

Vibration of an Elastic Tensegrity Structure

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1 Introduction

Following sculptures first created by Snelson in 1948, in 1961 Buckminster Fuller patented a class of cable-bar structures which he called tensegrity structures [Ful76, Sne73]. These consisted of arrangements of bars and cables, with the bars not connected to one another, so that structural integrity was maintained by the tension in the cables. Hence “tension-integrity”, compressed to “tensegrity”. A notable example is Snelson’s *Needle Tower* at the Hirshhorn Gallery in Washington, DC. Later work of both engineers (see, *eg.*, [Cal78, Cal82, PC86, CP91, Kuz84]) and mathematicians ([Con80, RW81, Con82, Whi87, CW96]) generalized the term “tensegrity structure” to include any pin-connected structural framework in which some of the elements are tension-only cables or compression-only struts. The simplest three-dimensional example of this type of structure is shown in Fig. 1. This example possesses the distinguishing characteristics of tensegrity structures: It is a form-finding structure, an under-constrained structural system, prestressible while displaying an infinitesimal flexure even when the constituent elements are undeformable. It forms only at nodal geometries in which the statics matrix becomes rank deficient.

Since tensegrity structures have been proposed to be used in various constructions where weight is at premium, it is of importance to consider their dynamical response in the vicinity of their equilibrium positions. Usually they are constructed with (essentially) rigid rods and (more) elastic cables, so presuming that the pin-connections at the nodes are efficient, the elastic response and intrinsic damping of the cables determine the vibrational behavior of such a structure. However, these structures, by the nature of their construction must show an infinitesimal flexibility in the equilibrium

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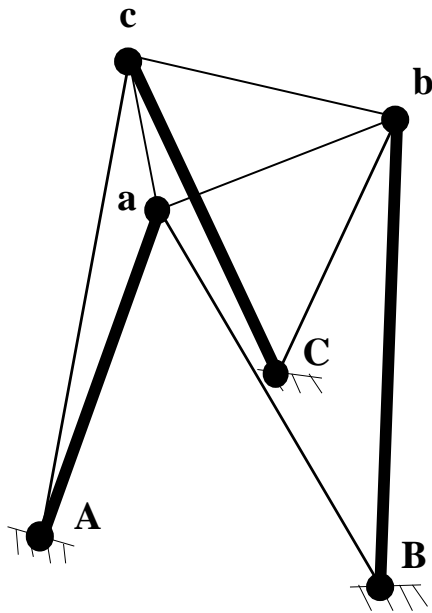


Figure 1: Simple Tensegrity Structure

position, and it is easy to suspect that this will lead to a less efficient mobilization of the cable's damping than would be true in a more conventional structure. In [OW00b] we have noted this fact, and presented some numerical calculations and a sketch, for a two-dimensional example structure, of the computations which verify it. This note gives the underlying computations for the structure of Fig. 1, quantifying the claims both analytically and numerically.

2 The Model

The problem will be examined for the tensegrity structure as pictured in Fig. 1, further simplified by assumptions of symmetry and uniformity. The structure is taken as a right regular prism, in which the end faces are equilateral triangles, perpendicular to the axis joining their centroids. The unique equilibrium position then is known to require a $5\pi/6$ relative rotation between the triangles.

We have examined the response to external loading of this structure in [OW00a], and we will follow the notations and calculations which we used there. The nodes A , B , and C are pinned to ground, the legs Aa , Bb , and Cc are inextensible bars, the elements ab , bc , and ca are inextensible cables, while the cross-cables Ac , Ba , and Cb are taken to be linearly elastic, but with damping. Considering only the cross-cables as elastic simplifies the calculations, but captures the essence of the elastic response of such systems.

We assume that

- The triangles ABC and abc are congruent equilateral triangles.
- The legs have the same length.
- The cross-cables are identical in length and in elastic modulus and damping coefficient.
- The mass of the system is localized in three equal masses at each of the nodes a , b , and c .

As a result of these simplifications the motion of the system can be described by a single parameter. We choose θ , the angle of rotation of the upper triangle about its centroid measured from the position of equilibrium.

We can use the calculations in [OW00b]: if the radius of the circumscribed circle of the base and upper triangles is a and the length of the rigid legs is L , then we find that the length of the cross-cables λ is given as

$$\lambda(\theta)^2 = L^2 - 2\sqrt{3}a^2 \cos \theta, \quad (2.1)$$

and the height h of the upper triangle as

$$h(\theta)^2 = L^2 - a^2(2 + \sqrt{3}\cos\theta + \sin\theta). \quad (2.2)$$

Note that from (2.1) it is implicitly assumed that $L > \sqrt{2\sqrt{3}}a$.

If we assume that the cross-cables are linearly elastic, with a spring constant of κ , then the **elastic energy** stored in the three totals to

$$\Phi(\theta) = \frac{3}{2}\kappa(\lambda(\theta) - \lambda_N)^2, \quad (2.3)$$

where λ_N is the natural length of the cable. The **kinetic energy** is given as

$$K(\theta, \dot{\theta}) = \frac{3}{2}ma^2(\dot{\theta})^2 + \frac{3}{2}m(\dot{h})^2, \quad (2.4)$$

accounting for the complementary rotational and vertical parts.

The final element is the **damping**. We assume that the damping force is isolated in the cross-cables and that it is linear in the rate of change of length of each cable. The total rate of dissipation of energy by these forces, then, is

$$\Delta(\dot{\theta}) = 3\gamma(\dot{\lambda})^2. \quad (2.5)$$

The equation of motion is obtained from the above as

$$\frac{\partial K}{\partial \dot{\theta}}\ddot{\theta} + \frac{\partial K}{\partial \theta}\dot{\theta} + \frac{d\Phi}{d\theta}\dot{\theta} + \Delta(\dot{\theta}) = 0. \quad (2.6)$$

We may divide the common term $\dot{\theta}$ from the equation and expand some terms to arrive at a non-linear second-order equation which governs the vibrations of the system:

$$\begin{aligned} & ma^2 \left[1 + \frac{a^2}{h^2} \sin^2\left(\frac{5\pi}{6} + \theta\right) \right] \ddot{\theta} + \\ & + \frac{ma^4}{h^2} \sin\left(\frac{5\pi}{6} + \theta\right) \left[\cos\left(\frac{5\pi}{6} + \theta\right) + \frac{a^2}{h^2} \sin^2\left(\frac{5\pi}{6} + \theta\right) \right] \dot{\theta} + \\ & + 3\gamma \frac{a^4}{\lambda^2} \sin^2\theta \dot{\theta} + \sqrt{3}\kappa a^2 \frac{\lambda - \lambda_N}{\lambda^2} \sin\theta = 0 \end{aligned} \quad (2.7)$$

The equation is rather complicated, but its essential behavior can be seen by looking at the leading terms in an expansion:

$$\ddot{\theta} + \alpha(\lambda(0) - \lambda_N)\theta + \beta\theta^3 + \tau\theta^2\dot{\theta} = 0. \quad (2.8)$$

The term $\alpha(\lambda(0) - \lambda_N)$ is the initial elastic modulus. If the cross-cables are slack in the equilibrium position then this term is zero and the elastic response is cubic, modulated by β , which always is positive. For us, the most important point is that the damping, with coefficient τ , is modulated by θ^2 ($\sin^2 \theta$ in the full equation (2.7)). This leads, as we make more precise in Section 4, to a strongly reduced damping effect.

By considering a quadrature for the solution of (2.8) with α and τ zero, one can show (*cf.*[Sto50]) that the product of the period and the amplitude of that solution is a constant, so that the period is inversely proportional to the amplitude. We shall see below that the solutions of (2.7) show corresponding period-lengthening with decay. (Of course the exact inverse proportionality of the special case does not apply even in the general case of (2.8).)

3 Numerical Calculations

In order to visualize the behavior of the system (2.7), we set parameters to nominal but consistent values:

$$a = 1, L = \sqrt{6 + \sqrt{3}}, m = 1, \gamma = 40, \kappa = 100, \lambda_N = \sqrt{6 - \sqrt{3}}. \quad (3.1)$$

The last condition ensures that the cross-cables are unstressed at equilibrium. A plot of a numerical solution of an initial value problem for the equation with these values is shown in Figure 2.

Notice the two significant phenomena: the period increases as the amplitude decays, and the decay is relatively slow. The decay is highlighted by plotting the change in energy of the system in Figure 3. In the same plot, we show the energy of the system, as altered by replacing the $\sin^2 \theta$ damping by a linear damping based on an average value of $\sin^2 \theta$ over the interval between zero and the starting value of θ . Finally, in an insert we show a fit of the energy function to $e(0)/(1 + At)^2$, which is the estimate arrived at in the next section.

To illustrate the effect of pre-stress in the cables, we leave all parameters as before, except that λ_N is reduced to a value which corresponds to a 1% strain in the cross-cables in the equilibrium position. The oscillation due to the same initial displacement is shown in Figure 4. Of course the frequency increases due to the prestress. Similar period-lengthening and decay of amplitude occur, although both are less pronounced than in the previous case.

The reduced efficiency of damping is illustrated first in Figure 5 by contrasting the response for the given system and one with linear damping

introduced as before. Second, in Figure 6 we make the same sort of energy-decay comparison as for the previous case, along with a fit of a curve of the form $e(0)/(1 + At)$ per our estimates in the next section.

4 Dissipation Calculations

To substantiate the relative slowness of the decay of the vibrations, we will estimate the energy of the system as it evolves in time. The total energy of the system is $K(\theta(t), \dot{\theta}(t)) + \Phi(\theta(t))$, where these are given by (2.4,2.3). The base stored energy is

$$\Phi(0) = \frac{3}{2} \kappa (\lambda_0 - \lambda_N), \quad (4.1)$$

where $\lambda_0 = \lambda(0)$ is the extension of the cables due to prestrain. We will consider the evolution of the **transient energy**

$$\begin{aligned} e &= K(\theta, \dot{\theta}) + \Phi(\theta) - \Phi(0) \\ &= \frac{3}{2} m a^2 \dot{\theta}^2 + \frac{3}{2} m \dot{h}^2 + \frac{3}{2} \kappa [(\lambda - \lambda_N)^2 - (\lambda_0 - \lambda_N)^2] \end{aligned} \quad (4.2)$$

through the energy equation (2.6):

$$\dot{e} = -3\gamma \left(\frac{d\lambda}{d\theta}\right)^2 \dot{\theta}^2. \quad (4.3)$$

Our main estimates are

Proposition 4.1 *For solutions of (4.3) within the feasible range $(-\frac{\pi}{6} \leq \theta(0) \leq \frac{\pi}{6})$*

- *if $\lambda(0) > \lambda_N$ (prestressed equilibrium) there is a constant μ such that*

$$e(t) \geq \frac{e(0)}{1 + \frac{\mu}{(\lambda_0 - \lambda_N)} e(0) t}. \quad (4.4)$$

- *if $\lambda(0) = \lambda_N$ (slack equilibrium) there is a constant ν such that*

$$e(t) \geq \frac{e(0)}{(1 + \nu \sqrt{e(0)} t)^2}. \quad (4.5)$$

Thus the decay is at best like $1/t^2$ as t increases, in contrast to the exponential decay characteristic of linearly damped systems.

To verify these estimates, note that from (4.2)

$$e \geq \frac{3ma^2}{2} \dot{\theta}^2, \quad (4.6)$$

and

$$e \geq \frac{3\kappa}{2} [(\lambda - \lambda_N)^2 - (\lambda_0 - \lambda_N)^2]. \quad (4.7)$$

Equation (4.6) yields an estimate for $\dot{\theta}^2$. We use (4.7) to obtain an estimate for $(d\lambda/d\theta)^2$.

In the feasible range,

$$\lambda \geq \lambda_0 \geq \lambda_N, \quad (4.8)$$

and we use (2.1) to see that

$$\left(\frac{d\lambda}{d\theta}\right)^2 = \frac{3a^4 \sin^2 \theta}{\lambda^2} \leq \frac{3a^4}{\lambda_0^2} \sin^2 \theta. \quad (4.9)$$

To bound the last term, note that its derivative with respect to θ is

$$\frac{6a^4}{\lambda_0^2} \cos \theta \sin \theta \leq \frac{6a^4}{\lambda_0^2} \sin \theta, \quad (4.10)$$

and similarly that, in the feasible range,

$$\frac{d}{d\theta} (\lambda - \lambda_0) = \frac{\sqrt{3}a^2 \sin \theta}{\sqrt{L^2 - 2\sqrt{3}a^2 \cos \theta}} \geq \frac{\sqrt{3}a^2}{\sqrt{L^2 - 3a^2}} \sin \theta. \quad (4.11)$$

Hence

$$\frac{d}{d\theta} (\lambda - \lambda_0) \geq \frac{d}{d\theta} \left(\frac{1}{A} \frac{3a^4}{\lambda_0^2} \sin^2 \theta \right), \quad (4.12)$$

where $A = 2\sqrt{3}a^2\sqrt{L^2 - 3a^2}/\lambda_0^2$.

Since each of the functions is zero when $\theta = 0$,

$$(\lambda - \lambda_0) \geq \frac{1}{A} \frac{3a^4}{\lambda_0^2} \sin^2 \theta \geq \frac{1}{A} \left(\frac{d\lambda}{d\theta}\right)^2, \quad (4.13)$$

which we rewrite as

$$(\lambda - \lambda_N) - (\lambda_0 - \lambda_N) \geq \frac{1}{A} \left(\frac{d\lambda}{d\theta}\right)^2. \quad (4.14)$$

Henceforth, we consider two cases. First, if $\lambda_0 > \lambda_N$ (prestressed case), we use (4.14) to deduce that

$$\left(\frac{d\lambda}{d\theta}\right)^2 \leq A \frac{(\lambda - \lambda_N)^2 - (\lambda_0 - \lambda_N)^2}{(\lambda - \lambda_N) + (\lambda_0 - \lambda_N)}$$

$$\leq \frac{A}{2(\lambda_0 - \lambda_N)} [(\lambda - \lambda_N)^2 - (\lambda_0 - \lambda_N)^2]. \quad (4.15)$$

Thus, from (4.7),

$$\left(\frac{d\lambda}{d\theta}\right)^2 \leq \frac{A}{3\kappa(\lambda_0 - \lambda_N)} e. \quad (4.16)$$

Substituting this and (4.6) into the energy equation (4.3) yields

$$\dot{e} \geq -\frac{2A\gamma}{3\kappa m a^2 (\lambda_0 - \lambda_N)} e^2 = -\frac{\mu}{(\lambda_0 - \lambda_N)} e^2. \quad (4.17)$$

Integration of this equation yields the first bound for the transient energy.

In the second case, if there is no prestress, $\lambda(0) = \lambda_N$, and we find from (4.14)

$$\left(\frac{d\lambda}{d\theta}\right)^2 \leq A(\lambda - \lambda_0). \quad (4.18)$$

and (4.7) leads immediately to

$$\left(\frac{d\lambda}{d\theta}\right)^2 \leq A \sqrt{\frac{2}{3\kappa}} \sqrt{e}. \quad (4.19)$$

This can be substituted into the energy equation together with (4.6) to obtain

$$\dot{e} \geq -\gamma A \sqrt{\frac{2}{3\kappa}} \frac{2}{m a^2} e^{3/2}, \quad (4.20)$$

or

$$\dot{e} \geq -2\nu e^{3/2}, \quad (4.21)$$

which we integrate to obtain the second estimate.

5 Angular Damping

In order to obtain more efficient damping in the system, one can introduce a linear damping term by enhancing the natural damping in the pin connections of the frame. For example, we introduce a damping which resists the rate-of-angulation of the legs to the upper cables as follows. The angle α which the legs make to the xy -plane is given through the equation

$$\begin{aligned} \sin^2 \alpha &= \frac{h^2}{L^2} = 1 - \frac{a^2}{L^2} (2 + \sqrt{3} \cos \theta + \sin \theta) \\ &= 1 - 2\rho^2 (1 - \cos \phi) \end{aligned} \quad (5.1)$$

where ϕ is the total angle of rotation, $\theta + \frac{5\pi}{6}$, and $\rho = a/L$. We suppose that the damping mechanism is to be applied between the leg and the nearest upper cable, and one can show that the angle τ between these elements obeys

$$\cos \tau = \cos \alpha \cos (\phi/2). \quad (5.2)$$

From (5.1) and (5.2) it follows that

$$\dot{\tau} = -\frac{\rho \cos \phi}{\sqrt{1 - \rho^2 (1 - \cos^2 \phi)}} \dot{\theta}. \quad (5.3)$$

If we assume that the damping is linear in $\dot{\tau}$, the energy dissipates at a rate $3\Gamma (\dot{\tau})^2$ and so the additional term in (2.7) is

$$3\Gamma \frac{\rho^2 \cos^2 \phi}{1 - \rho^2 (1 - \cos^2 \phi)} \dot{\theta}. \quad (5.4)$$

The linear approximation to this near $\theta = 0$ is

$$3\Gamma \left[\frac{3\rho^2}{4 - \rho^2} + \frac{8\sqrt{3}\rho^2(1 - \rho^2)}{(4 - \rho^2)^2} \theta \right] \dot{\theta}, \quad (5.5)$$

which confirms the linear nature of the damping, and ensures exponential decay of solutions of the equation.

6 Conclusions

We have demonstrated that the natural geometric flexibility inherent to a tensegrity structure at equilibrium leads to inefficient mobilization of the natural damping in the elastic cables of the structure, leading to a much slower rate of decay of amplitude of vibration than might be expected (order at best $1/t^2$ as opposed to exponential decay). This effect, readily apparent in models, could be a serious drawback in practical usages.

To control the effective damping, it seems that augmenting the natural damping would be inefficient. However a natural mode of damping, by ensuring damping of the angular motion between the structural elements leads to linearly damped equations and hence exponential decay of free vibrations.

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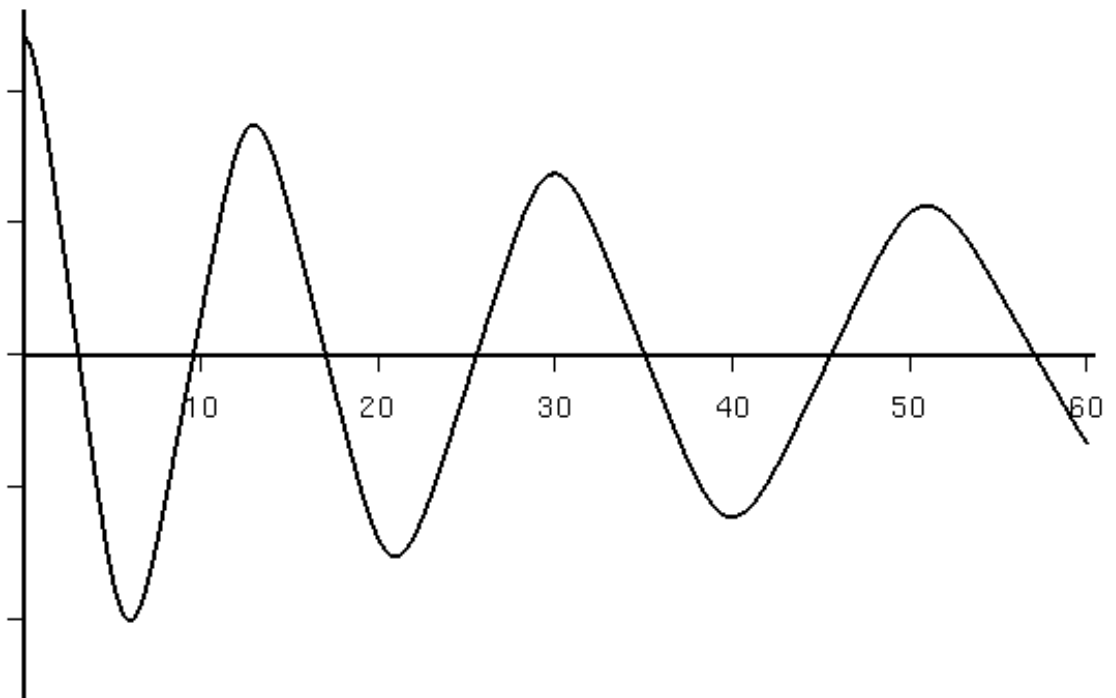


Figure 2: Theta versus time for no prestress

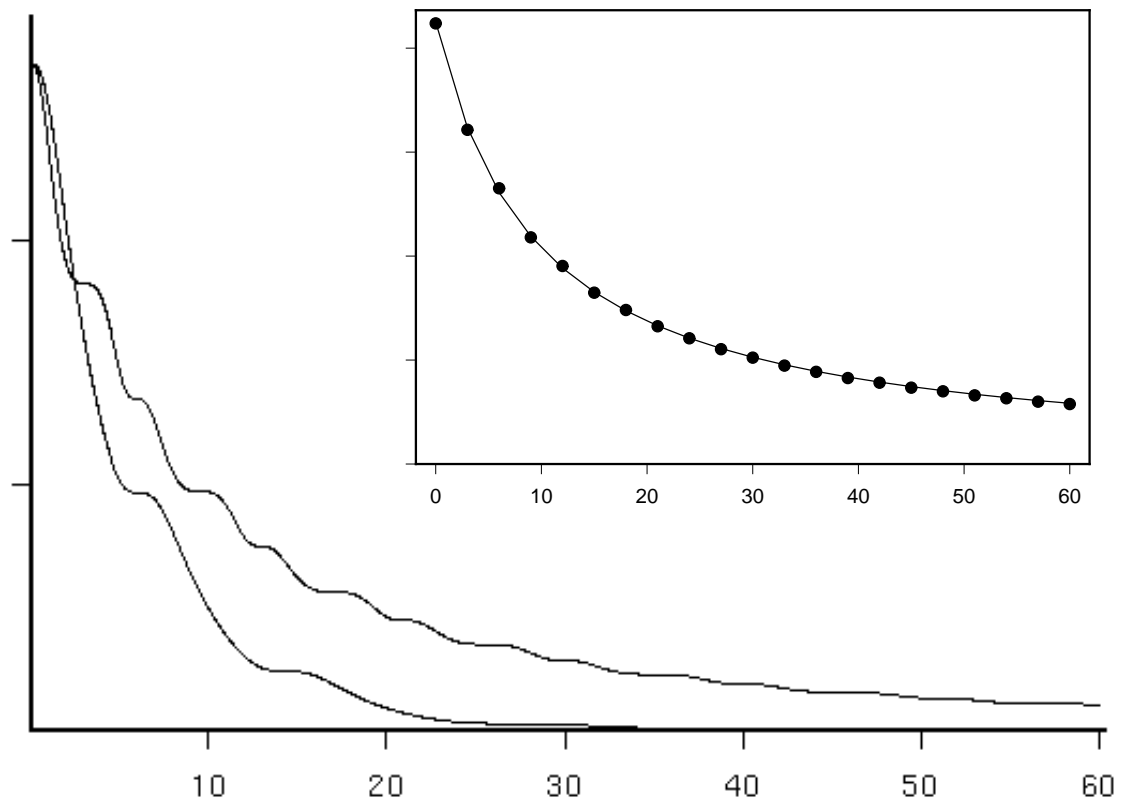


Figure 3: Energy decay. Upper curve is for the given system, lower is equivalent linearly damped system. Inset: fit to data of $e(0)/(1 + At)^2$

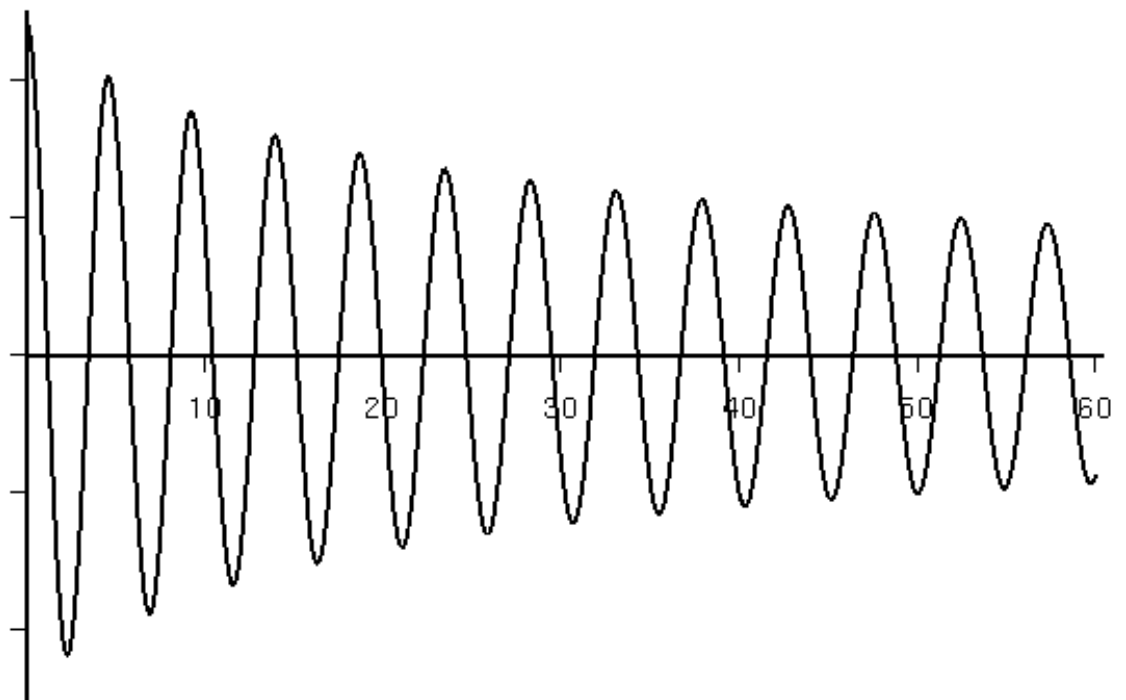


Figure 4: Theta versus time with prestress

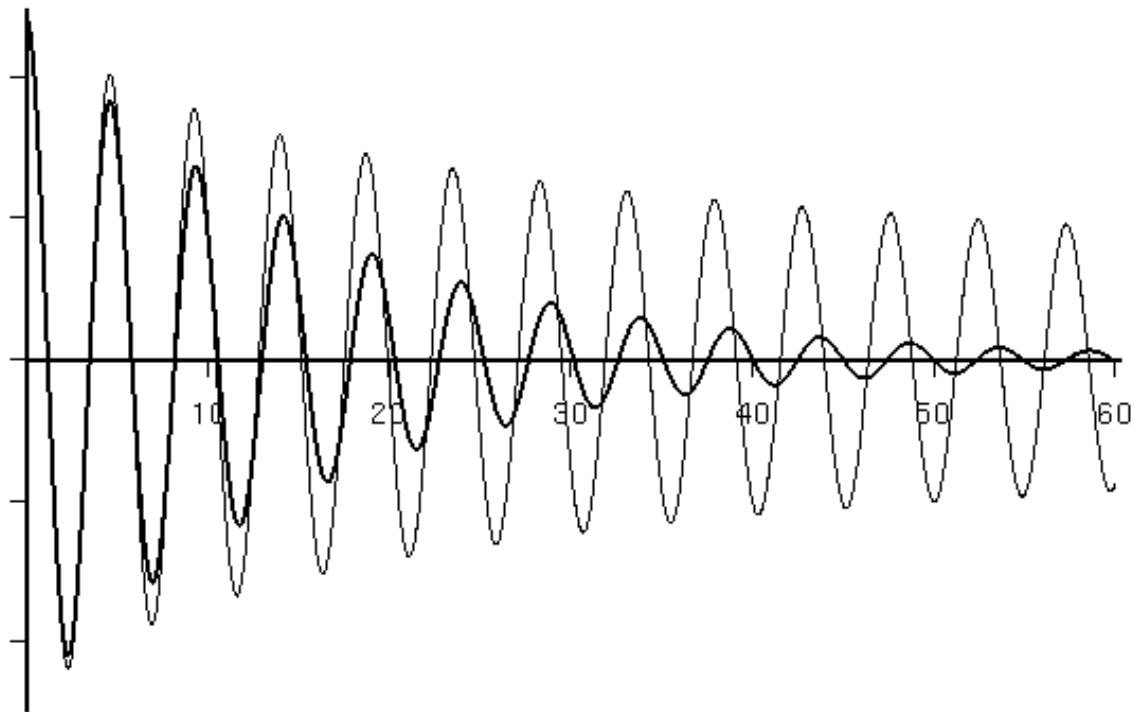


Figure 5: Theta versus time for equivalent inearly damped system with prestress, contrasted with naturally damped system.

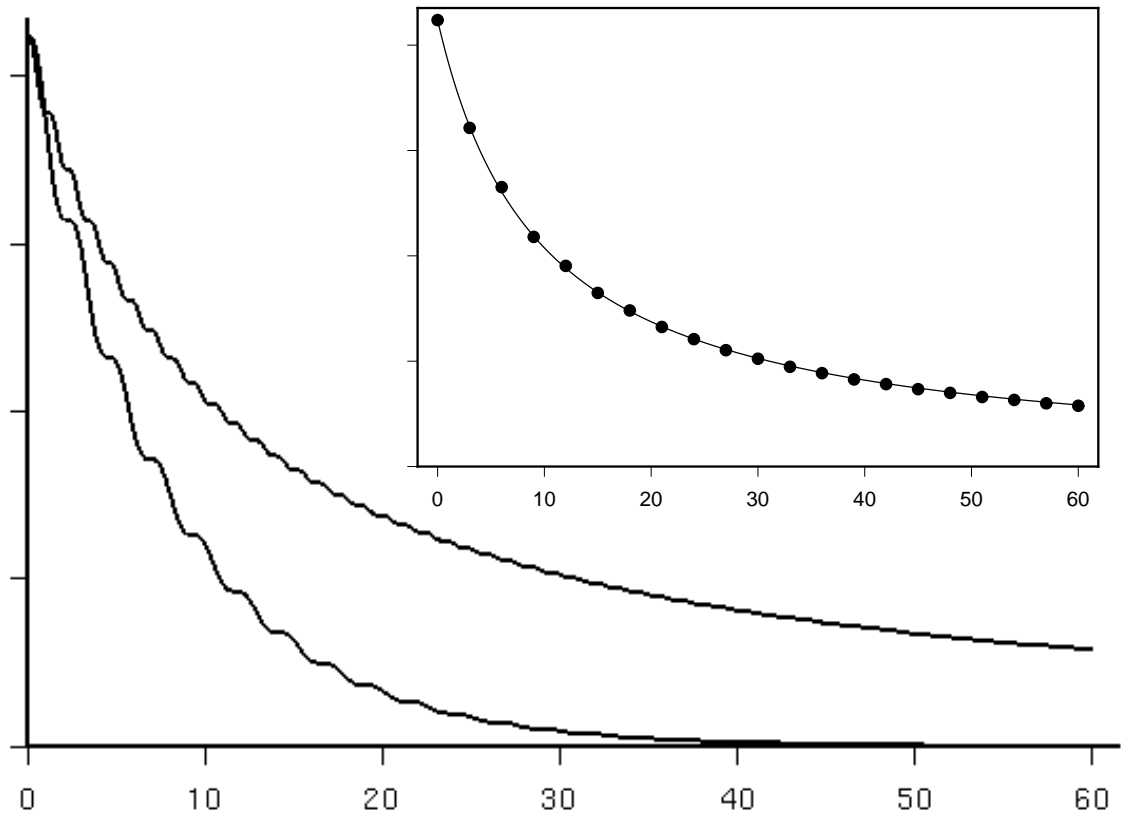


Figure 6: Energy decay with 1% prestrain. Upper curve is for the given system, lower is equivalent linearly damped system. Inset: fit to data of $e(0)/(1 + At)$