances. We also give an example of a graph with $c_1(F)$ strictly greater than $\min(\pi_F, \theta_F)$.

4.1 $K_{r+2} - e$.

Here we prove Theorem 1.6.

Let $r \geq 2$ and let $F = K_{r+2} - e$ be obtained from the complete graph K_{r+2} by deleting one edge. Clearly, F is r-critical. In addition, if uv is the edge removed from K_{r+2}, we may further reduce the chromatic number by removing an edge xy where $\{x, y\} \cap \{u, v\} = \emptyset$. It follows that

$$c(n_1, \ldots, n_r; F) = \sum_{i=2}^{r} \binom{n_i}{2} \prod_{\substack{2 \leq j \leq r \\ j \neq i}} n_j = \frac{(n - n_1 - r + 1)}{2} \prod_{i=2}^{r} n_i.$$

Therefore, $\alpha_F = \frac{r-1}{2r^r}$, $\zeta_F = \frac{1}{2r^{r-2}}$, and $\pi_F = \frac{r-1}{r^2}$.

On the other hand,

$$P_F(\boldsymbol{\xi}) = \frac{1}{2} \sum_{i=1}^{r} \xi_i^2 \prod_{\substack{1 \leq j \leq r \\ j \neq i}} \xi_j = \frac{1}{2} \left(\sum_{i=1}^{r} \xi_i \right) \prod_{i=1}^{r} \xi_i.$$

Therefore, if $\sum_{i=1}^{r} \xi_i = \rho$ is fixed, then by convexity $P_F(\boldsymbol{\xi})$ is minimized by picking $\boldsymbol{\xi} = (\rho - \frac{r-1}{r}, 1/r, \ldots, 1/r)$, implying that $\rho_F = \infty$.

Theorem 3.6 now implies that $c_1(F) \geq \pi_F$, so we only prove the upper bound $c_1(F) \leq \pi_F$. In fact, it suffices to show that $c_{1,0}(F) \leq \pi_F$ and we consider only large n that are multiples of r (to simplify computations, we will actually require that n be divisible by $2r$). As $\rho_F = \infty$, it follows that $M(H) = \emptyset$ for any $H \in \mathcal{H}_F(n, q)$; otherwise, (12) is violated. Therefore, we need only to compare graphs obtained from a complete r-partite graph by adding extra edges.

First, for $q \leq n/r$, we identify a graph $H^* \in \mathcal{T}_r^q(n)$ for which $\#F(H) = t_F(n, q)$. We then show that this value may be beaten by using a non-equitable partition. A key observation is that all the bad edges in H^* are contained in one part.

Let us now estimate the number of copies of F formed by pairs of bad edges in a graph $H \in \mathcal{T}_r^q(n)$. Let $V(H) = V_1 \cup \ldots \cup V_r$ with $|V_i| = n/r$ for all $i \in [r]$ and let B_i be the set of bad edges both of whose endpoints lie in the part V_i. Let $u_1v_1 \in B_i$ and $u_2v_2 \in B_j$.

1. If $i = j$ and the edges u_1v_1 and u_2v_2 have a common endpoint, we may create a copy of F by picking one vertex each from V_k where $k \neq i$. Therefore, we have $(n/r)^{r-1}$ copies of F containing both bad edges.

2. If $i \neq j$, we form a copy of K_{r+2} by picking a vertex from each of the parts V_k where $k \notin \{i, j\}$. We may then choose any of the $\binom{r+2}{2} - 2$ edges (except for u_1v_1 and u_2v_2) to be the one missing in F. In addition, for any choice of $k_1, k_2 \notin \{i, j\}$, we may pick 2 vertices from V_{k_1}, no vertices from V_{k_2} and one vertex each from V_l where $l \notin \{i, j, k_1, k_2\}$ to form a copy of F. So, the number of copies of F containing both edges is

$$\left(\binom{r+2}{2} - 2 \right) (n/r)^{r-2} + (r-2) \binom{n/r}{2} (r-3)(n/r)^{r-4}.$$

We now form $H^* \in \mathcal{T}_r^q(n)$ by placing all q bad edges in V_1 as follows: enumerate the bad edges as $e_1, e_2, \ldots, e_q$ and the vertices in V_1 as $v_1, v_2, \ldots, v_{n/r}$. Then, H^* contains the bad edges $e_i = v_{2i-1}v_{2i}$ for $i \leq n/(2r)$ and $e_{n/2r+j} = v_{2j}v_{2j+1}$ for $1 \leq j \leq q - n/(2r)$. That is, if possible, all bad edges in H^* form a matching. However, if $q > n/(2r)$, the bad edges form a path and disjoint edges.

Claim 4.1. *If $q \le n/r$, then $t_F(n,q) = \#F(H^*)$.*

Proof of Claim. We need to show that $\#F(H^*) \le \#F(H)$ for all $H \in \mathcal{T}_r^q(n)$. If $q \le n/(2r)$, we note that $\#F(H^*) = qc(n,F)$, which is a trivial lower bound for all $H \in \mathcal{T}_r^q(n)$. So we consider the case $n/(2r) \le q \le n/r$. In this case, we observe that $\#F(H^*) = qc(n,F) + 2(q - (n/2r))(n/r)^{r-1}$.

Now assume $\#F(H) = t_F(n,q)$ for some graph $H \in \mathcal{T}_r^q(n)$. Assume, without loss of generality, that $|B_1| \ge |B_i|$ for al $i \in [r]$. If $|B_1| \le n/(2r) - 1$, then $B \setminus B_1 \neq \emptyset$. Say $B_2 \neq \emptyset$, and consider removing an edge in B_2 and adding an isolated edge in B_1. Then, the number of copies of F is reduced by at least $(|B_1| - |B_2| + 1)(n/r)^{r-2}(\binom{r+2}{2} - 2)$, contradicting optimality of H.

On the other hand, if $|B_1| \ge n/(2r)$, then every edge $uv \in B \setminus B_1$ forms at least

$$c(n,F) + \left(\binom{r+2}{2} - 2\right)(n/r)^{r-2}|B_1| \ge c(n,F) + 2(n/r)^{r-1}$$

copies of F. In addition, by convexity, $\sum_{v \in V_1} \binom{d_B(v)}{2}$ is minimized when there are exactly $2|B_1| - n/r$ vertices of degree 2 and all remaining vertices have degree 1. As each vertex of degree 2 gives $(n/r)^{r-1}$ copies of F that use both edges incident to it. It follows that

$$\begin{aligned}
\#F(H) &\ge qc(n,F) + 2(n/r)^{r-1}\left(|B \setminus B_1| + |B_1| - n/(2r)\right) \\
&= qc(n,F) + (2q - n/r)(n/r)^{r-1} \\
&= \#F(H^*). \blacksquare
\end{aligned}$$

Now consider a graph H on partition $n = n_1 + n_2 + \ldots + n_r$ where $n_1 = n/r + 1$, $n_2 = n/r - 1$ and $n_i = n/r$ for $i \ge 3$ with $K(V_1, \ldots, V_r) \subseteq H$ and all $q+1$ bad edges contained in V_1 as in H^*. Then

$$\#F(H) \le (q+1)(c(n,F) - \zeta_F n^{r-1}) + (2q + 2 - n/r)(n/r)^{r-1} + \delta_0 n^{r-1}.$$

In particular, if $q \ge (\pi_F + \delta_0)n$, then

$$\#F(H) - t_F(n,q) \le \alpha_F n^r - (\pi_F + \delta_0)\zeta_F n^r + o(n^r) < -\delta_0 \zeta_F n^r/2,$$

thus proving the upper bound $c_{1,0}(F) \le \pi_F$, as required.

4.2 Non-tightness of Theorem 3.6

We now exhibit a graph for which $c_1(F) > \min(\pi_F, \theta_F)$. Let F be the graph in Figure 2. Interestingly, for this graph, $\rho_F = \hat{\rho}_F = \infty$, so we only have to show that $c_1(F) > \pi_F$. This inequality is strict because, for not too large q, we can reduce the number of copies of F by distributing the bad edges among the two parts of $K_{n/2,n/2}$ instead of placing them all into one part.

Theorem 4.2. $c_{1,0}(F) = \frac{3-\sqrt{5}}{4} > 1/6 = \pi_F$ *and* $c_{1,1}(F) = 1/3$.

Proof. First note that F is 2-critical and ab is the unique critical edge. There is a unique (up to isomorphism) 2-coloring χ of $F - ab$ with $\chi^{-1}(1) = \{a,b,f\}$ and $\chi^{-1}(2) = \{c,d,e,g\}$. It readily follows that

$$c(n_1, n_2; F) = \binom{n_2}{3}(n_1 - 2)(n_2 - 3)$$

and $\alpha_F = (3! \cdot 2^5)^{-1}$. Taking derivatives, we observe that $\zeta_F = 2^{-5}$ and $\pi_F = 1/6$.

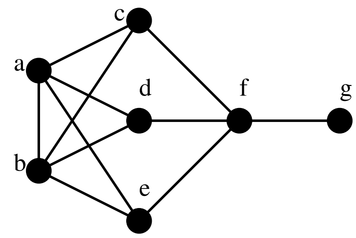

Figure 2: Example for non-tightness of Theorem 3.6.

We also have

$$P_F(\boldsymbol{\xi}) = \frac{1}{4 \cdot 3!}(\xi_1\xi_2^3 + \xi_1^3\xi_2),$$

which, if we fix $\xi_1 + \xi_2$, is minimized by maximizing the difference. Hence, $\rho_F = \infty$.

However, note that there exists a 2-coloring χ^* of $F - ab - fg$ with $\chi^*(a) = \chi^*(b) = \chi^*(f) = \chi^*(g) = 1$. In fact, if u_1v_1, u_2v_2 are two distinct edges in F, there is no 2-coloring χ'' of $F - u_1v_1 - u_2v_2$ with $\chi''(u_1) = \chi''(v_1)$ and $\chi''(u_2) = \chi''(v_2)$ unless $\{u_1v_1, u_2v_2\} = \{ab, fg\}$ and χ'' is isomorphic to χ^*. That is, for any $H \in \mathcal{T}_r^q(n)$, the only copies of F in H that use exactly two bad edges correspond to χ^*.

Once again, as $\rho_F = \infty$, (12) is violated unless $M(H) = \emptyset$ for any $H \in \mathcal{H}_F(n, q)$. In addition, by Claim 3.7, if $H \supseteq K(V_1, V_2)$ with $|V_1| \geq |V_2|$, then

$$|B_1| \geq |B_1| - |B_2| \geq (1 - \delta_0)(|V_1| - |V_2| - 1)\pi_F n.$$

In fact, we will show if n is even, $|V_1| \geq |V_2| + 2$, and $q < n/5$ then $B_2 = \emptyset$.

As a result of Claim 3.7 we may initially assume that $|B_1| \geq (1 - \delta_0)n/6$. Now, if $B_2 \neq \emptyset$, an edge $uv \in B_2$ is contained in at least $c(n_2, n_1; F) > c(n_1, n_2; F) + 2(\zeta_F - \delta_0)n^4$ copies of F. However, if we remove uv and replace it with an edge xy where $x, y \in V_1$ have $d_B(x), d_B(y) < 3$, we form at most $c(n_1, n_2; F) + 4q\binom{n/2}{3} + \delta_0 n^4$ copies of F. As

$$2\zeta_F n^4 = 2^{-4}n^4 > n^4/60 \geq qn^3/12,$$

this alteration reduces the number of copies of F. So, $\#F(H)$ is minimized by making $B_2 = \emptyset$. Therefore, we have at least $(1 - 2\delta_0)q^2/2$ disjoint pairs of edges in B_1, each of which forms $4\binom{|V_2|}{3}$ copies of F. It follows that

$$\#F(H) \geq (q + a^2)(c(n, F) - a\zeta_F n^4) + q^2\frac{n^3}{24} - \delta_0 n^5, \tag{14}$$

where $a = |V_1| - n/2 = n/2 - |V_2| \geq 1$. We note that (14) is minimized when $a = 1$.

On the other hand, if $H^* \in \mathcal{T}_2^q(n)$, we may place $q/2$ edges in each of B_1 and B_2, thereby forming at most $q^2/4$ pairs of bad edges that lie in the same part. Thus,

$$\#F(H^*) \leq qc(n, F) + q^2\frac{n^3}{48} + \delta_0 n^5. \tag{15}$$

Comparing the above quantities, and solving the resulting quadratic inequality, we see that $c_{1,0}(F) \geq \frac{3-\sqrt{5}}{4}$.

The upper bound follows by noting that inequalities (14) and (15) may be changed to equations by replacing the last term with $\pm\delta_0 n^5$.

On the other hand, if n is odd, and $H \in \mathcal{T}_r^q(n)$ contains q_1 bad edges in B_1 and q_2 edges in B_2, we have

$$\#F(H) = (q_1+q_2)\alpha_F n^5 + q_2\zeta_F n^4 + (q_1^2+q_2^2)\binom{n/2}{3} \pm \delta_0 n^5.$$

For $q = q_1 + q_2 < 3n/8$, this is minimized by letting $q_2 = 0$. By a similar argument as above, it follows that $c_{1,1}(F) \leq 2\pi_F = 1/3$, thereby completing the proof. ∎

4.3 Pair-free graphs

One property of the graph in Figure 2 is that there exists a 2-coloring of the vertices that would be a proper 2-coloring with the deletion of exactly two edges. We now consider graphs which do not have this property.

Definition 4.3. *Let F be an r-critical graph. We say that F is* pair-free *if there do not exist two (different, but not necessarily disjoint) edges u_1v_1, u_2v_2 and a proper r-coloring χ of $F - u_1v_1 - u_2v_2$ such that $\chi(u_1) = \chi(u_2) = \chi(v_1) = \chi(v_2)$.*

Many interesting graphs belong to this class, e.g., odd cycles and cliques. In addition, graphs obtained from the complete r-partite graph $K_{s_1,\ldots,s_r}$ by adding an edge to the part of size s_1 are pair-free if $s_i \geq 3$ for all $i \geq 2$.

Theorem 4.4. *Let F be pair-free and let $t = \mathrm{sgn}(\zeta_F)$. Then $c_{1,t}(F) \leq 2\pi_F$ and $c_{1,i}(F) \leq \pi_F$ for $i \not\equiv t \pmod r$.*

Proof. Let n be large and $q = (\pi_F + \delta_0)n$. We prove the case $n \not\equiv t \pmod r$; the other case follows in a similar manner. Write $n = n_1 + \ldots + n_r$, where $c(n,F) = c(n_1,\ldots,n_r;F)$ and consider the partition $n = n_1' + n_2' + \ldots + n_r'$ where $n_1' = n_1 + t$, $n_2' = n_2 - t$ and $n_i' = n_i$ for $i = 3, \ldots r$. Construct H' as follows: $H' \supseteq K(V_1', \ldots, V_r')$ with $|V_i'| = n_i'$. Next place $q+1$ bad edges in V_1' to form an almost regular bipartite graph. We claim that $\#F(H') < \#F(H)$ for any $H \in \mathcal{T}_r^q(n)$.

First of all, each bad edge in H' is contained in at most $c(n,F) - |\zeta_F|n^{f-3} + O(n^{f-4})$ copies of F that contain only one bad edge. As F is pair-free, no copy of F contains exactly two bad edges. In addition, we may bound the number of copies of F that use at least three bad edges by $O(n^{f-3})$.

On the other hand, $\#F(H) \geq qc(n,F)$. Therefore,

$$\begin{aligned} \#F(H') - \#F(H) &\leq (q+1)\big(c(n,F) - |\zeta_F|n^{f-3}\big) + O(n^{f-3}) - qc(n,F) \\ &< \alpha_F n^{f-2} - (\pi_F + \delta_0)n|\zeta_F|n^{f-3} + O(n^{f-3}) \ < \ 0, \end{aligned}$$

proving the theorem. ∎

For odd cycles, this implies that $c_{1,0}(C_{2k+1}) = 1/2$. In fact, with more effort, it is possible to show that $c_{1,1}(C_{2k+1}) = 1$. However, the proof is quite involved as one has to account for copies of C_{2k+1} that may appear in various configurations. We direct the interested reader to [21].

If $\deg(P_F) \geq r+1$, Theorem 3.5 implies that $c_{1,i}(F) \leq \hat{\rho}_F - (1 - 1/r)$. So, if F is pair-free and $\rho_F = \hat{\rho}_F$, we have the exact value of $c_{1,i}(F)$. This is the case for $F = K_{s,t}^+$ where $t \geq 3$.

Lemma 4.5. *Let $s, t \geq 2$ and $F = K_{s,t}^+$ be obtained from the complete bipartite graph $K_{s,t}$ by adding an edge to the part of size s. Then $\rho_F = \hat{\rho}_F$.*

Proof. Clearly, $F = K_{s,t}^+$ is 2-critical and

$$c(n_1, n_2; K_{s,t}^+) = \binom{n_2}{t}\binom{n_1-2}{s-2}.$$

It readily follows that

$$\alpha_F = \frac{2^{-(t+s-2)}}{t!(s-2)!}, \quad \zeta_F = (t-s+2)\frac{2^{-(t+s-3)}}{t!(s-2)!}, \text{ and}$$

$$\pi_F = \begin{cases} \infty, & \text{if } t = s-2, \\ (2(t-s+2))^{-1}, & \text{otherwise.} \end{cases}$$

On the other hand,

$$P_F(\boldsymbol{\xi}) = \frac{2^{-s+2}}{t!(s-2)!}(\xi_1\xi_2^t + \xi_1^t\xi_2).$$

As P_F is a homogeneous polynomial, we restrict ourselves to $\xi_1 + \xi_2 = 1$. Namely, let

$$\varphi_{s,t}(y) = P_F(1/2+y, 1/2-y) = \frac{2^{-s+2}}{t!(s-2)!}\Big((1/2+y)(1/2-y)^t + (1/2+y)^t(1/2-y)\Big).$$

We observe that $\varphi_{s,t}(y)$ is an even function with $\varphi_{s,t}(1/2) = \varphi_{s,t}(-1/2) = 0$ and $\varphi_{s,t}(0) = \alpha_F$. Routine calculations show that the coefficient s_k of y^k in $\varphi'_{s,t}(y)$ is

$$\Big((-1)^k - 1\Big)(2k+1-t)\binom{t}{k}\frac{2^{(s-2)+(k-t)}}{t!(s-2)!}.$$

It follows that $s_k = 0$ when $k = (t-1)/2$ or k is even. Otherwise, if $k < (t-1)/2$ (resp. $k > (t-1)/2$), then s_k is positive (resp. negative).

That is, for $t \geq 4$, the coefficients of $\varphi'_{s,t}(y)$ change sign exactly once. So, $\varphi'_{s,t}(y)$ has exactly one positive root and, by symmetry, exactly one negative root. As $\varphi''_{s,t}(0) = \frac{2^{-s+2}}{t!(s-2)!}\frac{4t(t-3)}{2^t} > 0$ for $t \geq 4$, it follows that $(0, \alpha_F)$ is the unique local minimum for $\phi_{s,t}$ with the two roots of p'_t providing local maxima.

In addition, if $t = 2, 3$, $\phi'_{s,t}(y)$ is a decreasing odd polynomial. So, $(0, \alpha)$ is the unique maximum point of $\phi_{s,t}(y)$ and no other local maxima or minima exist. It follows that $\hat{\rho}_F = \rho_F = \infty$ in these two cases.

If $t \geq 4$, we may solve for ρ_F as the root of a polynomial equation. In particular, if $\boldsymbol{\xi} \in \mathcal{S}_\rho$ with $\xi_1 = \xi_2 = \rho/2$, we have $P_F(\boldsymbol{\xi}) = \alpha_F\rho^{t+1}$. Comparing this quantity with $\alpha_F(\rho - (1-1/r))$, we observe that ρ_F is the smallest positive root of the equation $\rho^{t+1} = \rho - 1/2$. We also obtain the bounds $2^{-t-1} < \theta_F < 2^{-t}$ on θ_F for $t \geq 4$.

Now, if $\rho_F \neq \hat{\rho}_F$, then the two curves $\alpha_F\rho^{t+1}$ and $\alpha_F(\rho - (1-1/2))$ must be tangent at ρ_F. Therefore, ρ_F is not only a root of $g_1(\rho) = \rho^{t+1} - \rho + 1/2$, but also of its derivative $g'_1(\rho) = (t+1)\rho^t - 1$. However, as $(t+1) < (5/3)^t$ and $2^{-t} < 1/10$ for $t \geq 4$, we have

$$\rho_F = (t+1)^{-1/t} > 3/5 > (1/2 + 2^{-t}),$$

resulting in a contradiction. Hence, $\rho_F = \hat{\rho}_F$. ∎

Theorem 4.6. *Let $s, t \geq 2$ and $F = K_{s,t}^+$. Then $c_1(F) = c_{1,0}(F) = \min(\pi_F, \theta_F)$ and $c_{1,1}(F) = \min(2\pi_F, \theta_F)$.*

Proof. Theorems 3.6 and 4.4 and Lemma 4.5 imply the result for $t \geq 3$. On the other hand, if $t = 2$, we note that $K_{s,2}^+$ is not pair-free. However, the case $K_{2,2}^+ = K_4 - e$ is covered in Section 4.1 and the argument in Claim 4.1 can be extended to the cases where $s \geq 3$. ∎

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