Chapter 4

Gradients.

In this chapter, we assume a linear-space bundle $(\mathcal{B}, \tau, \mathcal{M})$ of class C^s, $s \geq 2$, is given. We also assume that both $\mathcal{M}$ and $\mathcal{B}$ have constant dimensions, and put $n := \dim \mathcal{M}$ and $m := \dim \mathcal{B} - \dim \mathcal{M}$. Then we have, as in (32.1), $m = \dim \mathcal{B}_x$ for all $x \in \mathcal{M}$.

41. Shift Gradients

Let $x \in \mathcal{M}$ be fixed.

Let $\mathbf{\Phi}$ be an analytic tensor functor and let $\mathbf{H} : \mathcal{M} \to \mathbf{\Phi}(\mathcal{B})$ be a cross section of $\mathbf{\Phi}(\mathcal{B})$ that is differentiable at x. We define the mapping

$$\widehat{\mathbf{H}} : \mathrm{Tlis}_x \mathcal{B} \to \mathbf{\Phi}(\mathcal{B}_x) \tag{41.1}$$

by

$$\widehat{\mathbf{H}}(\mathbf{T}) := \mathbf{\Phi}(\mathbf{T})^{-1} \mathbf{H}(\pi_x(\mathbf{T})) \quad \text{for all} \quad \mathbf{T} \in \mathrm{Tlis}_x \mathcal{B}, \tag{41.2}$$

where π_x is defined by (32.3). Since $\mathbf{\Phi}$ is analytic, it is clear that $\widehat{\mathbf{H}}$ is differentiable at $\mathbf{1}_{\mathcal{B}_x}$.

Difinition: *The* **shift-gradient** *of* $\mathbf{H}$ *at* x *is the linear mapping*

$$\Box_x \mathbf{H} \in \mathrm{Lin}\left(\mathrm{S}_x \mathcal{B}, \mathbf{\Phi}(\mathcal{B}_x)\right)$$

defined by

$$\Box_x \mathbf{H} := \nabla_{\mathbf{1}_{\mathcal{B}_x}} \widehat{\mathbf{H}}, \tag{41.3}$$

where $\widehat{\mathbf{H}}$ *is given by* (41.2).

For every bundle chart $\phi \in \mathrm{Ch}_x(\mathcal{B}, \mathcal{M})$, the spaces $\mathrm{Rng}\, \mathbf{I}_x$ and $\mathrm{Rng}\, \mathbf{A}_x^{\phi}$ are supplymentary in $\mathrm{S}_x \mathcal{B}$. Hence, for every $\mathbf{s} \in \mathrm{S}_x \mathcal{B}$ there is exactly one pair $(\mathbf{M}, \mathbf{t}) \in \mathrm{Lin}\, \mathcal{B}_x \times \mathrm{T}_x \mathcal{M}$ such that $\mathbf{s} = \mathbf{I}_x \mathbf{M} + \mathbf{A}_x^{\phi} \mathbf{t}$ and thus

$$(\Box_x \mathbf{H}) \mathbf{s} = (\Box_x \mathbf{H}) \mathbf{I}_x \mathbf{M} + (\Box_x \mathbf{H}) \mathbf{A}_x^{\phi} \mathbf{t}.$$

Proposition 1: *We have*

$$(\Box_x \mathbf{H}) \mathbf{I}_x \mathbf{M} = -(\mathbf{\Phi}_x^{\bullet} \mathbf{M}) \mathbf{H}(x) \quad \text{for all} \quad \mathbf{M} \in \mathrm{Lin}\, \mathcal{B}_x, \tag{41.4}$$

where $\mathbf{\Phi}_x^{\bullet} \in \mathrm{Lin}\,(\mathrm{Lin}\, \mathcal{B}_x, \mathrm{Lin}\, \mathbf{\Phi}(\mathcal{B}_x))$ *is defined to be the gradient of the mapping* $(\mathbf{L} \mapsto \mathbf{\Phi}(\mathbf{L})) : \mathrm{Lis}\, \mathcal{B}_x \to \mathrm{Lis}\,(\mathbf{\Phi}(\mathcal{B}_x))$ *at* $\mathbf{1}_{\mathcal{B}_x}$.

Proof: In view of (32.4) and (41.2) we have $\widehat{\mathbf{H}} \circ \iota_x : \mathrm{Lis}\,\mathcal{B}_x \to \mathbf{\Phi}(\mathcal{B}_x)$ and

$$(\widehat{\mathbf{H}} \circ \iota_x)(\mathbf{L}) = \mathbf{\Phi}(\mathbf{L})^{-1}\mathbf{H}(x) \quad \text{for all} \quad \mathbf{L} \in \mathrm{Lis}\,\mathcal{B}_x.$$

Taking the gradient of $(\widehat{\mathbf{H}}_x \circ \iota_x)$ at $\mathbf{1}_{\mathcal{B}_x}$ and using (32.11) and (41.3), we obtain the desired result (41.4). ∎

Example 1: Let $\mathcal{B}^* := \mathrm{Dl}\,(\mathcal{B})$, where Dl is the duality functor.

Let $\mathbf{h}$ be a cross section of $\mathcal{B}$, let $\boldsymbol{\omega}$ be a cross section of $\mathcal{B}^*$, let $\mathbf{L}$ be a cross section of $\mathrm{Lin}\,\mathcal{B}$, let $\mathbf{G}$ be a cross section of $\mathrm{Lin}\,(\mathcal{B}, \mathcal{B}^*) \cong \mathrm{Lin}_2(\mathcal{B}^2,)$ and

let $\mathbf{T}$ be a cross section of $\mathrm{Lin}\,(\mathcal{B}, \mathrm{Lin}\,\mathcal{B}) \cong \mathrm{Lin}_2(\mathcal{B}^2, \mathcal{B})$. Assume that all of these cross sections are differentiable at x. Then

$$(\Box_x \mathbf{h})\mathbf{I}_x\mathbf{M} = -\mathbf{M}\mathbf{h}(x); \tag{41.5}$$

$$(\Box_x \boldsymbol{\omega})\mathbf{I}_x\mathbf{M} = \boldsymbol{\omega}(x)\mathbf{M}; \tag{41.6}$$

$$(\Box_x \mathbf{L})\mathbf{I}_x\mathbf{M} = \mathbf{L}(x)\mathbf{M} - \mathbf{M}\mathbf{L}(x); \tag{41.7}$$

$$(\Box_x \mathbf{G})\mathbf{I}_x\mathbf{M} = \mathbf{G}(x) \circ (\mathbf{M} \times \mathbf{1}_{\mathcal{B}_x}) + \mathbf{G}(x) \circ (\mathbf{1}_{\mathcal{B}_x} \times \mathbf{M}) \tag{41.8}$$

and

$$(\Box_x \mathbf{T})\mathbf{I}_x\mathbf{M} = \mathbf{T}(x) \circ (\mathbf{M} \times \mathbf{1}_{\mathcal{B}_x}) + \mathbf{T}(x) \circ (\mathbf{1}_{\mathcal{B}_x} \times \mathbf{M}) - \mathbf{M}\mathbf{T}(x) \tag{41.9}$$

for all $\mathbf{M} \in \mathrm{Lin}\,\mathcal{B}_x$.

Let a bundle chart $\phi \in \mathrm{Ch}_x(\mathcal{B}, \mathcal{M})$ be given. We define the mapping

$$\mathbf{H}^\phi : \mathcal{O}_\phi \to \mathbf{\Phi}(\mathcal{V}_\phi)$$

by

$$\mathbf{H}^\phi(y) := \mathbf{\Phi}(\phi\rfloor_y)\mathbf{H}(y), \quad \text{for all} \quad y \in \mathcal{O}_\phi. \tag{41.10}$$

Proposition 2: *We have*

$$(\Box_x \mathbf{H})\mathbf{A}_x^\phi = \nabla_x^\phi \mathbf{H} = \mathbf{\Lambda}\big(\mathbf{A}_{\mathbf{H}(x)}^{\mathbf{\Phi}(\phi)}\big)\nabla_x \mathbf{H} \tag{41.11}$$

where $\mathbf{\Phi}(\phi)$ *is defined by* (24.5), $\nabla_x^\phi \mathbf{H}$ *is described in* (24.9) *and* $\mathbf{A}_{\mathbf{H}(x)}^{\mathbf{\Phi}(\phi)}$ *is defined in terms of* (31.19).

Proof: Let $y \in \mathcal{O}_\phi$ be given. Substituting $\mathbf{T} := (\phi\rfloor_y)^{-1}\phi\rfloor_x$ in (41.2) gives

$$\begin{aligned}\widehat{\mathbf{H}}((\phi\rfloor_y)^{-1}\phi\rfloor_x) &= \mathbf{\Phi}((\phi\rfloor_y)^{-1}\phi\rfloor_x)^{-1}\mathbf{H}(y) \\ &= \mathbf{\Phi}(\phi\rfloor_x)^{-1}\mathbf{\Phi}(\phi\rfloor_y)\mathbf{H}(y) = \mathbf{\Phi}(\phi\rfloor_x)^{-1}\mathbf{H}^\phi(y).\end{aligned}$$

Since $\mathrm{tlis}_x^{\phi\leftarrow}(y, \phi\rfloor_x) = (\phi\rfloor_y)^{-1}\phi\rfloor_x$ by (32.7), we obtain

$$(\widehat{\mathbf{H}} \circ \mathrm{tlis}_x^{\phi\leftarrow})(y, \phi\rfloor_x) = \mathbf{\Phi}(\phi\rfloor_x)^{-1}\mathbf{H}^{\phi}(y) \qquad \text{for all} \quad y \in \mathcal{O}_{\phi}.$$

Taking the gradient with respect to y at x and observing (51.2) gives

$$(\nabla_{\mathbf{1}_{\mathcal{B}_x}}\widehat{\mathbf{H}})(\nabla_{\mathbf{1}_{\mathcal{B}_x}}\mathrm{tlis}_x^{\phi})^{-1}(\mathbf{t}, \mathbf{0}) = \mathbf{\Phi}(\phi\rfloor_x)^{-1}(\nabla_x \mathbf{H}^{\phi})\,\mathbf{t}$$

for all $\mathbf{t} \in \mathrm{T}_x\mathcal{M}$. In view of definition (32.19) and (24.9) we obtain the first equality of the desired result (41.11).

It follows from (41.2), (41.3) and (31.29) with ϕ replaced by $\mathbf{\Phi}(\phi)$ that

$$\begin{aligned}
(\Box_x \mathbf{H})\mathbf{A}_x^{\phi} &= (\nabla_{\mathbf{1}_{\mathcal{B}_x}}\widehat{\mathbf{H}})\nabla_x(\phi\rfloor^{-1}\phi\rfloor_x) \\
&= \nabla_x\big(y \mapsto \mathbf{\Phi}(\phi\rfloor_x^{-1}\phi\rfloor_y)\mathbf{H}(y)\big) \\
&= \big(\mathbf{\Phi}(\phi)\big)\rfloor_x^{-1}\big(\mathrm{ev}_2 \circ \nabla_{\mathbf{H}(x)}\mathbf{\Phi}(\phi)\big)\nabla_x\mathbf{H} \\
&= \mathbf{\Lambda}\big(\mathbf{A}_{\mathbf{H}(x)}^{\mathbf{\Phi}(\phi)}\big)\nabla_x\mathbf{H}.
\end{aligned}$$

Since $\phi \in \mathrm{Ch}_x(\mathcal{B}, \mathcal{M})$ was arbitrary, the second part of (41.11) follows. ∎

The results of Props. 1 and 2 give the following commutative diagram

$$\begin{tikzcd}
\mathrm{Lin}\,\mathcal{B}_x \arrow[r, "\mathbf{I}_x"] \arrow[d, "-\left(\mathbf{\Phi}_x^{\bullet}\right)^{\sim}\mathbf{H}(x)"'] \arrow[dr, phantom, "(1)" near start] & \mathrm{S}_x\mathcal{B} \arrow[dl, "\Box_x\mathbf{H}"] \arrow[dr, phantom, "(2)"] & \mathrm{T}_x\mathcal{M} \arrow[l, "\mathbf{A}_x^{\phi}"'] \arrow[d, equal] \\
\mathbf{\Phi}(\mathcal{B}_x) & \mathrm{T}_{\mathbf{H}(x)}\mathbf{\Phi}(\mathcal{B}) \arrow[l, "\mathbf{\Lambda}\left(\mathbf{A}_{\mathbf{H}(x)}^{\mathbf{\Phi}(\phi)}\right)"] & \mathrm{T}_x\mathcal{M} \arrow[l, "\nabla_x\mathbf{H}"]
\end{tikzcd} \quad . \tag{41.12}$$

Prop. 1 and Prop. 2 are illustrated by (1) and (2) in the diagram, respectively.

Let tensor functors $\mathbf{\Phi}_1$, $\mathbf{\Phi}_2$ and $\mathbf{\Psi}$ and a natural bilinear assignment $B : (\mathbf{\Phi}_1, \mathbf{\Phi}_2) \to \mathbf{\Psi}$ be given. Also, let $\mathbf{H}_1 : \mathcal{M} \to \mathbf{\Phi}_1(\mathcal{B})$ be a cross section of $\mathbf{\Phi}_1(\mathcal{B})$ and let $\mathbf{H}_2 : \mathcal{M} \to \mathbf{\Phi}_2(\mathcal{B})$ be a cross section of $\mathbf{\Phi}_2(\mathcal{B})$. Then the mapping $B(\mathbf{H}_1, \mathbf{H}_2) : \mathcal{M} \to \mathbf{\Psi}$ defined by

$$B(\mathbf{H}_1, \mathbf{H}_2)(x) := B_{\mathcal{B}_x}(\mathbf{H}_1(x), \mathbf{H}_2(x)) \quad \text{for all} \quad x \in \mathcal{M} \tag{41.13}$$

is a cross section of $\mathbf{\Psi}(\mathcal{B})$.

General Product Rule

If $\mathbf{H}_1$ and $\mathbf{H}_2$ are differentiable at x, then $B(\mathbf{H}_1, \mathbf{H}_2)$ is also differentiable at x and we have

$$\big(\Box_x B(\mathbf{H}_1, \mathbf{H}_2)\big)\mathbf{s} = B_{\mathcal{B}_x}\big((\Box_x\mathbf{H}_1)\mathbf{s}, \mathbf{H}_2(x)\big) + B_{\mathcal{B}_x}\big(\mathbf{H}_1(x), (\Box_x\mathbf{H}_2)\mathbf{s}\big) \tag{41.14}$$

for all $\mathbf{s} \in \mathrm{S}_x\mathcal{B}$.

Proof: Put $\mathbf{H} := B(\mathbf{H}_1, \mathbf{H}_2)$ in (41.2), we have

$$\begin{aligned}\widehat{\mathbf{H}}(\mathbf{T}) &= B_{\mathcal{B}_x}\big(\mathbf{\Phi}_1(\mathbf{T}^{-1})\mathbf{H}_1(\pi_x(\mathbf{T})), \mathbf{\Phi}_2(\mathbf{T}^{-1})\mathbf{H}_2(\pi_x(\mathbf{T}))\big)\\ &= B_{\mathcal{B}_x}\big(\widehat{\mathbf{H}}_1(\mathbf{T}), \widehat{\mathbf{H}}_2(\mathbf{T})\big)\end{aligned}$$

for all $\mathbf{T} \in \mathrm{Tlis}_x\mathcal{B}$. Since B is bilinear, the desired result (41.14) follows from (41.3) together with the General Product Rule in flat spaces [FDS]. ∎

Example 2:

Let f be a scalar field, and let $\mathbf{h} : \mathcal{M} \to \mathcal{B}$ be a cross section of $\mathcal{B}$ and $\mathbf{H} : \mathcal{M} \to \mathrm{Lin}\,\mathcal{B}$ be a cross section of $\mathrm{Lin}\,\mathcal{B}$ that are differentiable at x. Then $f\mathbf{H}$ and $\mathbf{Hh}$ defined value-wise are also differentiable at x, and we have

$$(\Box_x f\mathbf{H})\mathbf{s} = ((\Box_x f)\mathbf{s})\mathbf{H}(x) + f(x)\,(\Box_x\mathbf{H})\mathbf{s} \tag{41.15}$$

and

$$\Box_x(\mathbf{Hh})\mathbf{s} = ((\Box_x\mathbf{H})\mathbf{s})\mathbf{h}(x) + \mathbf{H}(x)(\Box_x\mathbf{h})\mathbf{s} \tag{41.16}$$

for all $\mathbf{s} \in \mathrm{S}_x\mathcal{B}$. ∎

Example 3:

Let $\boldsymbol{\omega} : \mathcal{M} \to \mathrm{Skw}_p(\mathcal{B}^{\,p},)$ be a skew-p-form field and $\boldsymbol{\tau} : \mathcal{M} \to \mathrm{Skw}_q(\mathcal{B}^{\,q},)$ a skew-q-form field that are differentiable at x. Then $\boldsymbol{\omega} \wedge \boldsymbol{\tau}$ is a skew-$(p+q)$-form field which is also differentiable at x and we have

$$(\Box_x(\boldsymbol{\omega} \wedge \boldsymbol{\tau}))\mathbf{s} = (\Box_x\boldsymbol{\omega})\mathbf{s} \wedge \boldsymbol{\tau} + \boldsymbol{\omega} \wedge (\Box_x\boldsymbol{\tau})\mathbf{s} \tag{41.17}$$

for all $\mathbf{s} \in \mathrm{S}_x\mathcal{B}$. ∎

Let $\mathcal{L}$, and $\mathcal{L}'$ be linear-space bundles over $\mathcal{M}$. For every $x \in \mathcal{M}$, we denote the fiber product bundle (see Sect.22) of $(\mathrm{Tlis}_x\mathcal{L}, \pi_x, \mathcal{M})$ and $(\mathrm{Tlis}_x\mathcal{L}', \pi'_x, \mathcal{M})$ by

$$\Big(\mathrm{Tlis}_x\mathcal{L} \times_{\mathcal{M}} \mathrm{Tlis}_x\mathcal{L}'\ ,\ \pi_x \times_{\mathcal{M}} \pi'_x\ ,\ \mathcal{M}\Big). \tag{41.18}$$

Taking the gradient of the mapping

$$\pi_x \times_{\mathcal{M}} \pi'_x\ :\ \mathrm{Tlis}_x\mathcal{L} \times_{\mathcal{M}} \mathrm{Tlis}_x\mathcal{L}' \quad\longrightarrow\quad \mathcal{M} \tag{41.19}$$

at $\mathbf{1}_{\mathcal{L}_x} \times \mathbf{1}_{\mathcal{L}'_x}$, we have

$$\mathbf{P}_x \times_{\mathrm{T}_x\mathcal{M}} \mathbf{P}'_x\ :\ \mathrm{S}_x\mathcal{L} \times_{\mathrm{T}_x\mathcal{M}} \mathrm{S}_x\mathcal{L}' \quad\longrightarrow\quad \mathrm{T}_x\mathcal{M} \tag{41.20}$$

where $\mathbf{P}_x = \nabla_{\mathbf{1}_{\mathcal{L}_x}}\pi_x$ and $\mathbf{P}'_x = \nabla_{\mathbf{1}_{\mathcal{L}'_x}}\pi'_x$. It follows from

$$\pi_x \times_{\mathcal{M}} \pi'_x = \pi_x \circ \mathrm{ev}_1 = \pi'_x \circ \mathrm{ev}_2$$

that

$$(\mathbf{P}_x \times_{\mathrm{T}_x\mathcal{M}} \mathbf{P}'_x)(\mathbf{s}, \mathbf{s}') = \mathbf{P}_x\mathbf{s} = \mathbf{P}'_x(\mathbf{s}') \tag{41.21}$$

for all $(\mathbf{s}, \mathbf{s}') \in \mathrm{S}_x\mathcal{L} \times_{\mathrm{T}_x\mathcal{M}} \mathrm{S}_x\mathcal{L}'$.

Let $\boldsymbol{\Upsilon}$ be a tensor bifunctor and let $\mathbf{H}$ be a cross section of $\boldsymbol{\Upsilon}(\mathcal{L} \times_{\mathcal{M}} \mathcal{L}')$ which is differentiable at x. We define a mapping

$$\widehat{\mathbf{H}} : \mathrm{Tlis}_x\mathcal{L} \times_{\mathcal{M}} \mathrm{Tlis}_x\mathcal{L}' \to \boldsymbol{\Upsilon}(\mathcal{L}_x \times \mathcal{L}'_x) \tag{41.22}$$

by

$$\begin{aligned} \widehat{\mathbf{H}}\Big(\mathbf{T} \times \mathbf{T}'\Big) &:= \boldsymbol{\Upsilon}(\mathbf{T} \times \mathbf{T}')^{-1}\, \mathbf{H}(y) \\ &\quad \text{where} \quad y := \pi_x(\mathbf{T}) = \pi'_x(\mathbf{T}') \end{aligned} \tag{41.23}$$

for all $\mathbf{T} \times \mathbf{T}' \in \mathrm{Tlis}_x\mathcal{L} \times_{\mathcal{M}} \mathrm{Tlis}_x\mathcal{L}'$. The shift-gradient of $\mathbf{H}$ at x is the linear mapping

$$\square_x\mathbf{H} : \mathrm{S}_x\mathcal{L} \times_{\mathrm{T}_x\mathcal{M}} \mathrm{S}_x\mathcal{L}' \to \boldsymbol{\Upsilon}(\mathcal{L}_x \times \mathcal{L}'_x) \tag{41.24}$$

defined in (41.3); i.e.

$$\square_x\mathbf{H} = \nabla_{\mathbf{1}_{\mathcal{P}_x}} \widehat{\mathbf{H}}, \tag{41.25}$$

where $\mathbf{1}_{\mathcal{P}_x} := \mathbf{1}_{\mathcal{L}_x} \times \mathbf{1}_{\mathcal{L}'_x}$. We also use the following notations

$$\mathbf{I}_x := \nabla_{\mathbf{1}_{\mathcal{L}_x}} \mathrm{in}_x \quad \text{and} \quad \mathbf{I}'_x := \nabla_{\mathbf{1}_{\mathcal{L}'_x}} \mathrm{in}'_x$$

where $\mathrm{in}_x := \mathbf{1}_{\mathcal{L}_x \subset \mathcal{L}}$ and $\mathrm{in}'_x := \mathbf{1}_{\mathcal{L}'_x \subset \mathcal{L}'}$ are inclusion mappings.

Proposition 3: *We have*

$$(\square_x\mathbf{H})(\mathbf{I}_x\mathbf{M}, \mathbf{I}'_x\mathbf{M}') = -\overset{\bullet}{\boldsymbol{\Upsilon}}_x(\mathbf{M} \times \mathbf{M}')\mathbf{H}(x) \tag{41.26}$$

for all $\mathbf{M} \in \mathrm{Lin}\,\mathcal{L}_x$ *and all* $\mathbf{M}' \in \mathrm{Lin}\,\mathcal{L}'_x$, *where* $\overset{\bullet}{\boldsymbol{\Upsilon}}_x$ *is the gradient of the mapping* $\big(\mathbf{L} \times \mathbf{L}' \mapsto \boldsymbol{\Upsilon}(\mathbf{L} \times \mathbf{L}')\big)$ *at* $\mathbf{1}_{\mathcal{L}_x} \times \mathbf{1}_{\mathcal{L}'_x}$.

Example 4:

Let $\boldsymbol{\Phi}$ be a analytic tensor functor and let $\mathcal{L} := \mathrm{T}\mathcal{M}$ and $\mathcal{L}' := \mathcal{B}$. If $\mathbf{L} : \mathcal{M} \to \mathrm{Lin}\,(\mathrm{T}\mathcal{M}, \boldsymbol{\Phi}(\mathcal{B}))$ and $\mathbf{T} : \mathcal{M} \to \mathrm{Lin}_2\,(\mathrm{T}\mathcal{M}^2, \boldsymbol{\Phi}(\mathcal{B}))$ are cross sections that are differentiable at x, we have

$$\begin{aligned} \square_x\mathbf{L} &: \mathrm{S}_x\mathrm{T}\mathcal{M} \times_{\mathrm{T}_x\mathcal{M}} \mathrm{S}_x\mathcal{B} \to \mathrm{Lin}\,(\mathrm{T}_x\mathcal{M}, \boldsymbol{\Phi}(\mathcal{B}_x)) \\ \square_x\mathbf{T} &: \mathrm{S}_x\mathrm{T}\mathcal{M} \times_{\mathrm{T}_x\mathcal{M}} \mathrm{S}_x\mathcal{B} \to \mathrm{Lin}_2\,(\mathrm{T}_x\mathcal{M}^2, \boldsymbol{\Phi}(\mathcal{B}_x)) \end{aligned}$$

and

$$\begin{aligned} (\square_x\mathbf{L})(\mathbf{I}_x\mathbf{M}, \mathbf{I}'_x\mathbf{M}') &= \mathbf{L}(x)\mathbf{M} - \overset{\bullet}{\boldsymbol{\Phi}}_x(\mathbf{M}')\mathbf{L}(x) \\ (\square_x\mathbf{T})(\mathbf{I}_x\mathbf{M}, \mathbf{I}'_x\mathbf{M}') &= \mathbf{T}(x)\mathbf{M} + \mathbf{T}(x)^{\sim}\mathbf{M} - \overset{\bullet}{\boldsymbol{\Phi}}_x(\mathbf{M}')\mathbf{T}(x) \end{aligned} \tag{41.27}$$

for all $\mathbf{M} \in \operatorname{Lin} \mathrm{T}_x\mathcal{M}$ and $\mathbf{M}' \in \operatorname{Lin} \mathcal{B}_x$.

Proposition 4: *We have*

$$(\Box_x \mathbf{H})(\mathbf{A}_x^\theta, \mathbf{A}_x^\phi) = \nabla_x^{\phi_1,\phi_2}\mathbf{H}, \tag{41.28}$$

where $\nabla_x^{\phi_1,\phi_2}\mathbf{H}$ *is described in* (24.12)*, for all bundle charts* $\theta \in \mathrm{Ch}_x(\mathcal{L}, \mathcal{M})$ *and* $\phi \in \mathrm{Ch}_x(\mathcal{L}', \mathcal{M})$*.*

42. Covariant Gradients

Let $x \in \mathcal{M}$ and a connector $\mathbf{K} \in \mathrm{Con}\,_x\mathcal{B}$ be given.

Let $\mathbf{\Phi}$ be a tensor functor and $\mathbf{H} : \mathcal{M} \to \mathbf{\Phi}(\mathcal{B})$ be a cross section of $\mathbf{\Phi}(\mathcal{B})$ that is differentiable at x.

<u>**Definition**</u> **:** *We define the* **covariant gradient** *of* $\mathbf{H}$ *relative to* $\mathbf{K}$ *by*

$$\nabla_{\mathbf{K}}\mathbf{H} := (\Box_x\mathbf{H})\mathbf{K} \in \operatorname{Lin}\big(\mathrm{T}_x\mathcal{M}, \mathbf{\Phi}(\mathcal{B}_x)\big), \tag{42.1}$$

where $\Box_x\mathbf{H}$ *is the shift-gradient of* $\mathbf{H}$ *at* x *as defined by* (41.3)*.*

Given a bundle chart $\phi \in \mathrm{Ch}_x(\mathcal{B}, \mathcal{M})$. It follows from (41.11) and (42.1) that

$$\nabla_{\mathbf{A}_x^\phi}\mathbf{H} = \nabla_x^\phi\mathbf{H}.$$

If $f : \mathcal{M} \to$ is a scalar field differentiable at x, then we have $\Box_x f = \nabla_x f\,\mathbf{P}_x$ and hence

$$\nabla_{\mathbf{K}} f = \nabla_x f \qquad \text{for all} \quad \mathbf{K} \in \mathrm{Con}\,_x\mathcal{B}. \tag{42.2}$$

Proposition 1: For every bundle chart $\phi \in \mathrm{Ch}_x(\mathcal{B}, \mathcal{M})$ we have

$$(\nabla_{\mathbf{K}}\mathbf{H})\mathbf{t} = (\nabla_x^\phi\mathbf{H})\mathbf{t} + \overset{\bullet}{\mathbf{\Phi}}_x\big(\mathbf{\Gamma}_x^\phi(\mathbf{K})\mathbf{t}\big)\mathbf{H}(x) \quad \text{for all} \quad \mathbf{t} \in \mathrm{T}_x\mathcal{M}, \tag{42.3}$$

where $\overset{\bullet}{\mathbf{\Phi}}_x \in \operatorname{Lin}\,(\operatorname{Lin}\mathcal{B}_x, \operatorname{Lin}\mathbf{\Phi}(\mathcal{B}_x))$ is defined as in Prop. 1 of Sect.41.

Proof: By (32.27), we have

$$\begin{aligned}(\Box_x\mathbf{H})\mathbf{K}\mathbf{t} &= (\Box_x\mathbf{H})\mathbf{A}_x^\phi\mathbf{t} + \Box_x\mathbf{H}(\mathbf{K} - \mathbf{A}_x^\phi)\mathbf{t} \\ &= (\Box_x\mathbf{H})\mathbf{A}_x^\phi\mathbf{t} - \Box_x\mathbf{H}\big(\mathbf{I}_x\mathbf{\Gamma}_x^\phi(\mathbf{K})\mathbf{t}\big)\end{aligned}$$

for all $\mathbf{t} \in \mathrm{T}_x\mathcal{M}$. Using (32.4), we obtain

$$(\Box_x\mathbf{H})\mathbf{K}\mathbf{t} = (\Box_x\mathbf{H})\mathbf{A}_x^\phi\mathbf{t} + \overset{\bullet}{\mathbf{\Phi}}_x\big(\mathbf{\Gamma}_x^\phi(\mathbf{K})\mathbf{t}\big)\mathbf{H}(x).$$

The result (42.3) follows from the definition (42.1). ∎

Example 1:

Let $\mathbf{h}$ be a cross section of $\mathcal{B}$, let $\boldsymbol{\omega}$ be a cross section of $\mathcal{B}^*$, let $\mathbf{L}$ be a cross section of $\operatorname{Lin}\mathcal{B}$, let $\mathbf{G}$ be a cross section of $\operatorname{Lin}(\mathcal{B},\mathcal{B}^*) \cong \operatorname{Lin}_2(\mathcal{B}^2,)$, and

let $\mathbf{T}$ be a cross section of $\operatorname{Lin}(\mathcal{B},\operatorname{Lin}\mathcal{B}) \cong \operatorname{Lin}_2(\mathcal{B}^2,\mathcal{B})$. If these cross sections are differentiable at x, we have

$$(\nabla_{\mathbf{K}}\mathbf{h})\mathbf{t} = (\nabla_x^{\phi}\mathbf{h})\mathbf{t} + \boldsymbol{\Gamma}_x^{\phi}(\mathbf{K})(\mathbf{t},\mathbf{h}(x)); \tag{42.4}$$

$$(\nabla_{\mathbf{K}}\boldsymbol{\omega})\mathbf{t} = (\nabla_x^{\phi}\boldsymbol{\omega})\mathbf{t} - \boldsymbol{\omega}(x)\boldsymbol{\Gamma}_x^{\phi}(\mathbf{K})\mathbf{t}; \tag{42.5}$$

$$(\nabla_{\mathbf{K}}\mathbf{L})\mathbf{t} = (\nabla_x^{\phi}\mathbf{L})\mathbf{t} - \mathbf{L}(x)\big(\boldsymbol{\Gamma}_x^{\phi}(\mathbf{K})\mathbf{t}\big) + \big(\boldsymbol{\Gamma}_x^{\phi}(\mathbf{K})\mathbf{t}\big)\mathbf{L}(x); \tag{42.6}$$

$$\nabla_{\mathbf{K}}\mathbf{G}(\mathbf{t},\mathbf{b}) = (\nabla_x^{\phi}\mathbf{G})(\mathbf{t},\mathbf{b}) - \big(\mathbf{G}(x)\mathbf{b}\big)\big(\boldsymbol{\Gamma}_x^{\phi}(\mathbf{K})\mathbf{t}\big) - \mathbf{G}(x)\big(\boldsymbol{\Gamma}_x^{\phi}(\mathbf{K})(\mathbf{t},\mathbf{b})\big) \tag{42.7}$$

and

$$\begin{aligned}\nabla_{\mathbf{K}}\mathbf{T}(\mathbf{t},\mathbf{b}) = (\nabla_x^{\phi}\mathbf{T})(\mathbf{t},\mathbf{b}) &- \big(\mathbf{T}(x)\mathbf{b}\big)\big(\boldsymbol{\Gamma}_x^{\phi}(\mathbf{K})\mathbf{t}\big) - \mathbf{T}(x)\big(\boldsymbol{\Gamma}_x^{\phi}(\mathbf{K})(\mathbf{t},\mathbf{b})\big)\\ &+ \big(\boldsymbol{\Gamma}_x^{\phi}(\mathbf{K})\mathbf{t}\big)\big(\mathbf{T}(x)\mathbf{b}\big)\end{aligned} \tag{42.8}$$

for all $\mathbf{t} \in \mathrm{T}_x\mathcal{M}$ and all $\mathbf{b} \in \mathcal{B}_x$.

General Product Rule

Let $\mathbf{H}_1, \mathbf{H}_2$ *be cross sections as given in the General Product Rule of Sect. 41, then we have*

$$\nabla_{\mathbf{K}}B(\mathbf{H}_1,\mathbf{H}_2)\mathbf{t} = B_{\mathcal{B}_x}\big((\nabla_{\mathbf{K}}\mathbf{H}_1)\mathbf{t},\mathbf{H}_2(x)\big) + B_{\mathcal{B}_x}\big(\mathbf{H}_1(x),(\nabla_{\mathbf{K}}\mathbf{H}_2)\mathbf{t}\big) \tag{42.9}$$

for all $\mathbf{t} \in \mathrm{T}_x\mathcal{M}$*.*

Proof: Substituting $\mathbf{s} := \mathbf{K}\mathbf{t}$ in (41.14) and observing (42.1), we obtain (42.9).

The formulas (41.15), (41.16) and (41.17) remain valid if the shift gradient $\square_x$ there is replaced by the covariant gradient $\nabla_{\mathbf{K}}$ and $\mathbf{s} \in \mathcal{S}_x\mathcal{B}$ by $\mathbf{t} \in \mathrm{T}_x\mathcal{M}$.

Let $\mathcal{L}$ and $\mathcal{L}'$ be linear-space bundles over $\mathcal{M}$. Let $\boldsymbol{\Upsilon}$ be a tensor bifunctor and let $\mathbf{H} : \mathcal{M} \to \boldsymbol{\Upsilon}(\mathcal{L}\times_{\mathcal{M}}\mathcal{L}')$ be a cross section of $\boldsymbol{\Upsilon}(\mathcal{L}\times_{\mathcal{M}}\mathcal{L}')$ which is differentiable at x. Let a pair of connectors $(\mathbf{K},\mathbf{K}') \in \operatorname{Con}_x\mathcal{L} \times \operatorname{Con}_x\mathcal{L}'$ be given.

Definition: *The* **covariant-gradient of H at** x **relative to** $(\mathbf{K},\mathbf{K}')$ *is defined by*

$$\nabla_{(\mathbf{K},\mathbf{K}')}\mathbf{H} := (\square_x\mathbf{H})(\mathbf{K},\mathbf{K}') \tag{42.10}$$

which is in $\operatorname{Lin}\big(\mathrm{T}_x\mathcal{M},\boldsymbol{\Upsilon}(\mathcal{L}_x \times \mathcal{L}'_x)\big)$.

Proposition 2: *For every* $(\mathbf{K},\mathbf{K}') \in \operatorname{Con}_x\mathcal{L} \times \operatorname{Con}_x\mathcal{L}'$ *and all bundle charts* $\phi \in \mathrm{Ch}_x(\mathcal{L},\mathcal{M})$ *and* $\phi' \in \mathrm{Ch}_x(\mathcal{L}',\mathcal{M})$ *we have*

$$(\nabla_{(\mathbf{K},\mathbf{K}')}\mathbf{H})\mathbf{t} = (\nabla_x^{\phi,\phi'}\mathbf{H})\mathbf{t} + \boldsymbol{\Upsilon}_x^{\bullet}\big(\boldsymbol{\Gamma}_x^{\phi}(\mathbf{K})\mathbf{t} \times \boldsymbol{\Gamma}_x^{\phi'}(\mathbf{K}')\mathbf{t}\big)\mathbf{H}(x) \tag{42.11}$$

for all $\mathbf{t} \in \mathrm{T}_x\mathcal{M}$*, where* $\boldsymbol{\Upsilon}_x^{\bullet}$ *is described in Prop. 3 of Sect. 41.*

Proof: Equation (42.11) follows from $\mathbf{K} = \mathbf{A}_x^{\phi} - \mathbf{I}_x\boldsymbol{\Gamma}_x^{\phi}(\mathbf{K})$, $\mathbf{K}' = \mathbf{A}_x^{\phi'} - \mathbf{I}_x\boldsymbol{\Gamma}_x^{\phi'}(\mathbf{K}')$, (42.10) and (41.28). ∎

43. Alternating Covariant Gradients

Let a number $p \in$, with $p \geq 1$, connections $\mathbf{C} : \mathcal{M} \to \operatorname{Con}\mathrm{T}\mathcal{M}$ and $\mathbf{D} : \mathcal{M} \to \operatorname{Con}\mathcal{B}$ of class C^1 be given.

Let $\boldsymbol{\Phi}$ be an analytic tensor functor. For every differentiable $\Phi(\mathcal{B})$-valued skew-p-linear field $\mathbf{S} : \mathcal{M} \to \mathrm{Skw}_p(\mathrm{T}\mathcal{M}^p, \Phi(\mathcal{B}))$, the covariant gradient of $\mathbf{S}$ at $x \in \mathcal{M}$ relative to $(\mathbf{C},\mathbf{D})$ is the mapping

$$\nabla_{(\mathbf{C}(x),\mathbf{D}(x))}\mathbf{S} : \mathcal{M} \to \mathrm{Lin}(\mathrm{T}_x\mathcal{M}, \mathrm{Skw}_p(\mathrm{T}_x\mathcal{M}^p, \Phi(\mathcal{B}_x)).$$

Taking the alternating part of $\nabla_{(\mathbf{C}(x),\mathbf{D}(x))}\mathbf{S}$, we obtain the skew $(p+1)$-linear mapping

$$\mathrm{Alt}\,(\nabla_{(\mathbf{C}(x),\mathbf{D}(x))}\mathbf{S}) \in \mathrm{Skw}_{p+1}(\mathrm{T}_x\mathcal{M}^{p+1}, \Phi(\mathcal{B}_x)). \tag{43.1}$$

Proposition 1: *Let* $x \in \mathcal{M}$ *be given. For every manifold chart* $\chi \in \mathrm{Ch}_x\mathcal{M}$ *and every bundle chart* $\phi \in \mathrm{Ch}_x(\mathcal{M},\mathcal{B})$*, we have*

$$\begin{aligned}(p+1)&\mathrm{Alt}\,(\nabla_{(\mathbf{C}(x),\mathbf{D}(x))}\mathbf{S})(\mathbf{v}) \\ &= (p+1)\mathrm{Alt}\left(\nabla_x^{\chi,\phi}\mathbf{S} + \big(\boldsymbol{\Phi}_x^{\bullet}(\boldsymbol{\Gamma}_x^{\phi}(\mathbf{D}(x)))^{\sim}\mathbf{S}(x)\big)\right)(\mathbf{v}) \\ &\qquad - \sum_{1<i<j<p+1}(-1)^{i+j-1}\mathbf{S}(x)\big(\mathbf{T}_x(\mathbf{C}(x))(\mathbf{v}_i,\mathbf{v}_j), \mathrm{del}_{(i,j)}\mathbf{v}\big)\end{aligned} \tag{43.2}$$

where $\mathrm{del}_{(i,j)} : \mathcal{V}^{p+1} \to \mathcal{V}^{p-1}$ *is defined by* $\mathrm{del}_{(i,j)} := \mathrm{del}_j \circ \mathrm{del}_i$*,* $i < j$*, for all* $\mathbf{v} \in \mathrm{T}_x\mathcal{M}^{p+1}$.

Proof: Let $\chi \in \mathrm{Ch}_x\mathcal{M}$ and $\phi \in \mathrm{Ch}_x(\mathcal{B},\mathcal{M})$ be given. We have

$$\mathbf{C}(x) = \mathbf{A}_x^{\chi} - \mathbf{I}_x\boldsymbol{\Gamma}_x^{\chi}(\mathbf{C}(x)) \quad \text{and} \quad \mathbf{D}(x) = \mathbf{A}_x^{\phi} - \mathbf{I}_x\boldsymbol{\Gamma}_x^{\phi}(\mathbf{D}(x)).$$

For every $i \in (p+1)^]$, (42.11) gives

$$\nabla_{(\mathbf{C}(x),\mathbf{D}(x))}\mathbf{S}(\mathbf{v}_i, \mathrm{del}_i\mathbf{v}) = \nabla_x^{\chi,\phi}\mathbf{S}(\mathbf{v}_i, \mathrm{del}_i\mathbf{v}) + \Phi_x^{\bullet}(\mathbf{\Gamma}_x^{\phi}(\mathbf{D}(x)\mathbf{v}_i)\mathbf{S}(x)(\mathrm{del}_i\mathbf{v}) - \sum_{j\in(p+1)^]\setminus\{i\}} \mathbf{S}(x)(\mathrm{del}_{(i,j)}\mathbf{v}).j)\mathbf{\Gamma}_x^{\chi}(\mathbf{C}(x))(\mathbf{v}_i, \mathbf{v}_j) \tag{43.2}$$

for all $\mathbf{v} \in (\mathrm{T}_x\mathcal{M})^{\times(p+1)}$. Sum up and rearrange all the terms, we obtain the desired formula by observing that $\mathbf{T}_x = \mathbf{\Gamma}_x^{\chi} - \mathbf{\Gamma}_x^{\chi\sim}$. ∎

Prop.1 has several applications. The first application is given in the following Prop.2. The second kind of applications are Bianchi identities in Sect.44 and the third application leads to the definition of exterior differential in Sect.45.

For every cross section $\mathbf{H} : \mathcal{M} \to \Phi(\mathcal{B})$ of class C^p, $p \geq 2$, we define the **covariant gradient-mapping of H relative to D**

$$\nabla_{\mathbf{D}}\mathbf{H} : \mathcal{M} \to \mathrm{Lin}(\mathrm{T}\mathcal{M}, \Phi(\mathcal{B}))$$

by

$$\nabla_{\mathbf{D}}\mathbf{H}(y) := \nabla_{\mathbf{D}(y)}\mathbf{H} \quad \text{for all} \quad y \in \mathcal{M}. \tag{43.3}$$

The **second covariant gradient-mapping of H relative to** $(\mathbf{C}, \mathbf{D})$ is defined by

$$\nabla^{(2)}_{(\mathbf{C},\mathbf{D})}\mathbf{H} := \nabla_{(\mathbf{C},\mathbf{D})}(\nabla_{\mathbf{D}}\mathbf{H}) : \mathcal{M} \to \mathrm{Lin}_2\big(\mathrm{T}\mathcal{M}^2\,,\, \Phi(\mathcal{B})\,\big). \tag{43.4}$$

The second covarient gradient-mapping $\nabla^{(2)}_{(\mathbf{C},\mathbf{D})}\mathbf{H}$ is not necessarily symmetric. Indeed, we have the following:

Proposition 2: *We have*

$$\nabla^{(2)}_{(\mathbf{C},\mathbf{D})}\mathbf{H} - (\nabla^{(2)}_{(\mathbf{C},\mathbf{D})}\mathbf{H})^{\sim} = \Phi^{\bullet}(\mathbf{R}(\mathbf{D})(\cdot,\cdot))\mathbf{H} - \big(\nabla_{\mathbf{D}}\mathbf{H}\big)\mathbf{T}(\mathbf{C}) \tag{43.5}$$

where, for each $x \in \mathcal{M}$, $\Phi^{\bullet}(x) := \Phi_x^{\bullet} \in \mathrm{Lin}\,(\mathrm{Lin}\,\mathcal{B}_x, \mathrm{Lin}\,\Phi(\mathcal{B}_x))$ is defined as in Prop. 1 of Sect. 42.

Proof: Let $x \in \mathcal{M}$ be given. Choose $\chi \in \mathrm{Ch}_x\mathcal{M}$ and $\phi \in \mathrm{Ch}_x(\mathcal{B}, \mathcal{M})$. Applying Prop. 1 with $\mathbf{H}$ replaced by $\nabla_{\mathbf{D}(x)}\mathbf{H}$ and Φ replaced by $\mathrm{Lin} \circ (\mathrm{Id}, \Phi)$ (see [N2]), we have

$$\begin{aligned}\nabla^{(2)}_{(\mathbf{C}(x),\mathbf{D}(x))}\mathbf{H}(\mathbf{u},\mathbf{v}) - \nabla^{(2)}_{(\mathbf{C}(x),\mathbf{D}(x))}\mathbf{H}(\mathbf{v},\mathbf{u}) + \big(\nabla_{\mathbf{D}(x)}\mathbf{H}\big)\mathbf{T}_x(\mathbf{C}(x))(\mathbf{u},\mathbf{v}) \\ = (\nabla_{(\mathbf{A}_x^{\chi},\mathbf{A}_x^{\phi})}\nabla_{\mathbf{D}}\mathbf{H})(\mathbf{u},\mathbf{v}) - (\nabla_{(\mathbf{A}_x^{\chi},\mathbf{A}_x^{\phi})}\nabla_{\mathbf{D}}\mathbf{H})(\mathbf{v},\mathbf{u}) \\ + \Phi_x^{\bullet}(\mathbf{\Gamma}_x^{\phi}(\mathbf{D}(x))\mathbf{u})(\nabla_{\mathbf{D}(x)}\mathbf{H})\mathbf{v} - \Phi_x^{\bullet}(\mathbf{\Gamma}_x^{\phi}(\mathbf{D}(x))\mathbf{v})(\nabla_{\mathbf{D}(x)}\mathbf{H})\mathbf{u}\end{aligned} \tag{43.6}$$

for all $\mathbf{u},\mathbf{v}\in \mathrm{T}_x\mathcal{M}$. Observing $\nabla_{\mathbf{D}}\mathbf{H} = \nabla_{\mathbf{C}^\phi}\mathbf{H} + \Phi_x^\bullet(\mathbf{\Gamma}^\phi(\mathbf{D}))$, we have

$$\nabla_{(\mathbf{A}_x^\chi,\mathbf{A}_x^\phi)}\nabla_{\mathbf{D}(x)}\mathbf{H}(\mathbf{u},\mathbf{v}) = \nabla^{(2)}_{(\mathbf{A}_x^\chi,\mathbf{A}_x^\phi)}\mathbf{H}(\mathbf{u},\mathbf{v}) + \nabla_{(\mathbf{A}_x^\chi,\mathbf{A}_x^\phi)}\Phi_x^\bullet(\mathbf{\Gamma}^\phi(\mathbf{D}))\tilde{}\,\mathbf{H}(\mathbf{u},\mathbf{v}). \tag{43.7}$$

for all $\mathbf{u},\mathbf{v}\in \mathrm{T}_x\mathcal{M}$. Since $\Phi_x^\bullet$ is a natural linear assignment, the second term on the right handside of the equality in (43.7) is

$$\begin{aligned}(\nabla_{(\mathbf{A}_x^\chi,\mathbf{A}_x^\phi)}&\Phi_x^\bullet(\mathbf{\Gamma}^\phi(\mathbf{D}))\tilde{}\,\mathbf{H})(\mathbf{u},\mathbf{v})\\ &= \Phi_x^\bullet(\nabla_{(\mathbf{A}_x^\chi,\mathbf{A}_x^\phi)}\mathbf{\Gamma}^\phi(\mathbf{D})(\mathbf{u},\mathbf{v}))\mathbf{H}(x) + \Phi_x^\bullet(\mathbf{\Gamma}_x^\phi(\mathbf{D}(x))\mathbf{v})(\nabla_{\mathbf{A}_x^\phi}\mathbf{H})\mathbf{u}.\end{aligned} \tag{43.8}$$

We also have, the third term on the right hand side of the equality (43.6) satisfies

$$\begin{aligned}\Phi_x^\bullet&(\mathbf{\Gamma}_x^\phi(\mathbf{D}(x))\mathbf{u})(\nabla_{\mathbf{D}(x)}\mathbf{H})\mathbf{v}\\ &= \Phi_x^\bullet(\mathbf{\Gamma}_x^\phi(\mathbf{D}(x))\mathbf{u})\big(\nabla_{\mathbf{A}_x^\phi}\mathbf{H} + \Phi_x^\bullet(\mathbf{\Gamma}_x^\phi(\mathbf{D}(x))\big)\mathbf{v}\\ &= \Phi_x^\bullet(\mathbf{\Gamma}_x^\phi(\mathbf{D}(x))\mathbf{u})\nabla_{\mathbf{C}^\phi}\mathbf{H}\mathbf{v} + \Phi_x^\bullet(\mathbf{\Gamma}_x^\phi(\mathbf{D}(x))\mathbf{u})\Phi_x^\bullet(\mathbf{\Gamma}_x^\phi(\mathbf{D}(x))\mathbf{v})\\ &= \Phi_x^\bullet(\mathbf{\Gamma}_x^\phi(\mathbf{D}(x))\mathbf{u})\nabla_{\mathbf{C}^\phi}\mathbf{H}\mathbf{v} + \Phi_x^\bullet(\mathbf{\Gamma}_x^\phi(\mathbf{D}(x))\mathbf{u}\mathbf{\Gamma}_x^\phi(\mathbf{D}(x))\mathbf{v}).\end{aligned} \tag{43.9}$$

Combining (43.6) to (43.9) with (43.2) and observing that

$$\nabla^{(2)}_{(\mathbf{A}_x^\chi,\mathbf{A}_x^\phi)}\mathbf{H} = \Phi(\phi\big\rfloor_x)^{-1}\left(\nabla^{(2)}_\chi\mathbf{H}^\phi\right)(\nabla_x\chi\times\nabla_x\chi) \tag{43.10}$$

is symmetric and $x\in\mathcal{M}$ was arbitrary, we obtain (43.5). ∎

Remark: When the given bundle $\mathcal{B}$ is the tangent bundle $\mathrm{T}\mathcal{M}$, then we only need one connection say; the connection $\mathbf{C}$. If this is the case, we have

$$\nabla^{(2)}_{\mathbf{C}}\mathbf{H} - (\nabla^{(2)}_{\mathbf{C}}\mathbf{H})^{\sim} = \Phi^\bullet(\mathbf{R}(\mathbf{C})(\cdot,\cdot))\mathbf{H} - \big(\nabla_{\mathbf{C}}\mathbf{H}\big)\mathbf{T}(\mathbf{C}). \tag{43.11}$$

44. Bianchi Identities

Let connections $\mathbf{C} : \mathcal{M} \to \mathrm{Con}\,\mathrm{T}\mathcal{M}$ and $\mathbf{D} : \mathcal{M} \to \mathrm{Con}\,\mathcal{B}$ of class C^1 be given. Both of the torsion field $\mathbf{T}(\mathbf{C}) : \mathcal{M} \to \mathrm{Skw}_2(\mathrm{T}\mathcal{M}^2, \mathrm{T}\mathcal{M})$ of the connection $\mathbf{C}$ and the curvature field $\mathbf{R}(\mathbf{D}) : \mathcal{M} \to \mathrm{Skw}_2(\mathrm{T}\mathcal{M}^2, \mathrm{Lin}\mathcal{B})$ of the connection $\mathbf{D}$ are skew-2-linear fields. Applying Prop.1 of Sect.43, the alternating part of $\nabla_{\mathbf{C}}\mathbf{T}(\mathbf{C})$ gives the **first Bianchi idetity** and the alternating part of $\nabla_{(\mathbf{C},\mathbf{D})}\mathbf{R}(\mathbf{D})$ gives the **second Bianchi idetity**.

Proposition 1: (**First Bianchi idetity**) *We have*

$$\mathrm{Alt}\,(\nabla_{\mathbf{C}}\mathbf{T}(\mathbf{C}) + \mathbf{T}(\mathbf{C})\mathbf{T}(\mathbf{C})) = \mathrm{Alt}\,(\mathbf{R}(\mathbf{C})) \tag{44.1}$$

where $\mathbf{T}(\mathbf{C})\mathbf{T}(\mathbf{C})$ *is regarded as a cross section of* $\mathrm{Skw}_2(\mathrm{T}\mathcal{M}^2, \mathrm{Lin}\mathrm{T}\mathcal{M})$.

Proof: Applying Prop.1 of Sect.43, we have

$$\mathrm{Alt}\,(\nabla_{\mathbf{C}}\mathbf{T}(\mathbf{C}) + \mathbf{T}(\mathbf{C})\mathbf{T}(\mathbf{C})) = \mathrm{Alt}\,(\nabla_{\mathbf{C}^\chi}\mathbf{T}(\mathbf{C}) + \mathbf{\Gamma}^\chi(\mathbf{C})^\sim\,\mathbf{T}(\mathbf{C})). \tag{44.2}$$

Using (33.8) and (34.30), we see that

$$\mathrm{Alt}\,(\nabla_{\mathbf{C}^\chi}\mathbf{T}(\mathbf{C}) + \mathbf{\Gamma}^\chi(\mathbf{C})^\sim\,\mathbf{T}(\mathbf{C})) = \mathrm{Alt}\,(\mathbf{R}(\mathbf{C})). \tag{44.3}$$

The desire result (44.1) follows from (44.2) and (44.3). ∎

Remark 1: When $\mathbf{C}$ is curvature-free (but not necessary torsion free), Eq. (44.1) reduces to

$$\mathrm{Alt}\,(\nabla_{\mathbf{C}}\mathbf{T}(\mathbf{C}) + \mathbf{T}(\mathbf{C})\mathbf{T}(\mathbf{C})) = \mathbf{0}. \tag{44.4}$$

If in addition that $\mathrm{Alt}\,(\nabla_{\mathbf{C}}\mathbf{T}(\mathbf{C})) = \mathbf{0}$, then

$$\mathrm{Alt}\,(\mathbf{T}(\mathbf{C})\mathbf{T}(\mathbf{C})) = \mathbf{0}; \tag{44.5}$$

that is $\mathbf{T}(\mathbf{C})$ satisfies Jacobi identity (cf. Lie Group, Prop.7 of Sect.44). ∎

Proposition 2: (**Second Bianchi idetity**) *We have*

$$\mathrm{Alt}\,(\nabla_{(\mathbf{C},\mathbf{D})}\mathbf{R}(\mathbf{D}) + \mathbf{R}(\mathbf{D})\mathbf{T}(\mathbf{C})) = \mathbf{0}. \tag{44.6}$$

where $\mathbf{R}(\mathbf{D})\mathbf{T}(\mathbf{C})$ *is regarded as a cross section of* $\mathrm{Skw}_2(\mathrm{T}\mathcal{M}^2, \mathrm{Lin}(\mathrm{T}\mathcal{M}, \mathrm{Lin}\mathcal{B}))$.

Proof: Applying Prop.1 of Sect.43, we have

$$\begin{aligned}&\mathrm{Alt}\,(\nabla_{(\mathbf{C},\mathbf{D})}\mathbf{R} + \mathbf{R}_x(\mathbf{C})(\mathbf{T}_x(\mathbf{C})))\\ &\quad = \mathrm{Alt}\,(\nabla_{(\mathbf{A}_x^\chi,\mathbf{A}_x^\phi)}\mathbf{R} + \mathbf{\Gamma}_x^\phi(\mathbf{D})^\sim\mathbf{R}_x(\mathbf{C}) - \mathbf{R}_x(\mathbf{C})(\cdot,\cdot)\mathbf{\Gamma}_x^\phi(\mathbf{D})).\end{aligned} \tag{44.7}$$

Applying Prop.5 of Sect.34, we obtain

$$\begin{aligned} &\mathrm{Alt}\,(\nabla_{(\mathbf{A}_x^\chi,\mathbf{A}_x^\phi)}\mathbf{R} + \mathbf{\Gamma}_x^\phi(\mathbf{D})^\sim \mathbf{R}_x(\mathbf{C}) - \mathbf{R}_x(\mathbf{C})(\cdot,\cdot)\mathbf{\Gamma}_x^\phi(\mathbf{D})) \\ &\quad = \mathrm{Alt}\left(\nabla^{(2)}_{(\mathbf{A}_x^\chi,\mathbf{A}_x^\phi)}\mathbf{\Gamma}^\phi(\mathbf{D}) - \big(\nabla^{(2)}_{(\mathbf{A}_x^\chi,\mathbf{A}_x^\phi)}\mathbf{\Gamma}^\phi(\mathbf{D})\big)^\sim\right). \end{aligned} \tag{44.8}$$

In view of (44.5), we observe that

$$\nabla^{(2)}_{(\mathbf{A}_x^\chi,\mathbf{A}_x^\phi)}\mathbf{\Gamma}^\phi(\mathbf{D}) - \big(\nabla^{(2)}_{(\mathbf{A}_x^\chi,\mathbf{A}_x^\phi)}\mathbf{\Gamma}^\phi(\mathbf{D})\big)^\sim = \mathbf{0}. \tag{44.9}$$

The desired result follows from (44.7), (44.8) and (44.9). ∎

Remark 2: When the given linear-space bundle is the tangent bundle $\mathcal{B} := \mathrm{T}\mathcal{M}$ of $\mathcal{M}$, the Bianchi identities can be found in literatures (see [P]) as

$$\begin{aligned} &(\nabla_{\mathbf{C}}\mathbf{T}(\mathbf{C}))(\mathbf{U},\mathbf{V},\mathbf{W}) + (\nabla_{\mathbf{C}}\mathbf{T}(\mathbf{C}))(\mathbf{V},\mathbf{W},\mathbf{U}) + (\nabla_{\mathbf{C}}\mathbf{T}(\mathbf{C}))(\mathbf{W},\mathbf{U},\mathbf{V}) \\ &+\mathbf{T}(\mathbf{C})(\mathbf{T}(\mathbf{C})(\mathbf{U},\mathbf{V}),\mathbf{W}) + \mathbf{T}(\mathbf{C})(\mathbf{T}(\mathbf{C})(\mathbf{V},\mathbf{W}),\mathbf{U}) + \mathbf{T}(\mathbf{C})(\mathbf{T}(\mathbf{C})(\mathbf{W},\mathbf{U}),\mathbf{V}) \\ &= \mathbf{R}(\mathbf{C})(\mathbf{U},\mathbf{V},\mathbf{W}) + \mathbf{R}(\mathbf{C})(\mathbf{V},\mathbf{W},\mathbf{U}) + \mathbf{R}(\mathbf{C})(\mathbf{W},\mathbf{U},\mathbf{V}) \end{aligned} \tag{44.10}$$

and

$$\begin{aligned} &(\nabla_{\mathbf{C}}\mathbf{R}(\mathbf{C}))(\mathbf{U},\mathbf{V},\mathbf{W}) + (\nabla_{\mathbf{C}}\mathbf{R}(\mathbf{C}))(\mathbf{V},\mathbf{W},\mathbf{U}) + (\nabla_{\mathbf{C}}\mathbf{R}(\mathbf{C}))(\mathbf{W},\mathbf{U},\mathbf{V}) \\ &+\mathbf{R}(\mathbf{C})(\mathbf{T}(\mathbf{C})(\mathbf{U},\mathbf{V}),\mathbf{W}) + \mathbf{R}(\mathbf{C})(\mathbf{T}(\mathbf{C})(\mathbf{V},\mathbf{W}),\mathbf{U}) + \mathbf{R}(\mathbf{C})(\mathbf{T}(\mathbf{C})(\mathbf{W},\mathbf{U}),\mathbf{V}) \\ &= \mathbf{0} \end{aligned} \tag{44.11}$$

for all vector fields $\mathbf{U},\mathbf{V},\mathbf{W} \in \mathfrak{X}\mathrm{T}\mathcal{M}$. ∎

Remark 3: Most of the literatures, especially in physics, only deal with the special case : in the absence of torsion. Under this assumption, the Bianchi identities becomes

$$\mathrm{Alt}\,(\mathbf{R}(\mathbf{C})) = \mathbf{0} \tag{44.12}$$

and

$$\mathrm{Alt}\,(\nabla_{\mathbf{C}}\mathbf{R}(\mathbf{C})) = \mathbf{0}. \tag{44.13}$$

45. Differential Forms

Let $p \in$ and a differentiable $\mathcal{W}$-valued skew p-linear field $\boldsymbol{\omega}$ be given.

In this section, we apply Prop.1 of Sect.43 with the tensor functor $\mathbf{\Phi} := \mathrm{Tr}_{\mathcal{W}}$, the trival functor for a linear space $\mathcal{W}$ (see Sect.13).

Proposition 1: *For every $x \in \mathcal{M}$, we have*

$$\mathrm{Alt}\,(\nabla_x^{\chi}\boldsymbol{\omega}) = \mathrm{Alt}\,(\nabla_x^{\gamma}\boldsymbol{\omega}) \tag{45.1}$$

for all manifold charts $\chi, \gamma \in \mathrm{Ch}_x\mathcal{M}$.

Proof: The desire result (45.1) follows from Prop.1 of Sect.43 with $(\mathrm{Tr}_{\mathcal{W}})^{\bullet}_x = \mathbf{0}$ and $\mathbf{T}_x(\mathbf{A}_x^{\chi}) = \mathbf{0} = \mathbf{T}_x(\mathbf{A}_x^{\gamma})$ (see Theorem in Sect.33) for all manifold charts $\chi, \gamma \in \mathrm{Ch}_x\mathcal{M}$.

<u>**Definition**</u> **:** *The p^{th}-***exterior differential at** $x \in \mathcal{M}$

$$\boldsymbol{d}_x^p : \mathfrak{X}(\mathrm{Skw}_p(\mathrm{T}\mathcal{M}^p,)) \to \mathrm{Skw}_{p+1}(\mathrm{T}_x\mathcal{M}^{p+1},) \tag{45.2}$$

is defined by

$$\boldsymbol{d}_x^p\boldsymbol{\omega} := \frac{1}{p!}\mathrm{Alt}\,(\nabla_x^{\chi}\boldsymbol{\omega}) \quad \text{for all} \quad \boldsymbol{\omega} \in \mathfrak{X}(\mathrm{Skw}_p(\mathrm{T}\mathcal{M}^p,)) \tag{45.3}$$

which is valid for all manifold chart $\chi \in \mathrm{Ch}_x\mathcal{M}$.

*The p^{th}-***exterior differential**

$$\boldsymbol{d}^p : \mathfrak{X}^s(\mathrm{Skw}_p(\mathrm{T}\mathcal{M}^p,)) \to \mathfrak{X}^{s-1}(\mathrm{Skw}_{p+1}(\mathrm{T}\mathcal{M}^{p+1},)) \tag{45.4}$$

is defined by

$$\boldsymbol{d}^p(x) := \boldsymbol{d}_x^p \quad \text{for all} \quad x \in \mathcal{M}. \tag{45.5}$$

Remark : If $\mathcal{M}$ be the underline manifold of a flat space $\mathcal{E}$, then $\nabla\boldsymbol{\omega} = \nabla^{\chi}\boldsymbol{\omega}$ for all manifold chart χ. The definition (45.3) of exterior differential at x becomes

$$\boldsymbol{d}^p\boldsymbol{\omega} = \frac{1}{p!}\mathrm{Alt}\,(\nabla\boldsymbol{\omega}). \tag{45.6}$$

Equation (45.6) can be found in Sect.2.3 of [CH] and in Sect.51 of [B-W]. ▮

Proposition 2: *Let $\mathcal{W}$ be a linear space and let $\boldsymbol{\omega} : \mathcal{M} \to \mathrm{Skw}_p(\mathrm{T}\mathcal{M}^p, \mathcal{W})$ be a differentiable $\mathcal{W}$-valued skew p-linear field. For every $x \in \mathcal{M}$,we have*

$$\begin{aligned} \boldsymbol{d}_x^p\boldsymbol{\omega}(\mathbf{v}) = (&\frac{1}{p!}\mathrm{Alt}\,(\nabla_{\mathbf{C}(x)}\boldsymbol{\omega}))\mathbf{v} \\ &+ \sum_{1\le i<j\le p+1}(-1)^{i+j-1}\boldsymbol{\omega}(x)\big(\mathbf{T}_x(\mathbf{C}(x))(\mathbf{v}_i, \mathbf{v}_j), \mathrm{del}_{(i,j)}\mathbf{v}\big) \end{aligned} \tag{45.7}$$

for all connection $\mathbf{C}$ and all $\mathbf{v} \in \mathrm{T}_x\mathcal{M}^{p+1}$.

Proposition 3: *We have*

$$\boldsymbol{d}^{p+1} \circ \boldsymbol{d}^{p} = \mathbf{0}. \tag{45.7}$$

46. Lie gradients, Lie brackets

In this section, we only deal with the tangent bundle of a given C^s-manifold $\mathcal{M}$, where $2 \leq s \in \tilde{}$.

We assume that a vector-field $\mathbf{h}$ is given and that $\mathbf{h}$ is differentiable at x.

Proposition 1: *There is exactly one shift, which is called the* **shift** *of* $\mathbf{h}$ *at* x *and is denoted by* $\rhd_x \mathbf{h} \in \mathrm{S}_x\mathrm{T}\mathcal{M}$*, such that*

$$\mathbf{B}_x (\rhd_x \mathbf{h}) = \Box_x \mathbf{h}, \tag{46.1}$$

where $\mathbf{B}_x$ *is given in* (33.6) *and* $\Box_x\mathbf{h} \in \mathrm{Lin}\,(\mathrm{S}_x\mathrm{T}\mathcal{M}, \mathrm{T}_x\mathcal{M})$ *is the shift-gradient of* $\mathbf{h}$ *as defined by* (41.3). *We have*

$$\mathbf{P}_x (\rhd_x \mathbf{h}) = \mathbf{h}(x) \tag{46.2}$$

Proof: The injectivity of $\mathbf{B}_x$ (see Prop. 2 of Sect.15) shows that there is at most one $\rhd_x \mathbf{h} \in \mathrm{S}_x\mathrm{T}\mathcal{M}$ with the property (46.1).

We now choose $\chi \in \mathrm{Ch}_x\mathcal{M}$ and define

$$\rhd_x \mathbf{h} := \mathbf{I}_x \big((\Box_x\mathbf{h})\, \mathbf{A}_x^\chi\big) + \mathbf{A}_x^\chi\, \mathbf{h}(x). \tag{46.3}$$

By $(15.6)_1$ and (32.23) we have

$$\begin{aligned} \mathbf{B}_x (\rhd_x \mathbf{h}) &= (\Box_x\mathbf{h})(\mathbf{A}_x^\chi\, \mathbf{P}_x) + \mathbf{B}_x \big(\mathbf{A}_x^\chi\, \mathbf{h}(x)\big) \\ &= \Box_x\mathbf{h}\,(\mathbf{1}_{\mathrm{S}_x\mathrm{T}\mathcal{M}} - \mathbf{I}_x\, \mathbf{\Lambda}(\mathbf{A}_x^\chi)) + \mathbf{B}_x \big(\mathbf{A}_x^\chi\, \mathbf{h}(x)\big). \end{aligned} \tag{46.4}$$

It follows from (41.4) and $(15.6)_2$ that

$$\begin{aligned} \Box_x\mathbf{h}\Big(\mathbf{I}_x\big(\mathbf{\Lambda}(\mathbf{A}_x^\chi)(\mathbf{s})\big)\Big) &= -\mathbf{\Lambda}(\mathbf{A}_x^\chi)(\mathbf{s})\,\mathbf{h}(x) \\ &= -\mathbf{B}_x\,(\mathbf{s})\big(\mathbf{A}_x^\chi\,\mathbf{h}(x)\big) = \Big(\mathbf{B}_x\,\big(\mathbf{A}_x^\chi\,\mathbf{h}(x)\big)\Big)(\mathbf{s}) \end{aligned}$$

holds for all $\mathbf{s} \in \mathrm{S}_x\mathrm{T}\mathcal{M}$. Hence (46.4) reduces to (46.1). Applying $\mathbf{P}_x$ to (46.3) and observing $\mathbf{P}_x\,\mathbf{I}_x = \mathbf{0}$ and $\mathbf{P}_x\,\mathbf{A}_x^\chi = \mathbf{1}_{\mathrm{T}_x\mathcal{M}}$ yields (46.2). ∎

Proposition 2: *Let* $\chi \in \mathrm{Ch}_x\mathcal{M}$ *be given. The shift* $\rhd_x \mathbf{h}$ *of* $\mathbf{h}$ *at* x *satisfies*

$$\mathbf{\Lambda}(\mathbf{A}_x^\chi)(\rhd_x \mathbf{h}) = \nabla_x^\chi \mathbf{h} \tag{46.5}$$

Proof: The equality follows by operating on (44.3) with $\mathbf{\Lambda}(\mathbf{A}_x^\chi)$ and observing $\mathbf{\Lambda}(\mathbf{A}_x^\chi)\mathbf{I}_x = \mathbf{1}_{\mathrm{Lin}\mathrm{T}_x\mathcal{M}}$ and $\mathbf{\Lambda}(\mathbf{A}_x^\chi)\mathbf{A}_x^\chi = \mathbf{0}$. ∎

For every manifold chart $\chi \in \mathrm{Ch}_x\mathcal{M}$, we have

$$\mathbf{A}_x^\chi \mathbf{h}(x) + \mathbf{I}_x \Box_x \mathbf{h} \mathbf{A}_x^\chi = \left(\nabla_{\mathbf{1}_{\mathrm{T}_x\mathcal{M}}} \mathrm{tlis}_x^\chi\right)^{-1}\left(\, \mathbf{h}^\chi(x)\,,\, \nabla_x \mathbf{h}^\chi\,\right). \tag{46.6}$$

In view of (46.3), we have

$$\rhd_x \mathbf{h} = \left(\nabla_{\mathbf{1}_{\mathrm{T}_x\mathcal{M}}} \mathrm{tlis}_x^\chi\right)^{-1}\left(\, \mathbf{h}^\chi(x)\,,\, \nabla_x \mathbf{h}^\chi\,\right)$$

for every manifold chart $\chi \in \mathrm{Ch}_x\mathcal{M}$.

Remark: By (46.1) and the injectivity of $\mathbf{B}_x$, we have

$$\rhd_x \mathbf{k} = \mathbf{0} \quad \text{if and only if} \quad \Box_x \mathbf{k} = \mathbf{0} \tag{46.7}$$

Proposition 3: *If $f : \mathcal{M} \to$ is differentiable at x, so is the vector-field $f\mathbf{h}$ and we have*

$$\rhd_x(f\,\mathbf{h}) = f(x) \rhd_x \mathbf{h} + \mathbf{I}_x\,(\mathbf{h}(x) \otimes \nabla_x f). \tag{46.8}$$

Proof: It follows from $(15.6)_1$ with $\mathbf{M} := \mathbf{h}(x) \otimes \nabla_x f$ that

$$\mathbf{B}_x\left(\mathbf{I}_x\,(\mathbf{h}(x) \otimes \nabla_x f)\right) = (\mathbf{h}(x) \otimes \nabla_x f)\,\mathbf{P}_x = \mathbf{h}(x) \otimes \mathbf{P}_x^\top \nabla_x f.$$

In view of (46.4) and (41.15), it follows that

$$\begin{aligned}\mathbf{B}_x\left(\rhd_x(f\,\mathbf{h})\right) = \Box_x(f\,\mathbf{h}) &= f(x)\Box_x \mathbf{h} + \mathbf{h}(x) \otimes \mathbf{P}_x^\top \nabla_x f \\ &= \mathbf{B}_x\left(f(x) \rhd_x \mathbf{h} + \mathbf{I}_x\,(\mathbf{h}(x) \otimes \nabla_x f)\right)\end{aligned}$$

Since $\mathbf{B}_x$ is injective, (46.8) follows. ▮

Let $\mathbf{\Phi}$ be a functor as described in Sect.13 and let $\mathbf{H} : \mathcal{M} \to \mathbf{\Phi}(\mathrm{T}\mathcal{M})$ be a tensor-field that is differentiable at x. Also, let $\mathbf{k}$ be a vector-field that is differentiable at x.

Definition: *The* **Lie-gradient** *of* **$\mathbf{H}$ with respect to $\mathbf{k}$ at** x *is defined by*

$$(\mathrm{Lie}_{\mathbf{k}}\mathbf{H})_x := \Box_x \mathbf{H}(\rhd_x \mathbf{k}), \tag{46.9}$$

where $\Box_x\mathbf{H}$ is the shift-gradient of $\mathbf{H}$ at x as defined by (41.3) *and where $\rhd_x \mathbf{k}$ is the shift of $\mathbf{k}$ at x as determined by* (46.1).

Proposition 4: *Let $f : \mathcal{M} \to$ and $\mathbf{H}$ be differentiable at x. We have*

$$\begin{aligned}\left(\mathrm{Lie}_{\mathbf{k}} f\,\mathbf{H}\right)_x &= f(x)\left(\mathrm{Lie}_{\mathbf{k}}\mathbf{H}\right)_x + \left((\nabla_x f)\,\mathbf{k}(x)\right)\mathbf{H}(x); \\ \left(\mathrm{Lie}_{f\mathbf{k}}\mathbf{H}\right)_x &= f(x)\left(\mathrm{Lie}_{\mathbf{k}}\mathbf{H}\right)_x + \left(\mathbf{\Phi}_x^\bullet\left(\mathbf{k}(x) \otimes \nabla_x f\right)\right)\mathbf{H}(x),\end{aligned} \tag{46.9}$$

*where $\mathbf{\Phi}_x^\bullet \in \mathrm{Lin}\left(\mathrm{Lin}\mathrm{T}_x, \mathrm{Lin}\mathbf{\Phi}(\mathrm{T}_x)\right)$ is defined as in Prop.1 of Sect.*41.

General Product Rule

Let $\mathbf{H}_1, \mathbf{H}_2$ *be cross sections as given in the General Product Rule of Sect.*41, *then we have*

$$(\mathrm{Lie}_{\mathbf{k}} B(\mathbf{H}_1, \mathbf{H}_2))_x = B_{\mathcal{B}_x}\big((\mathrm{Lie}_{\mathbf{k}}\mathbf{H}_1)_x, \mathbf{H}_2(x)\big) + B_{\mathcal{B}_x}\big(\mathbf{H}_1(x), (\mathrm{Lie}_{\mathbf{k}}\mathbf{H}_2)_x\big). \tag{46.10}$$

Remark: We have

$$(\mathrm{Lie}_{\mathbf{k}}\mathbf{H})_x = (\nabla_{\mathbf{K}}\mathbf{H})\mathbf{k}(x) + \mathbf{\Phi}^{\bullet}\big(\mathbf{T}_x(\mathbf{K})\mathbf{k}(x) + \nabla_{\mathbf{K}}\mathbf{k}\big)\mathbf{H}(x)$$

for all $\mathbf{K} \in \mathrm{Com}_x(\mathrm{T}\mathcal{M})$. ∎

We now assume that two vector-fields $\mathbf{h}$ and $\mathbf{k}$, both are differentiable at x, are given.

<u>Definition</u>: *The* **Lie-bracket** *of* $\mathbf{h}$ *with* $\mathbf{k}$ *at* x *is defined by*

$$[\![\, \mathbf{k}\, ,\ \mathbf{h}\,]\!]_x := \mathbf{B}_x(\rhd_x \mathbf{h}, \rhd_x \mathbf{k}). \tag{46.11}$$

It follows from (46.1), (46.9) and (46.11) that

$$[\![\, \mathbf{k}\, ,\ \mathbf{h}\,]\!]_x = (\mathrm{Lie}_{\mathbf{k}}\mathbf{h})_x \tag{46.12}$$

Proposition 5: *We have*

$$[\![\, \mathbf{k}\, ,\ \mathbf{h}\,]\!]_x = -[\![\, \mathbf{h}\, ,\ \mathbf{k}\,]\!]_x. \tag{46.13}$$

If $f : \mathcal{M} \to$ *is differentiable at* x, *then*

$$[\![\, f\,\mathbf{h}\, ,\ \mathbf{k}\,]\!]_x = f(x)[\![\, \mathbf{h}\, ,\ \mathbf{k}\,]\!]_x - \big((\nabla_x f)\,\mathbf{k}(x)\big)\,\mathbf{h}(x). \tag{46.14}$$

Proof: (46.13) follows from the skewness of $\mathbf{B}_x$. Substitution of $f\mathbf{h}$ for $\mathbf{h}$ in (46.11) and use of (46.8) gives

$$[\![\, f\,\mathbf{h}\, ,\ \mathbf{k}\,]\!]_x = f(x)[\![\, \mathbf{h}\, ,\ \mathbf{k}\,]\!]_x - \mathbf{B}_x\left(\mathbf{I}_x\,(\mathbf{h}(x) \otimes \nabla_x f), \rhd_x \mathbf{k}\right)$$

and hence, by $(15.6)_1$,

$$[\![\, f\,\mathbf{h}\, ,\ \mathbf{k}\,]\!]_x = f(x)[\![\, \mathbf{h}\, ,\ \mathbf{k}\,]\!]_x - (\mathbf{h}(x) \otimes \nabla_x f)(\mathbf{P}_x \rhd_x \mathbf{k})$$

The desired result (46.14) now follows from (46.2). ∎

Remark: Let $r = \infty$, let $\mathbf{h}, \mathbf{k} \in \mathfrak{X}^\infty \mathcal{M}$ and let $\overset{\triangledown}{\mathbf{h}}$ and $\overset{\triangledown}{\mathbf{k}}$ be the mappings from $C^\infty(\mathcal{M})$ to $C^\infty(\mathcal{M})$ defined by (24.6). One can easily show that the mapping $\overset{\triangledown}{[\![\,\mathbf{h}\,,\,\mathbf{k}\,]\!]} : C^\infty(\mathcal{M}) \to C^\infty(\mathcal{M})$ corresponding to $\overset{\triangledown}{[\![\,\mathbf{h}\,,\,\mathbf{k}\,]\!]}$ is given by

$$\overset{\triangledown}{[\![\,\mathbf{h}\,,\,\mathbf{k}\,]\!]} = \overset{\triangledown}{\mathbf{h}} \circ \overset{\triangledown}{\mathbf{k}} - \overset{\triangledown}{\mathbf{k}} \circ \overset{\triangledown}{\mathbf{h}} \tag{46.15}$$

If $f \in C^\infty(\mathcal{M})$, we then have

$$\overset{\triangledown}{[\![\,f\mathbf{h}\,,\,\mathbf{k}\,]\!]} = f\,[\![\,\overset{\triangledown}{\mathbf{h}}\,,\,\overset{\triangledown}{\mathbf{k}}\,]\!] - \overset{\triangledown}{\mathbf{k}}(f)\overset{\triangledown}{\mathbf{h}}, \tag{46.16}$$

which can be derived from (46.14) or directly from (46.15). ▮

Proposition 6: *If both* $\mathbf{h}$ *and* $\mathbf{k}$ *are vector-fields that are differentiable at* x, *then have*

$$[\![\,\mathbf{h}\,,\,\mathbf{k}\,]\!]_x = (\nabla_x^\chi \mathbf{k})\,\mathbf{h}(x) - (\nabla_x^\chi \mathbf{h})\,\mathbf{k}(x). \tag{46.17}$$

for every manifold chart $\chi \in \mathbf{Ch}_x\mathcal{M}$ *where* $\nabla_x^\chi \mathbf{k}$ *and* $\nabla_x^\chi \mathbf{h}$ *be defined according to* (23.26). *Moreover, we have*

$$(\nabla_{\mathbf{K}}\mathbf{k})\mathbf{h}(x) - (\nabla_{\mathbf{K}}\mathbf{h})\mathbf{k}(x) = [\![\,\mathbf{h}\,,\,\mathbf{k}\,]\!]_x + \mathbf{T}_x(\mathbf{K})(\mathbf{h},\mathbf{k}) \tag{46.18}$$

for all $\mathbf{K} \in \mathrm{Con}_x\mathrm{T}\mathcal{M}$.

Proof: If we substitute $\mathbf{s} := \triangleright_x \mathbf{h}$ and $\mathbf{s}' := \triangleright_x \mathbf{k}$ in (33.6) and (12.5) we obtain from (46.11) that

$$[\![\,\mathbf{h}\,,\,\mathbf{k}\,]\!]_x = -\mathbf{D}_x^\chi\,(\triangleright_x \mathbf{h})\mathbf{P}_x\,(\triangleright_x \mathbf{k}) + \mathbf{D}_x^\chi\,(\triangleright_x \mathbf{k})\mathbf{P}_x\,(\triangleright_x \mathbf{h})$$

The desired result (46.17) follows now from (46.5) and (46.2).

By (42.3) we have

$$(\nabla_{\mathbf{K}}\mathbf{h})\mathbf{k}(x) = (\nabla_x^\chi \mathbf{h})\mathbf{k}(x) + \mathbf{\Gamma}_x^\chi(\mathbf{K})\big(\mathbf{k}(x), \mathbf{h}(x)\big).$$

Interchanging $\mathbf{h}$ and $\mathbf{k}$ and taking the difference, we obtain (46.18) from (46.17) and (33.8). ▮

Let $s \in 1..(r-1)$ and $\mathbf{h}, \mathbf{k} \in \mathfrak{X}^s\mathrm{T}\mathcal{M}$ be given. Then the vector-field $[\![\,\mathbf{h}\,,\,\mathbf{k}\,]\!]$ is defined by

$$[\![\,\mathbf{h}\,,\,\mathbf{k}\,]\!](x) := [\![\,\mathbf{h}\,,\,\mathbf{k}\,]\!]_x \qquad \text{for all} \quad x \in \mathcal{M} \tag{46.19}$$

It is clear from **Proposition 5** that $[\![\,\mathbf{h}\,,\,\mathbf{k}\,]\!] \in \mathfrak{X}^{s-1}\mathrm{T}\mathcal{M}$. Using (23.6), it follows from (46.17) and the definition (23.35) that

$$[\![\,\mathbf{h}\,,\,\mathbf{k}\,]\!]^\chi = (\nabla_\chi \mathbf{k}^\chi)\,\mathbf{h}^\chi - (\nabla_\chi \mathbf{h}^\chi)\,\mathbf{k}^\chi. \tag{46.20}$$

Proposition 7: (**Jacobi identity**): *Let* $s \in 2..(r-1)$ *and* $\mathbf{h}_1, \mathbf{h}_2, \mathbf{h}_3 \in \mathfrak{X}^s \mathrm{T}\mathcal{M}$ *be given, then*

$$[\![\, [\![\, \mathbf{h}_1 \,,\, \mathbf{h}_2 \,]\!] \,,\, \mathbf{h}_3 \,]\!] + [\![\, [\![\, \mathbf{h}_2 \,,\, \mathbf{h}_3 \,]\!] \,,\, \mathbf{h}_1 \,]\!] + [\![\, [\![\, \mathbf{h}_3 \,,\, \mathbf{h}_1 \,]\!] \,,\, \mathbf{h}_2 \,]\!] = \mathbf{0} \qquad (46.21)$$

Proof: A straightforward but somewhat tedious calculation, using (46.20) and the Symmetry Theorem for Second Gradients, yields the desired result (46.21). ▮

If $\mathcal{M}$ is a C^∞ manifold, then $\mathfrak{X}^\infty \mathrm{T}\mathcal{M}$ together with the bilinear mapping

$$[\![\ ,\ ]\!] : \mathfrak{X}^\infty \mathrm{T}\mathcal{M} \times \mathfrak{X}^\infty \mathrm{T}\mathcal{M} \longrightarrow \mathfrak{X}^\infty \mathrm{T}\mathcal{M}$$

given in (46.21) is a Lie algebra, as defined in Sect.11.

47. Transport Systems

We assume that $r \in \tilde{\ }$ with $r \geq 2$ and a C^r-manifold $\mathcal{M}$ are given. Let $(\mathcal{B}, \tau, \mathcal{M})$ be a C^s linear-space bundle, $s \in 0..r$.

We define the **bundle of transfer isomorphisms** of $\mathcal{B}$ by

$$\mathrm{Tlis}\,\mathcal{B} := \bigcup_{x \in \mathcal{M}} \mathrm{Tlis}_x \mathcal{B} = \bigcup_{x,y \in \mathcal{M}} \mathrm{Lis}(\mathcal{B}_x, \mathcal{B}_y). \qquad (47.1)$$

It is endowed with the natural structure of a C^s-fiber bundle over $\mathcal{M} \times \mathcal{M}$ whose bundle projection $\pi : \mathrm{Tlis}\,\mathcal{B} \to \mathcal{M} \times \mathcal{M}$ is

$$\pi(\mathbf{T}) :\in \left\{ \ (x, y) \in \mathcal{M} \times \mathcal{M} \ \Big| \ \mathbf{T} \in \mathrm{Lis}(\mathcal{B}_x, \mathcal{B}_y) \ \right\}. \qquad (47.2)$$

__**Definition:**__ *A subset* $\mathfrak{T}$ *of* $\mathrm{Tlis}\,\mathcal{B}$ *is called a* C^s **transport structure for** $\mathcal{B}$ *if* $\mathfrak{T}$ *is a* C^s*-submanifold of* $\mathrm{Tlis}\,\mathcal{B}$ *such that*

(T1) *for all* $\mathbf{A} \in \mathfrak{T}$, $\mathbf{A}^{-1} \in \mathfrak{T}$,

(T2) *for all* $\mathbf{A}, \mathbf{B} \in \mathfrak{T}$ *such that* $\mathrm{Cod}\,\mathbf{A} = \mathrm{Dom}\,\mathbf{B}$, $\mathbf{BA} \in \mathfrak{T}$,

(T3) *for all* $x, y \in \mathcal{M}$, $\mathfrak{T} \cap \mathrm{Lis}(\mathcal{B}_x, \mathcal{B}_y) \neq \{\ \}$.

It can be shown that $\mathfrak{T}_x := \mathfrak{T} \cap \mathrm{Tlis}_x\,\mathcal{B}$ is a C^s-submanifold of $\mathrm{Tlis}_x\,\mathcal{B}$.

Theorem on Transport Structure and Parallelisms

Let $\mathbf{C} : \mathcal{M} \to \mathrm{Con}\mathcal{B}$ *be a connection of class* C^s. *Define*

$$\mathfrak{F} := \{\mathbf{A} \in \mathrm{Tlis}\,\mathcal{B} \mid \cdots\cdots\cdots\cdots\}.$$

Then $\mathfrak{F}$ *is a transport structure for* $\mathcal{B}$.

Proof:

A cross section $\mathbf{F} : \mathcal{M} \times \mathcal{M} \to \mathfrak{T}$ is called a (global) **transport system for** $\mathcal{B}$ if

$$\mathbf{F}(x,z) = \mathbf{F}(y,z)\mathbf{F}(x,y) \qquad \text{for all} \quad x,y,z \in \mathcal{M} \tag{47.3}$$

and

$$\mathbf{F}(x,x) = \mathbf{1}_{\mathcal{B}_x} \qquad \text{for all} \quad x \in \mathcal{M}. \tag{47.4}$$

Recall that a cross section $\mathbf{T} : \mathcal{M} \to \mathrm{Tlis}_x\mathcal{B}$ of the bundle $\mathrm{Tlis}_x\mathcal{B}$, $x \in \mathcal{M}$, with

$$\mathbf{T}(x) = \mathbf{1}_{\mathcal{B}_x} \tag{47.5}$$

is called a transport from x. It follows from (47.3), (47.4) and (47.5) that, for each $x \in \mathcal{M}$, the mapping $\mathbf{F}(x,\cdot) : \mathcal{M} \to \mathrm{Tlis}_x\mathcal{B}$ is a transport from x. Moreover, we have

$$\mathbf{F}(y,\cdot) = \mathbf{F}(x,\cdot)\mathbf{F}(y,x) \quad \text{for all} \quad x,y \in \mathcal{M}. \tag{47.6}$$

Conversely, let $x \in \mathcal{M}$ and a transport $\mathbf{F}_x : \mathcal{M} \to \mathrm{Tlis}_x\mathcal{B}$ from x be given. For each $y \in \mathcal{M}$, we obtain a transport $\mathbf{F}_y : \mathcal{M} \to \mathrm{Tlis}_y\mathcal{B}$ from y by

$$\mathbf{F}_y(z) := \mathbf{F}_x(z)\mathbf{F}_x(y)^{-1} \quad \text{for all} \quad z \in \mathcal{M}. \tag{47.7}$$

and, a transport system $\mathbf{F} : \mathcal{M} \times \mathcal{M} \to \mathrm{Tlis}\,\mathcal{B}$ by

$$\mathbf{F}(y,z) := \mathbf{F}_x(z)\mathbf{F}_x(y)^{-1} \quad \text{for all} \quad y,z \in \mathcal{M}. \tag{47.8}$$

We conclude that, for each $x \in \mathcal{M}$, there is one to one correspondent between the set of all transports from x for $\mathcal{B}$ and the set of all transport systems for $\mathcal{B}$.

Every transport system $\mathbf{F} : \mathcal{M} \times \mathcal{M} \to \mathrm{Tlis}\,\mathcal{B}$ induces a connection $\mathbf{C} : \mathcal{M} \to \mathrm{Con}\mathcal{B}$ by

$$\mathbf{C}(y) := \nabla_{\mathbf{1}_{\mathcal{B}_y}}\mathbf{F}(y,\cdot) \quad \text{for all} \quad y \in \mathcal{M}. \tag{47.9}$$

Let a transport system $\mathbf{F} : \mathcal{M} \times \mathcal{M} \to \mathrm{Tlis}\,\mathcal{B}$ for $\mathcal{B}$, a tensor functor $\mathbf{\Phi}$ and a cross section $\mathbf{H} : \mathcal{M} \to \mathbf{\Phi}(\mathcal{B})$ be given. We say that $\mathbf{H}$ is **parallel with respect to F** if

$$\mathbf{H}(y) = \mathbf{\Phi}(\mathbf{F}(x,y))\mathbf{H}(x) \quad \text{for all} \quad x,y \in \mathcal{M}. \tag{47.10}$$

Proposition 1: *Let* $\mathbf{C}$ *be the connection induced by a transport system* $\mathbf{F}$*, as given in* (47.9)*. Let* $\mathbf{H}:\mathcal{O}\to\mathbf{\Phi}(\mathcal{B})$ *be a cross section of class* C^1*. If* $\mathbf{H}$ *is parallel with respect to* $\mathbf{F}$*, then* $\nabla_{\mathbf{C}}\mathbf{H}=\mathbf{0}$*. Conversely, if* $\nabla_{\mathbf{C}}\mathbf{H}=\mathbf{0}$ *and if* $\mathcal{M}$ *is connected then* $\mathbf{H}$ *is parallel with respect to* $\mathbf{F}$*.*

Proof: Fix $x\in\mathcal{M}$ and let $\mathbf{T}:=\mathbf{F}(x,\cdot)$. Let $y\in\mathcal{M}$ be given and define $\widehat{\mathbf{H}}_y:\mathrm{Tlis}_y\mathcal{B}\to\mathcal{B}_y$ in accord with (41.2). Then

$$\widehat{\mathbf{H}}_y(\mathbf{T}(z)\mathbf{T}(y)^{-1})=\mathbf{\Phi}(\mathbf{T}(y)\mathbf{T}(z)^{-1})\mathbf{H}(z)\quad\text{for all}\quad z\in\mathcal{M}.$$

Differentiation with respect to z at y gives, using (42.1), (41.3), (47.9), and the chain rule,

$$(\nabla_{\mathbf{C}}\mathbf{H})(y)=(\Box_y\mathbf{H})\mathbf{C}(y)=\mathbf{\Phi}(\mathbf{T}(y))\nabla_y\widetilde{\mathbf{H}},\tag{47.11}$$

where $\widetilde{\mathbf{H}}:\mathcal{M}\to\mathbf{\Phi}(\mathcal{B}_x)$ is defined by $\widetilde{\mathbf{H}}(z):=\mathbf{\Phi}(\mathbf{T}(z)^{-1})\mathbf{H}(z)$ for all $z\in\mathcal{M}$. Since $y\in\mathcal{M}$ was arbitrary and since $\mathbf{\Phi}(\mathbf{T}(y))$ is invertible, we conclude from (47.11) that $\nabla_{\mathbf{C}}\mathbf{H}=\mathbf{0}$, if and only if $\nabla\widetilde{\mathbf{H}}=\mathbf{0}$. Now if $\mathbf{H}=\mathbf{\Phi}(\mathbf{T})\mathbf{v}$ for some $\mathbf{v}\in\mathbf{\Phi}(\mathcal{B}_x)$, then $\widetilde{\mathbf{H}}$ is a constant and hence $\nabla\widetilde{\mathbf{H}}=0$. Conversely if $\mathcal{M}$ is connected and $\nabla\widetilde{\mathbf{H}}=0$, then $\widetilde{\mathbf{H}}$ is a constant and hence $\mathbf{H}=\mathbf{\Phi}(\mathbf{T})\mathbf{v}$ for some $\mathbf{v}\in\mathbf{\Phi}(\mathcal{B}_x)$. ∎

Remark : Let a connection $\mathbf{C}$, not necessarily induced by a transport system, be given. Then the condition $\nabla_{\mathbf{C}}\mathbf{H}=\mathbf{0}$ does not equivalent to to the condition that $\mathbf{H}$ is parallel with respective to a transport system. ∎

Proposition 2: *Let* $\mathbf{T}:[0,d]\to\mathrm{Tlis}_x\mathcal{B}$ *be a differentiable transfer process from* x*, and put* $p:=\pi_x\circ\mathbf{T}:[0,d]\to\mathcal{M}$*. For every differentiable cross section* $\mathbf{H}:\mathcal{M}\to\mathbf{\Phi}(\mathcal{B})$*, we have*

$$(\Box_{p(t)}\mathbf{H})(\mathrm{sd}_t\mathbf{T})=\partial_t\big(\,s\mapsto\mathbf{\Phi}(\mathbf{T}(t)\mathbf{T}^{-1}(s))\mathbf{H}(p(s))\,\big)\tag{47.12}$$

for all $t\in[0,d]$*, the derivative* (47.12) *may be interpreted, roughly, as the* **rate of change of H at** $p(t)$ **relative to** *the transfer process* $\mathbf{T}$*.*

Let $\mathbf{C}:\mathcal{M}\to\mathrm{Con}\mathcal{B}$ be a continuous connection and $p:[0,d]\to\mathcal{M}$ be a process of class C^1, with $x=p(0)$. Let $\mathbf{T}$ be the parallelism along p for the connection $\mathbf{C}$. It follows from (35.23), $\mathrm{sd}\mathbf{T}=(\mathbf{C}\circ p)p^\bullet$, that

$$(\nabla_{\mathbf{C}(p(t))}\mathbf{H})p^\bullet(t)=(\Box_{p(t)}\mathbf{H})(\mathrm{sd}_t\mathbf{T}).\tag{47.13}$$

This result does not depend on the choice of the process p, and hence does not depend on the parallelism $\mathbf{T}$ along p.

Proposition 3: *Let* $\mathbf{C} : \mathcal{M} \to \mathrm{Con}\mathcal{B}$ *be a continuous connection and let the cross section* $\mathbf{H} : \mathcal{M} \to \mathbf{\Phi}(\mathcal{B})$ *be differentiable. Then* $\nabla_{\mathbf{C}}\mathbf{H} = \mathbf{0}$ *if and only if, for every differentiable process* $p : [0, d] \to \mathcal{M}$,

$$((\Box\,\mathbf{H}) \circ p)(\mathrm{sd}\mathbf{T}) = \mathbf{0} \tag{47.14}$$

where $\mathbf{T}$ *is the parallelism along* p *for* $\mathbf{C}$.

Let $x \in \mathcal{M}$ and a continuous vector field $\mathbf{k} : \mathcal{M} \to \mathrm{T}\mathcal{M}$ be given. By the maximum **local flow** for $\mathbf{k}$ at x we mean a mapping

$$\alpha : I \times \mathcal{D} \to \mathcal{M}$$

where I is an open interval containing 0,and $\mathcal{D}$ containing x, and $\mathcal{D}$ is an open subset of $\mathcal{M}$ containing x, such that for every $y \in \mathcal{D}$ the mapping $\alpha(\cdot, y) : I \to \mathcal{M}$ is the maximum integral process (integral curve) of $\mathbf{k}$ with the initial condition y; i.e. $\alpha(0, y) = y$ and $\mathbf{k}\big(\alpha(t, y)\big) = (\alpha^{\bullet}(\cdot, y))(t)$.

Let $x \in \mathcal{M}$ and a continuous vector field $\mathbf{k} : \mathcal{M} \to \mathrm{T}\mathcal{M}$ be given. It is a well known theorem in O.D.E. (see Sect.1 of Ch.4, [L]) that there is a maximum local flow

$$\alpha : I \times \mathcal{D} \to \mathcal{M}$$

for $\mathbf{k}$ at x. We may define a mapping $\mathbf{L}_{\mathbf{k}} : I \to \mathrm{Tlis}_x\mathcal{M}$ by

$$\mathbf{L}_{\mathbf{k}}(t) := \nabla_x \alpha(t, \cdot) \quad \text{for all} \quad t \in I.$$

It is clear that

$$\mathbf{L}_{\mathbf{k}>}(I) = \bigcup_{y \in \alpha(\cdot, x)_>(I)} \mathrm{Lis}(\mathrm{T}_x, \mathrm{T}_y).$$

Since $\mathbf{L}_{\mathbf{k}}(0) = \mathbf{1}_{\mathrm{T}_x}$, $\mathbf{L}_{\mathbf{k}}$ is a transfer process from x. We shall call $\mathbf{L}_{\mathbf{k}}$ the **Lie transfer process** from x of the vector-field $\mathbf{k}$.

Proposition 4: *Let* $x \in \mathcal{M}$ *and a vector field* $\mathbf{k} : \mathcal{M} \to \mathrm{T}\mathcal{M}$ *be given. Let* $\mathbf{L}_{\mathbf{k}}$ *be the Lie transfer process from* x *of* $\mathbf{k}$. *We have* $\mathrm{sd}_0\mathbf{L}_{\mathbf{k}} = \rhd_x \mathbf{k}$ *and*

$$(\mathrm{Lie}_{\mathbf{k}}\mathbf{H})(x) = \partial_0\big(t \mapsto \mathbf{\Phi}(\mathbf{L}_{\mathbf{k}}(t)^{-1})\mathbf{H}(p(t))\big). \tag{47.15}$$

Proof: Define the processes $\mathbf{H} : I \to \mathrm{Lis}\mathcal{V}_\chi$ and $\mathbf{V} : I \to \mathrm{Lis}\mathcal{V}_\chi$ by

$$\begin{aligned}
\mathbf{H}(t) :&= \nabla_{\alpha_x(t)}\chi \nabla_x \alpha_t (\nabla_x \chi)^{-1} = \nabla_{\alpha_x(t)}\chi \mathbf{L}_{\mathbf{k}}(t)(\nabla_x\chi)^{-1} \\
\mathbf{V}(t) :&= \nabla_{\alpha_x(t)}\chi(\mathbf{D}^{\chi}_{\alpha_x(t)} \underset{\alpha_x(t)}{\rhd} \mathbf{k})(\nabla_{\alpha_x(t)}\chi)^{-1}
\end{aligned}$$

Taking the gradient of $\mathbf{H}$ at 0 and observing $\mathbf{D}^{\chi}_{\alpha_x(t)} \triangleright_{\alpha_x(t)} \mathbf{k} = (\nabla_{\alpha_x(t)}\chi)^{-1}\nabla_{\alpha_x(t)}\mathbf{k}^{\chi}$, we have

$$\begin{aligned}
\mathbf{H}^{\cdot}(t) &= \partial_t\big(s \mapsto \nabla_{\alpha_x(s)}\chi\nabla_x\alpha_s(\nabla_x\chi)^{-1}\big)\\
&= \partial_t\big(s \mapsto (\nabla_x\alpha_s)\big)^{\chi}(\nabla_x\chi)^{-1}\\
&= \nabla_x\big(\partial_t(s \mapsto \alpha_s)\big)^{\chi}(\nabla_x\chi)^{-1}\\
&= \nabla_x(\mathbf{k}^{\chi}\circ\alpha_t)(\nabla_x\chi)^{-1}\\
&= \nabla_{\alpha_x(t)}\mathbf{k}^{\chi}\nabla_x\alpha_t(\nabla_x\chi)^{-1}\\
&= \big(\nabla_{\alpha_x(t)}\chi\big((\nabla_{\alpha_x(t)}\chi)^{-1}\nabla_{\alpha_x(t)}\mathbf{k}^{\chi}\big)(\nabla_{\alpha_x(t)}\chi)^{-1}\big)\big(\nabla_{\alpha_x(t)}\chi\nabla_x\alpha_t(\nabla_x\chi)^{-1}\big)\\
&= \big(\nabla_{\alpha_x(t)}\chi(\mathbf{D}^{\chi}_{\alpha_x(t)} \underset{\alpha_x(t)}{\triangleright} \mathbf{k})(\nabla_{\alpha_x(t)}\chi)^{-1}\big)\big(\nabla_{\alpha_x(t)}\chi\nabla_x\alpha_t(\nabla_x\chi)^{-1}\big)\\
&= (\mathbf{V}\mathbf{H})(t).
\end{aligned}$$

This shows that $\mathbf{L_k}$ is the only transfer process from x such that $\mathrm{sd}\mathbf{L_k} = (\triangleright\mathbf{k})\circ\alpha_x$. Since $\alpha_x(0) = x$, we have $\mathrm{sd}_0\mathbf{L_k} = \triangleright_x\mathbf{k}$. The assertion follows by applying Prop.2. ▮

48. Lie Group

Definition: *A* **Lie group** *is a set $\mathcal{G}$ endowed both with the structure of a group and with the structure of a C^{ω}-manifold in such a way that the group-operation and the group-inversion are analytic mappings.*

We use multiplicative notation and terminology for the group $\mathcal{G}$ and denote its unity by u.

For every $x \in \mathcal{G}$, we define the **left-multiplication** $\mathrm{le}_x : \mathcal{G} \to \mathcal{G}$ by

$$\mathrm{le}_x(y) := xy \qquad \text{for all} \qquad y \in \mathcal{G}. \tag{48.1}$$

$\mathrm{le}_x : \mathcal{G} \to \mathcal{G}$, is invertible for all $x \in \mathcal{G}$; in fact,

$$(x \mapsto \mathrm{le}_x) : \mathcal{G} \to \mathrm{Perm}\,\mathcal{G} \tag{48.2}$$

is an injective group-homomorphism, *i.e.* we have

$$\mathrm{le}_u = \mathbf{1}_{\mathcal{G}} \quad , \quad \mathrm{le}_{xy} = \mathrm{le}_x \circ \mathrm{le}_y \quad , \quad \mathrm{le}_{x^{-1}} = \mathrm{le}_x^{\leftarrow} \tag{48.3}$$

for all $x, y \in \mathcal{G}$. Also, when $x \in \mathcal{G}$ is given, le_x is analytic and we have

$$\nabla_y\mathrm{le}_x \in \mathrm{Lis}(\mathrm{T}_x\mathcal{M}, \mathrm{T}_{xy}\mathcal{M}) \subset \mathrm{Tlis}_y\mathcal{G} \tag{48.4}$$

for all $y \in \mathcal{G}$. We define the analytic mapping

$$\mathbf{G} : \mathcal{G} \to \mathrm{Tlis}_u\mathcal{G} \tag{48.5}$$

by

$$\mathbf{G}(x) := \nabla_u \mathrm{le}_x \qquad \text{for all} \qquad x \in \mathcal{G}. \tag{48.6}$$

Taking the gradient of $(48.18)_2$ at u gives

$$\mathbf{G}(xy) := (\nabla_y \mathrm{le}_x)\mathbf{G}(y) \qquad \text{for all} \qquad x, y \in \mathcal{G}. \tag{48.7}$$

For every $\mathbf{t} \in \mathrm{T}_u\mathcal{M}$, we define the analytic vector field $\mathbf{Gt} : \mathcal{G} \to \mathrm{T}\,\mathcal{G}$ by

$$(\mathbf{Gt})(y) = \mathbf{G}(y)\mathbf{t} \qquad \text{for all} \qquad y \in \mathcal{G}. \tag{48.8}$$

We have

$$\mathbf{G}(u) = \mathbf{1}_{\mathrm{T}_u\mathcal{M}} \quad \text{and} \quad (\mathbf{Gt})(u) = \mathbf{t} \quad \text{for all} \quad \mathbf{t} \in \mathrm{T}_u\mathcal{M}. \tag{48.9}$$

Proposition 5: *For all* $\mathbf{t}, \mathbf{s} \in \mathrm{T}_u\mathcal{M}$ *we have*

$$[\![\, \mathbf{Gt}\,,\ \mathbf{Gs} \,]\!] = \mathbf{G}\,[\![\, \mathbf{Gt}\,,\ \mathbf{Gs} \,]\!]_u \tag{48.10}$$

Proof: Let $\mathbf{t} \in \mathrm{T}_u\mathcal{M}$ and $x \in \mathcal{G}$ be given and choose $\chi \in \mathrm{Ch}_x\mathcal{G}$. Since le_x is analytic and invertible and $\mathrm{le}_x(u) = x$, we have $\chi \circ \mathrm{le}_x \in \mathrm{Ch}_u\mathcal{G}$. Using the chain rule and (48.22), we obtain

$$\nabla_y(\chi \circ \mathrm{le}_x) = (\nabla_{xy}\chi)\nabla_y \mathrm{le}_x = (\nabla_{xy}\chi)\mathbf{G}(xy)\mathbf{G}_(y)^{-1} \quad \text{for all} \quad y \in \mathcal{G}. \tag{48.11}$$

Using the definitions (48.23) and (23.25), we see that

$$(\mathbf{Gt})^{\chi \circ \mathrm{le}_x}(y) = \nabla_y(\chi \circ \mathrm{le}_x)\mathbf{G}(y)\mathbf{t} = (\nabla_{xy}\chi)\mathbf{G}(xy)\mathbf{t}$$

for all $y \in \mathcal{G}$ and hence

$$(\mathbf{Gt})^{\chi \circ \mathrm{le}_x} = (\mathbf{Gt})^{\chi} \circ \mathrm{le}_x. \tag{48.12}$$

Using the chain rule again, we find

$$\nabla_u(\mathbf{Gt})^{\chi \circ \mathrm{le}_x} = \nabla_x(\mathbf{Gt})^{\chi}\mathbf{G}(x) \qquad \text{for all} \qquad \mathbf{t} \in \mathbf{T}_u \tag{48.13}$$

Now let $\mathbf{s}, \mathbf{t} \in \mathrm{T}_u\mathcal{M}$ be given and put $\mathbf{h} := \mathbf{Gt}$, $\mathbf{k} := \mathbf{Gs}$. Using (43.17) with x replaced by u and χ by $\chi \circ \mathrm{le}_x$ we conclude from (48.28) that

$$[\![\, \mathbf{h}\,,\ \mathbf{k} \,]\!]_u = \nabla_u(\chi \circ \mathrm{le}_u)^{-1}\big((\nabla_x\mathbf{k}^{\chi})\mathbf{h}(x) - (\nabla_x\mathbf{h}^{\chi})\mathbf{k}(x)\big).$$

Using (48.26) with $y := u$ and observing (48.23), we obtain

$$[\![\mathbf{h} \,,\, \mathbf{k}]\!]_u = \mathbf{G}(x)^{-1}\nabla_x\chi^{-1}\big((\nabla_x\mathbf{k}^\chi)\mathbf{h}(x) - (\nabla_x\mathbf{h}^\chi)\mathbf{k}(x)\big).$$

Since $x \in \mathcal{G}$ was arbitrary, we obtain (48.25) by applying (43.17) again. ∎

Proposition 6: *Define*

$$\big((\mathbf{t},\mathbf{s}) \mapsto [\mathbf{t},\mathbf{s}]\big) : \mathrm{T}_u\mathcal{M}^2 \to \mathrm{T}_u\mathcal{M} \tag{48.14}$$

by

$$[\mathbf{t},\mathbf{s}] := [\![\,\mathbf{Gt}\, ,\, \mathbf{Gs}\,]\!]_u, \tag{48.15}$$

where $\mathbf{G}$ *is defined by* (48.21). *Then* (48.21) *endows* $\mathrm{T}_u\mathcal{M}$ *with the structure of a Lie-algebra, i.e. it is bilinear, skew, and satisfies the "Jacobi-identity"*

$$\big[[\mathbf{t}_1,\mathbf{t}_2],\mathbf{t}_3\big] + \big[[\mathbf{t}_2,\mathbf{t}_3],\mathbf{t}_1\big] + \big[[\mathbf{t}_3,\mathbf{t}_1],\mathbf{t}_2\big] = \mathbf{0} \tag{48.16}$$

for all $\mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3 \in \mathrm{T}_u\mathcal{M}$. *We use the notation* $\mathrm{La}\,\mathcal{G} := \mathrm{T}_u\mathcal{M}$ *for this Lie-algebra and call it the* **Lie-algebra** *of* $\mathcal{G}$.

Proof: It is clear from the definition (48.30) and from (43.13) that $(\mathbf{t},\mathbf{s}) \mapsto [\mathbf{t},\mathbf{s}]$ is bilinear and skew. The Jacobi-indendity (48.31) follows from Prop. 7 of Sect. 43, applied to $\mathbf{h}_i := \mathbf{Gt}_i$, $i \in 3^]$, and Prop. 5. ∎

For each $y \in \mathcal{G}$, define $\mathbf{C}(y) \in \mathrm{Lin}(\mathrm{T}_y\mathcal{M}, \mathrm{S}_y\mathrm{T}\,\mathcal{G})$ by

$$\mathbf{C}(y) := \nabla_y\big(z \mapsto \mathbf{G}(z)\mathbf{G}(y)^{-1}\big). \tag{48.17}$$

Then (48.32) defines, as described in (48.9), a natural connection $\mathbf{C} : \mathcal{G} \to \mathrm{Con}\,\mathcal{G}$ on $\mathcal{G}$. This connection is analytic.

Let a vector fuield $\mathbf{h} \in \mathfrak{X}^1(\mathrm{T}\,\mathcal{G})$ be given and let the lineon-field $\nabla_{\mathbf{C}}\mathbf{h}$ be defined according to (41.3). Then it follows from Prop.2 that $\nabla_{\mathbf{C}}\mathbf{h} = \mathbf{0}$ if $\mathbf{h} = \mathbf{Gt}$ for some $\mathbf{t} \in \mathrm{T}_u\mathcal{M}$, where $\mathbf{G}$ is defined by (48.21). Conversely, if $\nabla_{\mathbf{C}}\mathbf{h} = \mathbf{0}$ and if $\mathcal{G}$ is connected, then $\mathbf{h} = \mathbf{Gt}$ for some $\mathbf{t} \in \mathrm{T}_u\mathcal{M}$.

Proposition 7: *The Lie-algebra-operation of* $\mathrm{T}_u\mathcal{M}$ *is the opposite of the torsion* $\mathbf{T}_u(\mathbf{C}(u))$*, i.e.*

$$[\mathbf{t},\mathbf{s}] = \mathbf{T}_u(\mathbf{C}(u))(\mathbf{t},\mathbf{s}) \qquad \text{for all} \qquad \mathbf{t},\mathbf{s} \in \mathbf{T}_u. \tag{48.18}$$

Proof: Let $\mathbf{t},\mathbf{s} \in \mathbf{T}_u$ be given. Application of (43.18) to $\mathbf{h} := \mathbf{Gt}$, $\mathbf{k} := \mathbf{Gs}$, $x := u$ gives (48.33) if (48.30) is observed and $\nabla_{\mathbf{C}}\mathbf{h} = \mathbf{0} = \nabla_{\mathbf{C}}\mathbf{k}$, as described in above, is applied. ∎

Remark : The curvature field $\mathbf{R}(\mathbf{C}) = 0$???

Proposition 8: *Let $d \in {}^{\times}$ and $p \in [0,d] \to \mathcal{G}$, of class C^1 and with $p(0) = u$, be given. Then $\mathbf{G} \circ p : [0,d] \to \mathrm{Tlis}_u\mathcal{G}$ is the parallelism along p for $\mathbf{C}$.*

Proof: Put $\mathbf{T} := \mathbf{G} \circ p$. Then $\mathbf{T}(s)\mathbf{T}(t)^{-1} = \mathbf{G}(p(s))\mathbf{G}(p(t))^{-1}$ for all $s, t \in [0,d]$. Hence, by (48.32), (35.10), and the chain rule,

$$\mathrm{sd}_t\mathbf{T} = \mathbf{C}(p(t))p^{\cdot}(t) \qquad \text{for all} \qquad t \in [0,d],$$

i.e. $\mathrm{sd}\mathbf{T} = (\mathbf{C} \circ p)p^{\cdot}$. In view of (35.23) the assertion follows. ∎

An non-constant homomorphism $q : \;\to \mathcal{G}$ from the additive group of to $\mathcal{G}$ is called a **one-parameter subgroup** of $\mathcal{G}$ if it is of class C^1.

Proposition 9: *Let $d \in {}^{\times}$ and $p \in [0,d] \to \mathcal{G}$, of class C^1 and with $p(0) = u$, be given. Then p is geodesic if and only if $p = q|_{[0,d]}$ for some one-parameter subgroup q of $\mathcal{G}$.*

Proof: By Prop. 6 and (35.28), p is geodesic if and only if $p^{\cdot}(0) \neq \mathbf{0}$ and

$$\mathbf{G}(p(t))p^{\cdot}(0) = p^{\cdot}(t) \qquad \text{for all} \qquad t \in [0,d]. \tag{48.19}$$

Let q be a one-parameter subgroup of $\mathcal{G}$ and $p = q|_{[0,d]}$. Let $t \in [0,d[$ be given. Then

$$\mathrm{le}_{p(t)}p(s) = q(t)q(s) = q(t+s) = p(t+s)$$
$$\text{for all} \quad s \in [0,d] \cap ([0,d] - t) = [0, d-t[.$$

Differentiating with respect to s at 0 and using (48.21), we get

$$\mathbf{G}(p(t))p^{\cdot}(0) = p^{\cdot}(t).$$

Since $t \in [0.d[$ was arbitrary and since $p^{\cdot}$ is continuous at d, (48.34) follows.

Assume now that p is geodesic, *i.e.* that (48.34) holds. Let $q : I \to \mathcal{G}$ be the (unique) solution of the differential equation

$$? \quad q \in \mathrm{C}^1(I,\mathcal{G}) \qquad , \qquad (\mathbf{G} \circ q)p^{\cdot}(0) = q^{\cdot} \tag{48.20}$$

whose domain I is the maximal interval that contains $0 \in$. Then I is an open interval, $[0,d] \subset I$, and $p = q|_{[0,d]}$ by the standard uniqueness theorem for differential equations. Let $t \in I$ be given and define $u : I \to \mathcal{G}$ and $v : (I - t) \to \mathcal{G}$ by

$$u(s) := q(t)q(s) = \mathrm{le}_{q(t)}(q(s)) \qquad \text{for all} \qquad s \in I \tag{48.21}$$

and

$$v(s) := q(t+s) \qquad \text{for all} \qquad s \in I - t \tag{48.22}$$

Using the chain rule and (48.24), it follows from (48.36) that

$$u^{\cdot}(s) = (\nabla_{q(s)}\mathrm{le}_{q(t)})q^{\cdot}(s) = \mathbf{G}(q(t)q(s))\mathbf{G}(q(s))^{-1}q^{\cdot}(s)$$

for all $s \in I$ and hence, by (71.23) and (71.24), that

$$u^{\cdot} = (\mathbf{G} \circ u)p^{\cdot}(0) \quad , \quad u(0) = q(t). \tag{48.23}$$

On the other hand, it follows (48.35) and (48.36) that

$$v^{\cdot}(s) = q^{\cdot}(t+s) = \mathbf{G}(q(t+s))p^{\cdot}(0)$$

for all $s \in I - t$ and hence that

$$v^{\cdot} = (\mathbf{G} \circ v)p^{\cdot}(0) \quad , \quad v(0) = q(t). \tag{48.24}$$

Comparing (48.38) and (48.39), we see that u and v satisfiy the same differential equation and initial condition. Since the domain of q is the maximal interval containng 0, it is clear that the domains of u and v must both be the maximal interval containing 0. It follows that $I - t = I$, which can be valid for all $t \in I$ only if $I = $. The standard uniqueness theorem for differential equations shows that $u = v$ and hence, by (48.36) and (48.37), that $q(t+s) = q(t)q(s)$ for all $s \in$. Since $t \in$ was arbitrary, it follows that q must be a one-parameter subgroup of $\mathcal{G}$. ∎