Finite Dimesional Spaces Vol.II
by Walter Noll (~1990)

4. Volume Integrals.

41. Negligible sets, almost continuous functions.

We assume that a flat space $\mathcal{E}$ with translation space $\mathcal{V}$ is given and we put $n := \dim\mathcal{E} = \dim\mathcal{V}$.

Let $\mathcal{B}$ be a norming cell of $\mathcal{V}$ (see Vol.I, Sect.51). A concept of *volume* for the space $\mathcal{E}$ should have the following property: If the norming cell $\mathcal{B}$ has volume 1, then a cell in $\mathcal{E}$ modelled on $\mathcal{B}$ of a given scale $s \in \mathbb{P}$ has volume s^n. (Recall that such a cell is a set of the form $q + s\mathcal{B}$, with $q \in \mathcal{E}$.) Intuitively, a subset $\mathcal{S}$ of $\mathcal{E}$ will be *negligible* with respect to volume if it can be covered by a finite collection of cells whose total volume can be made arbitrarily small. This is the motivation for the following:

Definition 1. *A given subset $\mathcal{S}$ of $\mathcal{E}$ is said to be* **negligible** *(in $\mathcal{E}$) if, for every norming cell $\mathcal{B}$ and every $\epsilon \in \mathbb{P}^\times$ one can find a finite subset $\mathfrak{k}$ of $\mathcal{E}$ and a family $(\rho_p \mid p \in \mathfrak{k})$ in $\mathbb{P}^\times$ such that*

(i) *the collection $\{p + \rho_p\mathcal{B} \mid p \in \mathfrak{k}\}$ covers $\mathcal{S}$ and*
(ii)$\sum(\rho_p{}^n \mid \mathfrak{k}) \leq \epsilon$.

In Def.1, the phrase "for some norming cell $\mathcal{B}$" can be replaced by "for every neighborhood $\mathcal{B}$ of zero in $\mathcal{V}$ " without change of substance. More precisely:

Proposition 1. *If the given subset $\mathcal{S}$ of $\mathcal{E}$ is negligible and if $\mathcal{B} \in \mathrm{Nhd}_0\mathcal{V}$ and $\epsilon \in \mathbb{P}^\times$ are given, one can find a finite subset $\mathfrak{k}$ of $\mathcal{S}$ and a family $(\rho_p \mid p \in \mathfrak{k})$ in $\mathbb{P}^\times$ with the properties (i) and (ii) of Def.1.*

Proof: We choose a norming cell $\mathcal{B}_0$ that satisfies the conditions (i) and (ii) of Def.1. By Cor.1 to the Norm-Equivalence Theorem (see Sect.51 of Vol.I) we can choose $\sigma \in \mathbb{P}^\times$ such that $2\sigma\mathcal{B}_0 \subset \mathcal{B}$. By Def.1, we can determine a finite subset $\mathfrak{h}$ of $\mathcal{S}$ and a family $(\tau_r \mid r \in \mathfrak{h})$ in $\mathbb{P}^\times$ such that

$$\mathcal{S} \subset \bigcup\{r + \tau_r\mathcal{B}_0 \mid r \in \mathfrak{h}\} \quad \text{and} \quad \sum_{\mathfrak{r}\in\mathfrak{h}} \tau_{\mathfrak{r}}^{\mathfrak{n}} \leq \sigma^{\mathfrak{n}}\epsilon. \tag{41.1}$$

We modify $\mathfrak{h}$ by omitting those points r from $\mathfrak{h}$ for which $r + \tau_r\mathcal{B}_0$ is disjoint from $\mathcal{S}$. Then (41.1) remains valid. Let $r \in \mathfrak{h}$ be given. We may then choose $z_r \in \mathcal{S}$ such that $z_r \in r + \tau_r\mathcal{B}_0$. Since $\mathcal{B}_0$ is a norming cell, we have $r \in z_r + \tau_r\mathcal{B}_0$ and hence $r + \tau_r\mathcal{B}_0 \subset z_r + 2\tau_r\mathcal{B}_0$. Since $2\sigma\mathcal{B}_0 \subset \mathcal{B}$ we obtain $r + \tau_r\mathcal{B}_0 \subset z_r + \frac{\tau_r}{\sigma}\mathcal{B}_0$. Since $r \in \mathfrak{h}$ was arbitrary, it follows from (41.1) that

$$\mathcal{S} \subset \bigcup\{z_r + \frac{\tau_r}{\sigma}\mathcal{B} \mid r \in \mathfrak{h}\} \quad \text{and} \quad \sum_{\mathfrak{r}\in\mathfrak{h}} (\frac{\tau_{\mathfrak{r}}}{\sigma})^{\mathfrak{n}} \leq \epsilon.$$

The conclusion now follows when we put $\mathfrak{k} := \{z_r \mid r \in \mathfrak{h}\}$ and $\rho_p := \max\{\frac{\tau_r}{\sigma} \mid r \in \mathfrak{h}, z_r = p\}$ for each $p \in \mathfrak{k}$. ∎

The following facts are easy consequences of Def.1 and Prop.1.

Proposition 2. *(i) The empty set is negligible.*
(ii) Every subset of a negligible set is negligible.
(iii) The union of a finite collection of negligible sets is again negligible.
(iv) The closure of a negligible set is negligible.
(v) The image of a negligible set under a flat isomorphism is again negligible.
(vi) If $n > 0$, then all finite sets are negligible.

If $n = 0$, then $\mathcal{E}$ is a singleton and the only neglible subset of $\mathcal{E}$ is the empty set.

We now assume that a subset $\mathcal{D}$ of $\mathcal{E}$ is given. We use the **notation** $\mathrm{Bnb}\,\mathcal{D}$ for the collection of all bounded subsets of $\mathcal{D}$ that have a negligible boundary. We use the **notation** $\mathrm{Nb}\,\mathcal{D}$ for the collection of all subsets $\mathcal{C}$ of $\mathcal{D}$, not necessarily bounded, such that every bounded subset of the boundary $\mathrm{Bdy}\mathcal{C}$ of $\mathcal{C}$ is negligible. Of course, we have $\mathrm{Bnb}\,\mathcal{D} \subset \mathrm{Nb}\,\mathcal{D}$. In view of (iv) and (iii) of Prop.2, every negligible set belongs to $\mathrm{Bnb}\,\mathcal{D}$.

The following results are immediate consequences of Prop.2.

Proposition 3. *The interior of every set belonging to* $\mathrm{Bnb}\,\mathcal{D}$ $[\mathrm{Nb}\,\mathcal{D}]$ *also belongs to* $\mathrm{Bnb}\,\mathcal{D}$ $[\mathrm{Nb}\,\mathcal{D}]$*. If $\mathcal{D}$ is closed, then the closure of every set belonging to* $\mathrm{Bnb}\,\mathcal{D}$ $[\mathrm{Nb}\,\mathcal{D}]$ *also belongs to* $\mathrm{Bnb}\,\mathcal{D}$ $[\mathrm{Nb}\,\mathcal{D}]$*. The union and the intersection of a finite collection of sets in* $\mathrm{Bnb}\,\mathcal{D}$ $[\mathrm{Nb}\,\mathcal{D}]$ *again belongs to* $\mathrm{Bnb}\,\mathcal{D}$ $[\mathrm{Nb}\,\mathcal{D}]$*. If a flat isomorphism $\alpha : \mathcal{E} \longrightarrow \mathcal{E}'$ from $\mathcal{E}$ to a flat space $\mathcal{E}'$ and $\mathcal{A} \in \mathrm{Bnb}\,\mathcal{E}$ $[\mathrm{Nb}\,\mathcal{D}]$ are given, then $\alpha_{>}(\mathcal{A}) \in \mathrm{Bnb}\,\alpha_{>}(\mathcal{D})$ $[\mathrm{Nb}\,\alpha_{>}(\mathcal{D})]$.*

Definition 2. *Let a mapping f with $\mathrm{Dom}\,f = \mathcal{D}$ and $\mathrm{Cod}\,f$ included in some flat space be given. We say that f is* **almost continuous** *if every bounded subset of the set of all discontinuities of f is negligible (in $\mathcal{E}$).*

We use the **notation** $\mathrm{Ac}\,\mathcal{D}$ for the set of all functions $f : \mathcal{D} \longrightarrow \mathbf{R}$ that are almost continuous and the **notation** $\mathrm{Bbac}\,\mathcal{D}$ for the set of all functions $f : \mathcal{D} \longrightarrow \mathbf{R}$ that, in addition, have bounded range and bounded support. It is easily seen that both $\mathrm{Ac}\,\mathcal{D}$ and $\mathrm{Bbac}\,\mathcal{D}$ are stable under valuewise addition and multiplication with real numbers, i.e. that theyare *subspacec* of the linear space $\mathrm{Map}(\mathcal{D}, \mathbf{R})$. Moreover, $\mathrm{Bbac}\,\mathcal{D}$ is a subspace of $\mathrm{Ac}\,\mathcal{D}$.

It is clear that a given subset $\mathcal{A}$ of $\mathcal{E}$ belongs to $\mathrm{Nb}\,\mathcal{E}$ $[\mathrm{Bnb}\,\mathcal{E}]$ if and only if the characteristic function $\mathrm{ch}_{\mathcal{A}} : \mathcal{E} \longrightarrow \mathbf{R}$ of $\mathcal{A}$, defined by

$$\mathrm{ch}_{\mathcal{A}}(x) = \begin{cases} 1 & \text{if } x \in \mathcal{A} \\ 0 & \text{if } x \in \mathcal{E} \setminus \mathcal{A} \end{cases} \tag{41.2}$$

belongs to $\mathrm{Ac}\,\mathcal{E}$ $[\mathrm{Bbac}\,\mathcal{E}]$.

Let a function $f : \mathcal{D} \longrightarrow \mathbf{R}$ be given. We then define the **standard extension** $\bar{f} : \mathcal{E} \longrightarrow \mathbf{R}$ of f by

$$\bar{f}(x) := \begin{cases} f(x) & \text{if } x \in \mathcal{D} \\ 0 & \text{if } x \in \mathcal{E} \setminus \mathcal{D} \end{cases}. \tag{41.3}$$

The following result is easily proved.

Proposition 4. *If* $\mathcal{D}$ *belongs to* $\mathrm{Nb}\,\mathcal{E}$ *then*

$$f \in \mathrm{Ac}\,\mathcal{D} \quad \Longleftrightarrow \quad \bar{f} \in \mathrm{Ac}\,\mathcal{E}.$$

and

$$f \in \mathrm{Bbac}\,\mathcal{D} \quad \Longleftrightarrow \quad \bar{f} \in \mathrm{Bbac}\,\mathcal{E}\ .$$

The proof of the following assertion will be deferred to Sect.43.

Proposition 5. *Every box in* $\mathcal{E}$ *belongs to* $\mathrm{Bnb}\,\mathcal{E}$. *(For the definition of box, see Vol.I, Sect.51.)*

The proof of the following result depends on Prop.5.

Proposition 6. *Every bounded subset of a proper flat in* $\mathcal{E}$ *is negligeable.*

Proof: Let a bounded subset $\mathcal{S}$ of a proper flat in $\mathcal{E}$ be given. Then $\mathcal{S}$ is also a bounded subset of a hyperplane $\mathcal{F}$ and hence included in a box in $\mathcal{F}$. Every such box is a subset of the boundary of a box in $\mathcal{E}$. Since this boundary is negligible by Prop.5, it follows from (ii) of Prop.2 that $\mathcal{S}$ is negligible. ∎

Of course, a bounded subset of a given proper flat $\mathcal{F}$ in $\mathcal{E}$, though negligible in $\mathcal{E}$, need not be negligible when regarded as a subset of the flat space $\mathcal{F}$.

The following two results serve to describe a large class of negligible sets. Their proofs will be deferred to Sect.44.

Proposition 7. *Let* $\mathcal{D} \in \mathrm{Bnb}\,\mathcal{E}$ *and* $f \in \mathrm{Bbac}\,\mathcal{D}$ *be given. Then the Graph* $\mathrm{Gr}(f) := \{(x, f(x)) \mid x \in \mathcal{D}\}$ *of* f *is negligible in* $\mathcal{E} \times \boldsymbol{R}$.

Proposition 8. *Let* $\mathcal{S}_1$ *and* $\mathcal{S}_2$ *be subsets of given flat spaces* $\mathcal{E}_1$ *and* $\mathcal{E}_2$, *respectively. If one of* $\mathcal{S}_1$ *and* $\mathcal{S}_2$ *is bounded and the other negligible, then* $\mathcal{S}_1 \times \mathcal{S}_2$ *is negligible in the flat space* $\mathcal{E}_1 \times \mathcal{E}_2$.

The proof of the following result depends on Props.7 and 8.

Proposition 9. *Let a closed set* $\mathcal{D}$ *belonging to* $\mathrm{Bnb}\,\mathcal{E}$ *and continuous functions* $g : \mathcal{D} \longrightarrow \boldsymbol{R}$ *and* $h : \mathcal{D} \longrightarrow \boldsymbol{R}$ *be given such that* $g \le h$. *Then the set (see Fig.1)*

$$\mathcal{A} := \{(x, \xi) \mid x \in \mathcal{D},\ \xi \in [g(x), h(x)]\}$$

belongs to $\mathrm{Bnb}\,(\mathcal{E} \times \boldsymbol{R})$.

Proof: We use the Compactness Theorem and the Compact Image Theorem stated in Sect.51 of Vol.I. The first of these insures that $\mathcal{D}$ is compact and the second insures that $\mathrm{Rng}\,g$ and $\mathrm{Rng}\,h$ are compact and hence bounded. Thus, we may choose $\alpha, \beta \in \mathbf{R}$ such that $\alpha \le g \le h \le \beta$. It follows that $\mathcal{A}$ is a subset of $\mathcal{D} \times [\alpha, \beta]$ and hence is bounded. The boundary of $\mathcal{A}$ is the union of $\mathrm{Gr}(g)$, $\mathrm{Gr}(h)$, and a subset of $\mathrm{Bdy}\,\mathcal{D} \times [\alpha, \beta]$ (see Fig.1). The first two of these are negligible by Prop.7 and the last is negligible by Prop.8. Hence $\mathrm{Bdy}\mathcal{A}$ is negligible by (ii) of Prop.2. ∎

The proof of the following result is based on Prop.9 and on the Theorem on Continuity of Convex Functions (see Sect.00).

Proposition 10. *Every bounded convex subset of $\mathcal{E}$ has a negligible boundary and hence belongs to* $\mathrm{Bnb}\,\mathcal{E}$.

Proof: We proceed by induction over the dimension n of $\mathcal{E}$. The assertion is trivially valid when $n = 0$. We assume, then, that $n > 0$ and that the assertion is valid when $\mathcal{E}$ is replaced by a flat space of dimension $n - 1$. We choose a hyperplane $\mathcal{F}$ in $\mathcal{E}$ and a vector $\mathbf{e} \in \mathcal{V}^{\times}$ that does not belong to the direction space of $\mathcal{F}$, so that the mapping

$$((p, \xi) \mapsto (p + \xi\mathbf{e})) : \quad \mathcal{F} \times \mathbf{R} \longrightarrow \mathcal{E} \tag{41.4}$$

is a flat isomorphism. Now let a bounded convex subset $\mathcal{C}$ of $\mathcal{E}$ be given. Then $\mathrm{Clo}\,\mathcal{C}$ is also bounded and convex (see Cor.2 of the Half-Space Intersection Theorem in Sect.54 of Vol.I). We put

$$\mathcal{D} := \{p \in \mathcal{F} \mid p + \xi\mathbf{e} \in \mathrm{Clo}\,\mathcal{C} \text{ for some } \xi \in \mathbf{R}\}.$$

The convexity of $\mathrm{Clo}\,\mathcal{C}$ implies that there are functions $g : \mathcal{D} \longrightarrow \mathbf{R}$ and $h : \mathcal{D} \longrightarrow \mathbf{R}$ such that $\mathrm{Clo}\,\mathcal{C}$ is the image, under the isomorphism (41.4), of the set

$$\mathcal{A} := \{(p, \xi) \mid p \in \mathcal{D},\ \xi \in [g(p), h(p)]\}.$$

Morover, the set $\mathcal{D}$ is closed and convex, and the functions g and $-h$ are convex. By the induction hypothesis, $\mathcal{D}$ has a negligible boundary in $\mathcal{F}$. By the Theorem on Continuity of Convex Functions, the restrictions of h and g to the interior of $\mathcal{D}$ in $\mathcal{F}$ must be continuous. Moreover, since $\mathcal{C}$ and hence $\mathcal{A}$ is a bounded set, $\mathrm{Rng}\,h$ and $\mathrm{Rng}\,g$ must be bounded subsets of $\mathbf{R}$. It follows that h and g belong to $\mathrm{Bbac}\,\mathcal{D}$, and we may choose $\alpha, \beta \in \mathbf{R}$ such that $\alpha \leq g \leq h \leq \beta$. We apply Prop.7 to conclude that $\mathrm{Gr}(g)$ and $\mathrm{Gr}(h)$ are negligible in $\mathcal{F} \times \mathbf{R}$. The boundary of $\mathcal{A}$ is the union of $\mathrm{Gr}(g)$, $\mathrm{Gr}(h)$, and a subset of $\mathrm{Bdy}_{\mathcal{F}}\mathcal{D} \times [\alpha, \beta]$. The latter is a negligible subset of $\mathcal{F} \times \mathbf{R}$ by Prop.8. Using (ii) and (iii) of Prop.2, we conclude that $\mathrm{Bdy}\mathcal{A}$ is a neglible subset of $\mathcal{F} \times \mathbf{R}$ and hence that $\mathrm{BdyClo}\,\mathcal{C}$ is a negligible subset of $\mathcal{E}$. Since $\mathrm{BdyClo}\,\mathcal{C} = \mathrm{Bdy}\,\mathcal{C}$ (see Sect.54 of Vol.I) it follows that $\mathcal{C} \in \mathrm{Bnb}\,\mathcal{E}$. ∎

42. Integrals and volumes.

We assume again that a flat space $\mathcal{E}$ with translation space $\mathcal{V}$ is given.

We consider the linear space $\mathrm{Bbac}\,\mathcal{E}$ of functions as defined in the previous section. A linear mapping from this space to $\mathbf{R}$ is called a **linear functional**.

Let a translation $\mathbf{v} \in \mathcal{V}$ be given. Then, for every $f \in \mathrm{Bbac}\,\mathcal{E}$, the composite $f \circ \mathbf{v}$ clearly belongs to $\mathrm{Bbac}\,\mathcal{E}$. Thus, we may consider the mapping $f \mapsto f \circ \mathbf{v}$ from $\mathrm{Bbac}\,\mathcal{E}$ into itself. It is clear that this mapping is linear and isotone, which means that $f \leq g$ implies $f \circ \mathbf{v} \leq g \circ \mathbf{v}$ for all $f, g \in \mathrm{Bbac}\,\mathcal{E}$. For every $\mathcal{A} \in \mathrm{Bnb}\,\mathcal{E}$ we have $\mathrm{ch}_{\mathcal{A}} \in \mathrm{Bbac}\,\mathcal{E}$ (see Sect.41) and

$$\mathrm{ch}_{\mathcal{A}} \circ \mathbf{v} = \mathrm{ch}_{\mathcal{A}-\mathbf{v}}. \tag{42.1}$$

Remark 1: Given $\mathbf{v} \in \mathcal{V}$ and $f \in \mathrm{Map}\,(\mathcal{E}, \mathbf{R})$, the graph of $f \circ \mathbf{v}$ is obtained from the graph of f by the translation $(-\mathbf{v}, 0) \in \mathcal{V} \times \mathbf{R}$, i.e. we have

$$\mathrm{Gr}(f \circ \mathbf{v}) = \mathrm{Gr}(f) + (-\mathbf{v}, 0). \tag{42.2}$$

(see Fig.1). ∎

Remark 2: The mapping $\mathrm{Tr} : \mathcal{V} \longrightarrow \mathrm{Perm}\,(\mathrm{Bbac}\,\mathcal{E})$, defined by $\mathrm{Tr}_{\mathbf{v}}(f) = f \circ \mathbf{v}$ for all $f \in \mathrm{Bbac}\,\mathcal{E}$ and all $\mathbf{v} \in \mathcal{V}$, is an action of the additive group of $\mathcal{V}$ on $\mathrm{Bbac}\,\mathcal{E}$ in the sense of Def.1 of Sect.31 of Vol.I. ∎

Definition 1. *A non-zero linear functional* Igl *on the function-space* $\mathrm{Bbac}\,\mathcal{E}$ *is called an* **integral** *on* $\mathcal{E}$ *if it is isotone in the sense that*

$$f \geq g \quad \Longrightarrow \quad \mathrm{Igl}\,(f) \geq \mathrm{Igl}\,(g) \quad \text{for all} \quad f, g \in \mathrm{Bbac}\,\mathcal{E} \tag{42.3}$$

and translation-invariant in the sense that

$$\mathrm{Igl}\,(f \circ \mathbf{v}) = \mathrm{Igl}\,(f) \quad \text{for all} \quad f \in \mathrm{Bbac}\,\mathcal{E},\ \mathbf{v} \in \mathcal{V}. \tag{42.4}$$

It is easily follows from the linearity of Igl that the condition (42.3) can be replaced by the weaker condition

$$f \geq 0 \quad \Longrightarrow \quad \mathrm{Igl}\,(f) \geq 0 \quad \text{for all} \quad f \in \mathrm{Bbac}\,\mathcal{E}. \tag{42.5}$$

Definition 2. *Given an integral* Igl *on* $\mathcal{E}$, *the mapping* $\mathrm{vol} : \mathrm{Bnb}\,\mathcal{E} \longrightarrow \mathbb{P}$ *defined by*

$$\mathrm{vol}\,(\mathcal{A}) := \mathrm{Igl}\,(\mathrm{ch}_{\mathcal{A}}) \quad \text{for all} \quad \mathcal{A} \in \mathrm{Bnb}\,\mathcal{E} \tag{42.6}$$

is called the **volume**-*function associated with* Igl.

The following result is an easy consequence of Defs.1 and 2.

Proposition 1. *The volume-function associated with a given integral* Igl *is isotone in the sense that*

$$\mathcal{A} \subset \mathcal{B} \quad \Longrightarrow \quad \mathrm{vol}\,(\mathcal{A}) \leq \mathrm{vol}\,(\mathcal{B}) \quad \text{for all } \mathcal{A}, \mathcal{B} \in \mathrm{Bnb}\,\mathcal{E}, \tag{42.7}$$

translation-invariant in the sense that

$$\mathrm{vol}\,(\mathcal{A} - \mathbf{v}) = \mathrm{vol}\,(\mathcal{A}) \quad \text{for all } \mathcal{A} \in \mathrm{Bnb}\,\mathcal{E},\ \mathbf{v} \in \mathcal{V}, \tag{42.8}$$

and subadditive in the following sense: for every finite collection $\mathfrak{C}$ *of sets belonging to* $\mathrm{Bnb}\,\mathcal{E}$, *the union* $\bigcup \mathfrak{C}$ *belongs to* $\mathrm{Bnb}\,\mathcal{E}$ *and*

$$\mathrm{vol}\,(\bigcup \mathfrak{C}) \leq \sum_{\mathcal{C} \in \mathfrak{C}} \mathrm{vol}\,(\mathcal{C}). \tag{42.9}$$

The inequality in (42.9) becomes an equality if the collection $\mathfrak{C}$ is disjoint.

The following result, whose proof will be given in Sect.43, shows that there is an integral and that it is unique to within a strictly positive multiplicative factor.

Theorem on Existence and Uniqueness of Integrals. *There is an integral* Igl *on* $\mathcal{E}$. *A functional* $\mathrm{J} : \mathrm{Bbac}\,\mathcal{E} \longrightarrow \boldsymbol{R}$ *is also an integral if and only if* $\mathrm{J} = \mathrm{c}\,\mathrm{Igl}$ *for some* $c \in \mathbb{P}^{\times}$.

Given any integral Igl *on* $\mathcal{E}$ *and any norming box* $\mathcal{B}$ *of* $\mathcal{V}$ *(see Sect.51 of Vol.I), there is a* $\gamma \in \mathbb{P}^{\times}$ *such that*

$$\mathrm{vol}\,(\mathcal{C}) = s^n \gamma \tag{42.10}$$

for all boxes $\mathcal{C}$ *of scale* s *modelled on* $\mathcal{B}$.

From now on we assume that an integral Igl on $\mathcal{E}$ and hence the corresponding volume function vol on $\mathrm{Bnb}\,\mathcal{E}$ is fixed.

Proposition 2. *Let* $\mathcal{S} \in \mathrm{Bnb}\,\mathcal{E}$ *be given. Then the following conditions are equivalent:*

(i) $\mathcal{S}$ *is negligible.*

(ii) $\mathrm{vol}\,(\mathcal{S}) = 0$.

(iii) $\mathrm{Int}\,\mathcal{S} = \emptyset$.

Proof: We choose a norming box $\mathcal{B}$ and determine $\gamma \in \mathbb{P}^{\times}$ such that (42.10) holds for all boxes of scale s modelled on $\mathcal{B}$.

(i) $\Rightarrow$ (ii): Assume that $\mathcal{S}$ is negligible and let $\epsilon \in \mathbb{P}^{\times}$ be given. By Prop.1 of Sect.41, we can determine a finite subset $\mathfrak{k}$ of $\mathcal{S}$ and a family $(\mathcal{C}_p \mid p \in \mathfrak{k})$ of boxes modelled on $\mathcal{B}$ such that

$$\mathcal{S} \;\subset\; \bigcup\{\mathcal{C}_p \mid p \in \mathfrak{k}\} \quad \text{and} \quad \sum_{p\in\mathfrak{k}} {\rho_p}^n \leq \frac{\epsilon}{\gamma},$$

where, for each $p \in \mathfrak{k}$, ρ_p is the scale of the box $\mathcal{C}_p$. It follows from Prop.1 above and (42.10) that

$$\mathrm{vol}\,(\mathcal{S}) \leq \sum_{p\in\mathfrak{k}} \mathrm{vol}\,(\mathcal{C}_p) = \sum_{p\in\mathfrak{k}} ({\rho_p}^n \gamma) \leq \epsilon.$$

Since $\epsilon \in \mathbb{P}^{\times}$ was arbitrary, we conclude that $\mathrm{vol}\,(\mathcal{S}) = 0$.

(ii) $\Rightarrow$ (iii): Assume that $\mathrm{Int}\,\mathcal{S} \neq \emptyset$. We can then then choose a box $\mathcal{C}$ modelled on $\mathcal{B}$ such that $\mathcal{C} \subset \mathcal{S}$. By (42.7) and (42.10) we have $\mathrm{vol}\,(\mathcal{S}) \geq \mathrm{vol}\,(\mathcal{C}) = s^n\gamma$, where s is the scale of $\mathcal{C}$. It follows that $\mathrm{vol}\,(\mathcal{S}) \neq 0$.

(iii) $\Rightarrow$ (i): Assume that $\mathrm{Int}\,\mathcal{S} = \emptyset$. We then have $\mathcal{S} \subset \mathrm{Clo}\,\mathcal{S} = \mathrm{Bdy}\,\mathcal{S}$. Since $\mathcal{S} \in \mathrm{Bnb}\,\mathcal{E}$, it follows from (ii) of Prop.2 of Sect.41 that $\mathcal{S}$ is negligible. ∎

The following result is often useful for proving that a given set is negligible. Its proof will be deferred to Sect.43.

Proposition 3. *Suppose that a given subset* $\mathcal{S}$ *of* $\mathcal{E}$ *has the following property: For every* $\epsilon \in \mathbb{P}^\times$ *there is* $\mathcal{A} \in \operatorname{Bnb}\mathcal{E}$ *such that* $\mathcal{S} \subset \mathcal{A}$ *and* $\operatorname{vol}(\mathcal{A}) \leq \epsilon$. *Then* $\mathcal{S}$ *is negligible.*

We now assume that a set $\mathcal{D} \in \operatorname{Nb}\mathcal{E}$, as defined in Sect.41, is given. Then, for every $g \in \operatorname{Bbac}\mathcal{D}$, the standard extension $\bar{g}$ of g, as defined in accord with (41.3), belongs to $\operatorname{Bbac}\mathcal{E}$ and hence it is meaningful to consider the integral $\operatorname{Igl}(\bar{g})$.

Definition 3. *Let a real valued function* f *such that* $\mathcal{D} \subset \operatorname{Dom} f \subset \mathcal{E}$ *and* $f|_{\mathcal{D}} \in \operatorname{Bbac}\mathcal{D}$ *be given. Then the* **integral** *of* f **over** $\mathcal{D}$ *is defined by*

$$\int_{\mathcal{D}} f := \operatorname{Igl}(\overline{f|_{\mathcal{D}}}). \tag{42.11}$$

If $\mathcal{D} := \mathcal{E}$ and $f \in \operatorname{Bbac}\mathcal{E}$, then (42.11) becomes

$$\int_{\mathcal{E}} f := \operatorname{Igl}(f). \tag{42.12}$$

If f is a *continuous* real-valued function with $\mathcal{D} \subset \operatorname{Dom} f \subset \mathcal{E}$ and if $\mathcal{D}$ is compact (i.e. closed and bounded), then $f|_{\mathcal{D}}$ is bounded by the Compact-Image Theorem (see Sect.58 of Vol.I) and hence $f|_{\mathcal{D}}$ belongs to $\operatorname{Bbac}\mathcal{D}$. Therefore, the integral $\int_{\mathcal{D}} f$ is meaningful.

The following five results follow very easily from the defining properties of the integral and from Prop.3. The proofs are left to the reader.

Proposition 4. *If* $f|_{\mathcal{D}} \in \operatorname{Bbac}\mathcal{D}$ *we have* $|f||_{\mathcal{D}} \in \operatorname{Bbac}\mathcal{D}$ *and*

$$|\int_{\mathcal{D}} f| \ \leq \ \int_{\mathcal{D}} |f|. \tag{41.13}$$

Proposition 5. *Assume that* $\mathcal{D}$ *is bounded, i. e. that* $\mathcal{D} \in \operatorname{Bnb}\mathcal{E}$. *Then*

$$\inf(f_>(\mathcal{D}))\operatorname{vol}(\mathcal{D}) \ \leq \ \int_{\mathcal{D}} f \ \leq \ \sup(f_>(\mathcal{D}))\operatorname{vol}(\mathcal{D}) \ . \tag{42.14}$$

whenever $f|_{\mathcal{D}} \in \operatorname{Bbac}\mathcal{D}$.

Proposition 6. *Assume that* $\mathcal{D}$ *is bounded, i.e. that* $\mathcal{D} \in \operatorname{Bnb}\mathcal{E}$. *Let* $(h_k \mid k \in \mathbb{N}^\times)$ *be a sequence of functions* $h_k \in \operatorname{Bbac}\mathcal{D}$ *that converges uniformly to a function* $\lim_{k\to\infty} h_k$ *that belongs to* $\operatorname{Bbac}\mathcal{D}$. *Then*

$$\lim_{k\to\infty} \int_{\mathcal{D}} h_k \ = \ \int_{\mathcal{D}} (\lim_{k\to\infty} h_k). \tag{42.15}$$

Proposition 7. *The support of a given function* $f : \mathcal{D} \longrightarrow \mathbb{R}$ *with bounded range is negligible if and only if* $f \in \operatorname{Bbac}\mathcal{D}$ *and* $\int_{\mathcal{D}} f = 0$.

Proposition 8. *Let* $\mathcal{D}_1, \mathcal{D}_2 \in \operatorname{Nb}\mathcal{E}$ *be such that* $\operatorname{Int}(\mathcal{D}_1 \cap \mathcal{D}_2) = \emptyset$. *We then have*

$$\int_{\mathcal{D}_1 \cup \mathcal{D}_2} f = \int_{\mathcal{D}_1} f + \int_{\mathcal{D}_2} f \tag{42.16}$$

whenever $\mathcal{D}_1 \cup \mathcal{D}_2 \subset \mathrm{Dom}\, f \subset \mathcal{E}$ *and* $f|_{\mathcal{D}_1 \cup \mathcal{D}_2} \in \mathrm{Bbac}\,(\mathcal{D}_1 \cup \mathcal{D}_2)$.

The notation

$$\int_{\mathcal{D}} f(x)\mathrm{d}x = \int_{\mathcal{D}} f \tag{42.17}$$

is often used, especially if the function f is specified by an evaluation rule $x \mapsto f(x)$. In (42.17), the letter x is a dummy and can be replaced by any other letter that has not been given a previous meaning. (The "dx" should perhaps be read as "the dummy is x".)

If we apply the Theorem on Existence and Uniqueness of Integrals to the case when $\mathcal{E} := \mathbf{R}$, we see that there is exactly one integral Igl on $\mathbf{R}$ such that $\mathrm{vol}\,(]0,1[) = 1$. In this case, the term **length** rather than volume is used and we write le := vol. For every bounded interval H in $\mathbf{R}$ we have $\mathrm{le}\,(H) = \sup H - \inf H$. Let a continuous real-valued function f with $[a,b] \subset \mathrm{Dom}\, f \subset \mathbf{R}$ be given. Using the notations (42.11) and (42.17), we then have

$$\int_{[a,b]} f = \int_{[a,b]} f(t)\mathrm{d}t = \int_a^b f = \int_a^b f(t)\mathrm{d}t,$$

where the integrals from a to b on the right should be interpreped as integrals familiar from elementary real analysis (see Sect.08 of Vol.I).

Proposition 9. *Let* $\alpha : \mathcal{E} \longrightarrow \mathcal{E}'$ *be a flat isomarphism from* $\mathcal{E}$ *to a given flat space* $\mathcal{E}'$*. Then*

$$\mathrm{Bbac}\,\mathcal{E} = \{f \circ \alpha \mid f \in \mathrm{Bbac}\,\mathcal{E}'\}. \tag{42.19}$$

Moreover, recalling that the integral Igl *on* $\mathcal{E}$ *is given, we can define an integral* Igl$'$ *on* $\mathcal{E}'$ *by*

$$\mathrm{Igl}'(f) := \mathrm{Igl}\,(f \circ \alpha) \quad \text{for all} \quad f \in \mathrm{Bbac}\,\mathcal{E}'. \tag{42.20}$$

Proof: The equality (42.19) is an easy consequence of Prop.2 of Sect.41.

We now *define* the mapping $\mathrm{Igl}' : \mathrm{Bbac}\,\mathcal{E}' \longrightarrow \mathbf{R}$ by (42.20). It is clear that Igl$'$ is linear, isotone, and not zero. To show that Igl$'$ is translation-invariant, let $\mathbf{v}' \in \mathcal{V}' := \mathcal{E}' - \mathcal{E}'$ be given. Since $\alpha^{\leftarrow}$ is also flat, it follows from Def.1 of Sect.33 of Vol.I that we can choose $\mathbf{v} \in \mathcal{V}$ such that $\mathbf{v}' \circ \alpha = \alpha \circ \mathbf{v}$. Since Igl is translation-invariant, we have

$$\mathrm{Igl}'(f \circ \mathbf{v}') = \mathrm{Igl}\,(f \circ \mathbf{v}') \circ \alpha) = \mathrm{Igl}\,(f \circ \alpha \circ \mathbf{v}) = \mathrm{Igl}\,(f \circ \alpha) = \mathrm{Igl}'(f)$$

for all $f \in \mathrm{Bbac\,vol}\, E'$. ▮

We now choose a point $q \in \mathcal{E}$. Then $(x \mapsto x - q) : \mathcal{E} \longrightarrow \mathcal{V}$ is a flat isomophism and hence induces an intetgral Igl$'$ on $\mathcal{V}$ according to Prop.8. It is easily seen that Igl$'$ does not depend on the choice of $q \in \mathcal{E}$. We will continue to use the notation (42.12) when $\mathcal{E}$ is replaced by $\mathcal{V}$ and Igl by Igl$'$, respectively. Using also the notation (42.17), we have

$$\int_{\mathcal{V}} h = \int_{\mathcal{V}} h(\mathbf{v})\mathrm{d}\mathbf{v} = \int_{\mathcal{E}} h(x-q)\mathrm{d}x \quad \text{for all} \quad q \in \mathcal{E} \tag{42.21}$$

and all $h \in \operatorname{Bbac}\mathcal{V}$. If the given subset $\mathcal{D}$ of $\mathcal{E}$ is such that every bounded subset of $\operatorname{Bdy}\mathcal{D}$ is negligible, we have

$$\int_{\mathcal{D}} f = \int_{\mathcal{D}-q} f(q+\mathbf{v})\mathrm{d}\mathbf{v} \quad \text{for all} \quad q \in \mathcal{E} \tag{42.22}$$

and all $f \in \operatorname{Bbac}\mathcal{D}$.

Proposition 10. *Let an integral* Igl *on $\mathcal{E}$ and a flat automorphism α of $\mathcal{E}$ be given. Then there is $c \in \mathbb{P}^\times$ such that*

$$\operatorname{Igl}(f \circ \alpha) = c \operatorname{Igl}(f) \quad \text{for all} \quad f \in \operatorname{Bbac}\mathcal{E} \tag{42.23}$$

and

$$\operatorname{vol}(\alpha_>(\mathcal{A})) = \frac{1}{c}\operatorname{vol}(\mathcal{A}) \quad \text{for all} \quad \mathcal{A} \in \operatorname{Bnb}\mathcal{E}\ . \tag{42.24}$$

In addition, if $\nabla\alpha = d\mathbf{1}_{\mathcal{V}}$ for some $d \in \mathbb{P}^\times$, then $c = \frac{1}{d^n}$.

Proof: It follows from Prop.9 that $(f \mapsto \operatorname{Igl}(f \circ \alpha)) : \operatorname{Bbac}\mathcal{E} \longrightarrow \mathbb{R}$ is an integral on $\mathcal{E}$. Hence, by the Theorem on Existence and Uniqueness of Integrals, (42.23) must hold for a suitable $c \in \mathbb{P}^\times$.

Noting that $\operatorname{ch}_{\alpha_>(\mathcal{A})} \circ \alpha = \operatorname{ch}_{(\mathcal{A})}$ for all $\mathcal{A} \in \operatorname{Bnb}\mathcal{E}$, it follows from (42.6) and (42.23) that (42.24) is valid.

Assume now that $\nabla\alpha = d\,\mathbf{1}_{\mathcal{V}}$ for some $d \in \mathbb{P}^\times$. We choose a norming box $\mathcal{B}$ and a box $\mathcal{A}$ of scale 1 modelled on $\mathcal{B}$. It is easily seen that $\alpha_>(\mathcal{A})$ is a box of scale d modelled on $\mathcal{B}$. Hence, in view of (42.10), it follows from (42.24) that $d^n\gamma = \frac{1}{c}\gamma$ and hence $c = \frac{1}{d^n}$. ∎

43. Tiles and pavings.

We assume again that a flat space $\mathcal{E}$ with translation space $\mathcal{V}$ is given and we put $n := \dim\mathcal{E}$. In this section, we also fix a basis $\mathbf{b} := (\mathbf{b}_i \mid i \in I)$ of $\mathcal{V}$ and a point $q \in \mathcal{E}$. We use the **notation**

$$\mathcal{T} := (\operatorname{lnc}_{\mathbf{b}})_>([0,1[^I), \tag{43.1}$$

where $\operatorname{lnc}_{\mathbf{b}}$ is the linear combination mapping defined in Def.1 of Sect.15 of Vol.I.

Definition 1. *Given $x \in \mathcal{E}$ and $s \in \mathbb{P}^\times$, we call the set $\mathcal{J} := x + s\mathcal{T}$ a* **tile** *of* **scale** *s and* **corner** *x (see Fig.1).*

Given $s \in \mathbb{P}^\times$, we define the **paving** *Π_s of* **scale** *s to be the set of all tiles of scale s whose corners are of the form $q + \operatorname{lnc}_{\mathbf{b}}(sk)$ for some $k \in \mathbb{Z}^I$, i.e.*

$$\Pi_s := \{q + \operatorname{lnc}_{\mathbf{b}}(sk) + s\mathcal{T} \mid k \in \mathbb{Z}^I\}. \tag{43.2}$$

Let $s \in \mathbb{P}^\times$ be given. It is clear that the interior $\operatorname{Int}\mathcal{J}$ of a given tile $\mathcal{J}$ is a box of scale $\frac{s}{2}$ modelled on $\operatorname{Box}(\mathbf{b})$ (see Sect.51 of Vol.I). Moreover, the paving Π_s is a partition of $\mathcal{E}$.

The following **notations**, valid for any given subset $\mathcal{S}$ of $\mathcal{E}$, will be useful.

$$\Delta_s(\mathcal{S}) := \{\mathcal{J} \in \Pi_s \mid \mathcal{J} \subset \mathcal{S}\}, \tag{43.3}$$

$$\Delta_s^{\mathrm{Clo}}(\mathcal{S}) := \{\mathcal{J} \in \Pi_s \mid \mathrm{Clo}\,\mathcal{J} \subset \mathcal{S}\}, \tag{43.4}$$

$$\Gamma_s(\mathcal{S}) := \{\mathcal{J} \in \Pi_s \mid \mathcal{J} \cap \mathcal{S} \neq \emptyset\}. \tag{43.5}$$

$$\Gamma_s^{\mathrm{Clo}}(\mathcal{S}) := \{\mathcal{J} \in \Pi_s \mid \mathrm{Clo}\,\mathcal{J} \cap \mathcal{S} \neq \emptyset\}. \tag{43.6}$$

In words, $\Delta_s(\mathcal{S})$, $[\Delta_s(\mathcal{S})]$ is the collection of all tiles in Π_s which are included in [whose closures are included in] $\mathcal{S}$. The set $\Gamma_s(\mathcal{S})$, $[\Gamma_s^{\mathrm{Clo}}(\mathcal{S})]$ is the collection of all tiles that meet [whose closures meet] $\mathcal{S}$.

The following two facts are easily verified.

Proposition 1. *Given $s \in \mathbb{P}^\times$ and $m \in \mathbb{N}^\times$, the paving Π_s is a refinement of the paving Π_{ms}. More specifically, each tile in Π_{ms} is the union of a disjoint collection of m^n tiles in Π_s, so that $\#\Delta_s(\mathcal{K}) = m^n$ for all $\mathcal{K} \in \Pi_{ms}$.*

Proposition 2. *Let $\rho \in \mathbb{P}^\times$, a tile $\mathcal{K}$ of scale ρ, and $s \in]0, \frac{\rho}{2}]$ be given. Then we have*

$$(\frac{\rho}{s} - 2)^n \;\leq\; \#\Delta_s^{\mathrm{Clo}}(\mathrm{Int}\,\mathcal{K}) \;\leq\; \#\Gamma_s^{\mathrm{Clo}}(\mathrm{Clo}\,\mathcal{K}) \;\leq\; (\frac{\rho}{s} + 2)^n. \tag{43.7}$$

Prop.2 is illustrated in Fig.2 for the case when $n := 2$ and $3s \leq \rho \leq 4s$.

Proposition 3. *Let a subset $\mathcal{S}$ of $\mathcal{E}$ be given. If, for every $\epsilon \in \mathbb{P}^\times$ there is a $s \in \mathbb{P}^\times$ such that*

$$\#\Gamma_s(\mathcal{S}) \;\leq\; \frac{\epsilon}{s^n}, \tag{43.8}$$

then $\mathcal{S}$ is negligible. Conversely, if $\mathcal{S}$ is negligible, one can find, for each $\epsilon \in \mathbb{P}^\times$, a $\sigma \in \mathbb{P}^\times$ such that

$$\#\Gamma_s^{\mathrm{Clo}}(\mathcal{S}) \;\leq\; \frac{\epsilon}{s^n} \quad \text{for all} \quad s \in]0, \sigma]. \tag{43.9}$$

Proof: First assume that $\mathcal{S}$ satisfies the condition described and that $\epsilon \in \mathbb{P}^\times$ is given. Determine $s \in \mathbb{P}^\times$ such that (43.8) holds. Let $\mathcal{J} \in \Pi_s$ be given and assume that $\mathcal{J} \cap \mathcal{S} \neq \emptyset$. Denote the center of the box $\mathrm{Int}\,\mathcal{J}$ by p, so that $\mathrm{Int}\,\mathcal{J} = p + \frac{s}{2}\mathrm{Box}(\mathbf{b})$ and hence $\mathcal{J} \subset p + s\mathrm{Box}(\mathbf{b})$. In view of (43.8), the set $\mathfrak{k}$ of all centers obtained as just descibed has no more that $\frac{\epsilon}{2^n}$ elements. Clearly, the collection $\{p + s\mathrm{Box}(\mathbf{b}) \mid p \in \mathfrak{k}\}$ covers $\mathcal{S}$. The condition (ii) of Def.1 of Sect.41 holds when we put $\rho_p := s$ for all $p \in \mathfrak{k}$. Since $\epsilon \in \mathbb{P}^\times$ was arbitrary, it follows that $\mathcal{S}$ is negligible.

Assume now that $\mathcal{S}$ is negligible and that $\epsilon \in \mathbb{P}^\times$ is given. By Prop.1 of Sect.41, applied to the case when $\mathcal{B}$ is replaced by $\mathcal{T} - \frac{1}{2}\sum_{i \in I} \mathbf{b}_i \;\in\; \mathrm{Nhd}_0(\mathcal{V})$, we can determine a finite subset $\mathfrak{k}$ of $\mathcal{S}$ and a family $(\mathcal{K}_p \mid p \in \mathfrak{k})$ of tiles such that

$$\mathcal{S} \;\subset\; \bigcup\{\mathcal{K}_p \mid p \in \mathfrak{k}\} \;\;\text{and}\;\; \sum_{p \in \mathfrak{k}} {\rho_p}^n \;\leq\; \frac{\epsilon}{2^n}, \tag{43.10}$$

where, for each $p \in \mathfrak{k}$, ρ_p is the scale of the tile $\mathcal{K}_p$. We put $\sigma := \frac{1}{2}\min\{\rho_p \mid p \in \mathfrak{k}\}$. Let $s \in]0,\sigma]$ be given. By (43.10) we have $\mathrm{Clo}\,\mathcal{J} \cap \mathcal{S} \neq \emptyset$ for a given $\mathcal{J} \in \Pi_s$ if and only if $\mathrm{Clo}\,\mathcal{J} \cap \mathcal{K}_p \neq \emptyset$ for at least one $p \in \mathfrak{k}$. It follows that

$$\#\Gamma_s^{\mathrm{Clo}}(\mathcal{S}) \;\leq\; \sum_{p\in\mathfrak{k}} \#\Gamma_s^{\mathrm{Clo}}(\mathcal{K}_p),$$

and hence, by Prop.2, that

$$\#\Gamma_s^{\mathrm{Clo}}(\mathcal{S}) \;\leq\; \sum_{p\in\mathfrak{k}} (\frac{\rho_p}{s} + 2)^n.$$

Since $(\frac{\rho_p}{s} + 2) = \frac{\rho_p+2s}{s} \leq \frac{\rho_p+2\sigma}{s} \leq \frac{2\rho_p}{s}$ for all $p \in \mathfrak{k}$, the desired inequality (43.9) follows from (43.10). ▮

Proposition 4. *Every tile has a negligible boundary and hence belongs to* $\mathrm{Bnb}\,\mathcal{E}$.

Proof: Let a tile $\mathcal{K}$ be given and denote the scale of $\mathcal{K}$ by ρ. Clearly, since $\mathrm{Bdy}\,\mathcal{K} = \mathrm{Clo}\,\mathcal{K} \setminus \mathrm{Int}\,\mathcal{K}$, we have

$$\#\Gamma_s(\mathrm{Bdy}\,\mathcal{K}) = \#\Gamma_s(\mathrm{Clo}\,\mathcal{K}) - \#\Delta_s(\mathrm{Int}\,\mathcal{K}).$$

Using Prop.2, it follows that

$$s^n(\#\Gamma_s(\mathrm{Bdy}\,\mathcal{K})) \;\leq\; (\rho + 2s)^n - (\rho - 2s)^n$$

for all $s \in]0, \frac{\rho}{2}]$. The right side of this inequality can be made less than any given $\epsilon \in \mathbb{P}^\times$ by making s small enough. Hence $\mathrm{Bdy}\,\mathcal{K}$ is negligible by Prop.3 ▮

Proof of Prop.5 of Sect.41: Let a box $\mathcal{B}$ in $\mathcal{E}$ be given. We may assume that the given basis $\mathbf{b}$ is such that $\mathcal{B}$ is modelled on $\mathrm{Box}\,(\mathbf{b})$. It is clear that there is exactly one tile $\mathcal{K}$ such that $\mathcal{B} \subset \mathcal{K} \subset \mathrm{Clo}\,\mathcal{B}$ and hence $\mathrm{Bdy}\,\mathcal{K} = \mathrm{Bdy}\,\mathcal{B}$. It follows from Prop.4 that $\mathrm{Bdy}\,\mathcal{B}$ is negligible and hence that $\mathcal{B} \in \mathrm{Bnb}\,\mathcal{E}$. ▮

44. Existence of integrals.

The main purpose of this Section is to furnish a proof of the Theorem on Existence and Uniqueness of Integrals stated in Sect.42. To do so, we make the same assumptions and use the same definitions and notations as in Sect.43.

We now assume that $f \in \mathrm{Bbac}\,\mathcal{B}$ is given. Since f is bounded, $\inf f|_{\mathcal{J}}$ and $\sup f|_{\mathcal{J}}$ belong to $\mathbf{R}$ for all subsets $\mathcal{J}$ of $\mathcal{E}$. Given $s \in \mathbb{P}^\times$, we have $f|_{\mathcal{J}} = 0$ and hence $\inf f|_{\mathcal{J}} = \sup f|_{\mathcal{J}} = 0$ for all but a finite number of tiles $\mathcal{J} \in \Pi_s$ because f has bounded support. Therefore, the sums $\sum_{\mathcal{J}\in\Pi_s}(\inf f|_{\mathcal{J}})$ and $\sum_{\mathcal{J}\in\Pi_s}(\sup f|_{\mathcal{J}})$ are well defined because only a finite number of summands are not zero. These remarks show that the following definition is meaningful.

Definition 1. *For each* $k \in \boldsymbol{R}$ *the functionals* $\underline{S}_k : \mathrm{Bbac}\,\mathcal{E} \longrightarrow \boldsymbol{R}$ *and* $\overline{S}_k : \mathrm{Bbac}\,\mathcal{E} \longrightarrow \boldsymbol{R}$ *are defined by*

$$\underline{S}_k(f) := 2^{-kn} \sum_{\mathcal{J}\in\Pi_{2^{-kn}}} \inf f|_{\mathcal{J}}, \qquad \overline{S}_k(f) := 2^{-kn} \sum_{\mathcal{J}\in\Pi_{2^{-kn}}} \sup f|_{\mathcal{J}} \tag{44.1}$$

for all $f \in \mathrm{Bbac}\,\mathcal{B}$.

It is clear from Def.1 that

$$\underline{S}_k(f) \;\leq\; \overline{S}_k(f) \quad \text{for all} \quad k \in \mathbb{N}. \tag{44.2}$$

.

Lemma 1. *The sequence* $(\underline{S}_k(f) \mid k \in \mathbb{N})$ *is isotone and the sequence* $(\overline{S}_k(f) \mid k \in \mathbb{N})$ *is antitone.*

Proof: Let $k \in \mathbb{N}$ be given. We apply Prop.1 of Sect.43 to the case when $s := 2^{-(k+1)}$ and $m := 2$ and find that

$$2^n = \#\Delta_{2^{-(k+1)}}(\mathcal{K}) \quad \text{for all} \quad \mathcal{K} \in \Pi_{2^{-k}}.$$

Since $\inf f|_{\mathcal{K}} \leq \inf f|_{\mathcal{J}}$ when $\mathcal{J} \subset \mathcal{K}$ it follows that

$$2^{-kn} \sum_{\mathcal{K}\in\Pi_{2^{-kn}}} \inf f|_{\mathcal{K}} \;\leq\; 2^{-(k+1)n} \sum_{\mathcal{J}\in\Pi_{2^{-(k+1)n}}} \inf f|_{\mathcal{J}},$$

i.e., in view of (44.1), that $\underline{S}_k(f) \leq \underline{S}_{k+1}(f)$. Since $k \in \mathbb{N}$ was arbitrary, it follows that $(\underline{S}_k(f) \mid k \in \mathbb{N})$ is isotone. Applying this conclusion to the case when f is replaced by $-f$ and noting that $\underline{S}_k(-f) = -\overline{S}_k(f)$, we see that $(\overline{S}_k(f) \mid k \in \mathbb{N})$ is antitone. ∎

Lemma 2. *For every* $\epsilon \in \mathbb{P}^\times$ *there is a* $k \in \mathbb{N}$ *such that*

$$\underline{S}_k(f) \;-\; \overline{S}_k(f) \;\leq\; \epsilon. \tag{44.3}$$

Proof: We put $b := \sup f - \inf f$ and denote by $\mathcal{S}$ the negligible set of points at which f fails to be continuous.

Let $\epsilon \in \mathbb{P}^\times$ be given. By Prop.3 of Sect.43 we can determine $m \in \mathbb{N}$ such that

$$2^{-km}\,\#\Gamma^{\mathrm{Clo}}_{2^{-m}}(\mathcal{S}) \;\leq \frac{\epsilon}{2b}. \tag{44.4}$$

We put

$$\mathcal{D} := \bigcup \Gamma^{\mathrm{Clo}}_{2^{-m}}(\mathcal{S}). \tag{44.5}$$

Let $k \in m + \mathbb{N}$ be given. By Prop.1 of Sect.43, each tile in $\Pi_{2^{-m}}$ is the union of a disjoint collection of 2^{k-m} tiles in $\Pi_{2^{-k}}$. Hence we have

$$\#\Delta_{2^{-k}}(\mathcal{D}) = 2^{(k-m)n}\#\Gamma^{\mathrm{Clo}}_{2^{-m}}(\mathcal{S}).$$

Therefore, since $b \geq \sup f|_{\mathcal{J}} - \inf f|_{\mathcal{J}}$ for every $\mathcal{J} \in \operatorname{Sub}\mathcal{E}$, it follows from (44.4) that

$$2^{-kn} \sum_{\mathcal{J}\in\Pi_{2^{-k}}\cap\operatorname{Sub}\mathcal{D}} (\sup f|_{\mathcal{J}} - \inf f|_{\mathcal{J}}) \;\leq\; 2^{-kn} b\,(\#\Delta_{2^{-k}}(\mathcal{D})) \;\leq\; \frac{\epsilon}{2} \tag{44.6}$$

for all $k \in m + \mathbb{N}$. Next we put

$$\mathcal{C} := \bigcup\{\mathcal{K} \in \Pi_{2^{-m}} \mid f|_{\operatorname{Clo}\mathcal{K}} \text{ is continuous and not zero }\}. \tag{44.7}$$

Since f has bounded support $f|_{\operatorname{Clo}\mathcal{K}}$ can be non-zero for only a finite number of tiles $\mathcal{K} \in \Pi_{2^{-m}}$. Hence $\Delta_{2^{-m}}(\mathcal{C})$ is a finite set, so that the notation

$$\nu := 2^{-mn}(\#\Delta_{2^{-m}}(\mathcal{C}))$$

is meaningful. Again, since each tile in $\Pi_{2^{-m}}$ is the union of a disjoint collection of 2^{k-m} tiles in $\Pi_{2^{-k}}$, we have

$$\nu = 2^{-kn}(\#\Delta_{2^{-k}}(\mathcal{C})) \quad \text{for all} \;\; k \in m + \mathbb{N}. \tag{44.8}$$

Since $\mathcal{C}$ is bounded and hence $\operatorname{Clo}\mathcal{C}$ compact by the Compactness Theorem, it follows from the Uniform Continiuity Theorem (see Sect.58 of Vol.I) that $f|_{\operatorname{Clo}\mathcal{C}}$ and hence $f|_{\mathcal{C}}$ is uniformly continuous. Therefore, we can determine $\delta \in \mathbb{P}^{\times}$ such that $|f(x) - f(y)| \leq \frac{\epsilon}{2\nu}$ for all $x, y \in \mathcal{C}$ such that $x - y \in \delta\operatorname{Box}(\mathbf{b})$. Thus, given any tile $\mathcal{J}$ of scale $s \in]0, \delta]$ such that $\mathcal{J} \subset \mathcal{C}$, we have

$$\sup f|_{\mathcal{C}} - \inf f|_{\mathcal{C}} = \sup\{|f(x) - f(y)| \mid x, y \in \mathcal{J}\} \leq \frac{\epsilon}{2\nu}.$$

We determine $m' \in \mathbb{N}$ such that $2^{-m'} \leq \delta$. Observing (44.8), we conclude that

$$2^{-kn} \sum_{\mathcal{J}\in\Pi_{2^{-k}}\cap\operatorname{Sub}\mathcal{D}} (\sup f|_{\mathcal{J}} - \inf f|_{\mathcal{J}}) \;\leq \frac{\epsilon}{2} \qquad \text{for all} \qquad k \in \max\{m, m'\} + \mathbb{N}. \tag{44.9}$$

We now put $k := \max\{m, m'\}$. Let $\mathcal{J} \in \Pi_{2^{-k}}$ be given. It is clear that the sets $\mathcal{D}$ and $\mathcal{C}$ defined by (44.5) and (44.7) are disjoint and hence that $\mathcal{J} \subset \mathcal{D}$ and $\mathcal{J} \subset \mathcal{C}$ are mutually exclusive posssibilities. Moreover, if neither $\mathcal{J} \subset \mathcal{D}$ nor $\mathcal{J} \subset \mathcal{C}$ we must have $f|_{\mathcal{J}} = 0$ and hence $\sup f|_{\mathcal{J}} - \inf f|_{\mathcal{J}} = 0$. Therefore, by adding (44.6) and (44.9), we obtain

$$2^{-kn} \sum_{\mathcal{J}\in\Pi_{2^{-k}}} (\sup f|_{\mathcal{J}} - \inf f|_{\mathcal{J}}) \;\leq \epsilon,$$

which, by (44.1), proves the desired (44.3). ∎

Proposition 1. *The sequences* $(\underline{S}_k \mid k \in \mathbb{N})$ *and* $(\overline{S}_k \mid k \in \mathbb{N})$ *of functionals (see Def.1) both converge value-wise to one and the same functional* $S : \mathrm{Bbac}\,\mathcal{E} \longrightarrow \mathbb{R}$ *and we have*

$$\underline{S}_k \le S \le \overline{S}_k \qquad \text{for all} \;\; k \in \mathbb{N}. \tag{44.9}$$

The functional S *is isotone and linear and it satisfies*

$$S(\mathrm{ch}_{\mathcal{K}}) = \rho^n \tag{44.10}$$

for every tile $\mathcal{K}$ *of scale* ρ.

Proof: The fact that the sequences $(\underline{S}_k(f) \mid k \in \mathbb{N})$ and $(\overline{S}_k(f) \mid k \in \mathbb{N})$ both converge to the same limit

$$S(f) := \lim_{k\to\infty} \underline{S}_k(f) = \lim_{k\to\infty} \overline{S}_k(f) \qquad \text{for all} \;\; f \in \mathrm{Bbac}\,\mathcal{E} \tag{44.11}$$

is an immediate consequence of (44.2) and of Lemmas 1 and 2. So is the inequality (44.9).

It is clear from (44.1) that, for each $k \in \mathbb{N}$, $\underline{S}_k$ is a isotone functional. It follows from (44.11) that S is also isotone. The proof that S is linear is a fairly easy exercise, which is left to the reader.

Let a tile $\mathcal{K}$ of scale ρ and $s \in \mathbb{P}^\times$ be given. For every $\mathcal{J} \in \Pi_s$ we then have

$$\inf\,(\mathrm{ch}_{\mathcal{K}}|_{\mathcal{J}}) = \begin{cases} 1 & \text{if } \mathcal{J} \subset \mathcal{K} \\ 0 & \text{if } \mathcal{J} \not\subset \mathcal{K} \end{cases}$$

and

$$\sup\,(\mathrm{ch}_{\mathcal{K}}|_{\mathcal{J}}) = \begin{cases} 1 & \text{if } \mathcal{J} \cap \mathcal{K} \neq \emptyset \\ 0 & \text{if } \mathcal{J} \cap \mathcal{K} = \emptyset \end{cases}$$

Hence, using the notations (43.3) and (43.5), the definitions (44.1) yield

$$\underline{S}_k(\mathrm{ch}_{\mathcal{K}}) = 2^{-kn}(\#\Delta_{2^{-k}}(\mathcal{K})),$$

$$\overline{S}_k(\mathrm{ch}_{\mathcal{K}}) = 2^{-kn}(\#\Gamma_{2^{-k}}(\mathcal{K})),$$

for all $k \in \mathbb{N}$. Using Prop.2 of Sect.43, we obtain

$$(\rho - 2^{-k+1})^n \le \underline{S}_k(\mathrm{ch}_{\mathcal{K}}) \le \overline{S}_k(\mathrm{ch}_{\mathcal{K}}) \le (\rho + 2^{-k+1})^n$$

for all $k \in \mathbb{N}$. Taking the limit $k \to \infty$, we conclude from (44.11) that (44.10) holds. ∎

Proposition 2. *Let* $\mathfrak{S}$ *be a given subspace of* $\mathrm{Bbac}\,\mathcal{E}$ *that contains the characteristic function of every tile and is translation-invariant in the sense that*

$f \circ \mathbf{v} \in \mathfrak{S}$ for all $f \in \mathfrak{S}$ and all $\mathbf{v} \in \mathcal{V}$. Also, let $c \in \mathbb{P}^\times$, be given. Then $J := S|_{\mathfrak{S}}$ is the only linear functional on $\mathfrak{S}$ that is isotone and satisfies

$$J(\mathrm{ch}_{\mathcal{J}}) = c2^{-kn} \quad \text{for all } k \in \mathbb{N} \ \text{ and } \mathcal{J} \in \Pi_{2^{-k}}. \tag{44.12}$$

Moreover, if J' is a given isotone and translation-invariant linear functional on $\mathfrak{S}$, we must have $J' = c'S|_{\mathfrak{S}}$ for some $c' \in \mathbb{P}^\times$.

Proof: Let a functional J with the properties stated above be given. Also let $f \in \mathfrak{S}$ and $s \in \mathbb{P}^\times$ be given. It is clear that

$$\sum_{\mathcal{J} \in \Pi_s} (\inf f|_{\mathcal{J}})\,\mathrm{ch}_{\mathcal{J}} \ \le f \le \sum_{\mathcal{J} \in \Pi_s} (\sup f|_{\mathcal{J}})\,\mathrm{ch}_{\mathcal{J}}.$$

Noting that $\mathrm{ch}_{\mathcal{J}} \in \mathfrak{S}$ for all $\mathcal{J} \in \Pi_s$ and using the linearity and isotonicity of J, we obtain

$$\sum_{\mathcal{J} \in \Pi_s} (\inf f|_{\mathcal{J}}) J(\mathrm{ch}_{\mathcal{J}}) \ \le J(f) \le \sum_{\mathcal{J} \in \Pi_s} (\sup f|_{\mathcal{J}}) J(\mathrm{ch}_{\mathcal{J}}).$$

Putting $s := 2^{-k}$ and using (44.12) and Def.1, we find that

$$c\underline{S}_k(f) \le J(f) \le c\overline{S}_k(f) \quad \text{for all } k \in \mathbb{N}.$$

Taking the limit $k \to \infty$ we conclude, with the help of Prop.1, that $J(f) = cS(f)$. Since $f \in \mathfrak{S}$ was arbitrary, it follows that $J = S|_{\mathfrak{S}}$. On the other hand, if we *define* $J := S|_{\mathfrak{S}}$, then it follows from Prop.1 that J has the properties stated in the Proposition.

Now let an isotone and translation-invariant linear functional J' on $\mathfrak{S}$ be given. Also let $k \in \mathbb{N}$ and a tile $\mathcal{J} \in \Pi_{2^{-k}}$ be given. Since the partition $\Pi_{2^{-k}}$ of $\mathcal{E}$ is a refinement of the partition Π_1, we can determine exactly one $\mathcal{K} \in \Pi_1$ such that $\mathcal{J} \subset \mathcal{K}$, and we have

$$\mathrm{ch}_{\mathcal{K}} \ = \sum_{\mathcal{J} \in \Pi_{2^{-k}} \cap \mathrm{Sub}\mathcal{K}} \mathrm{ch}_{\mathcal{J}}. \tag{44.13}$$

Since J' was assumed to be translation-invariant, its value at all tiles of a given scale must be the same. In particular, we have $J'(\mathcal{I}) = J'(\mathcal{J})$ for all $\mathcal{I} \in \Pi_{2^{-k}}$. Let c' be the value of J' at all tiles of scale 1. Using (44.12), the linearity of $\mathcal{J}$, and Prop.1 of Sect.43, we conclude that $c' = J'(\mathrm{ch}_{\mathcal{K}}) = 2^{kn} J'(\mathrm{ch}_{\mathcal{J}})$ and hence that $J'(\mathrm{ch}_{\mathcal{J}}) = c'2^{-kn}$. It follows that J' satisfies the requirements of the first statement of Prop.2 and hence that $J' = c'S|_{\mathfrak{S}}$. ∎

Proposition 3. *The functional S described in Prop.1 is an integral and the corresponding volume-function (see Def.2 of Sect.42) satisfies*

$$\mathrm{vol}_S(\mathcal{C}) = (2s)^n \tag{44.14}$$

for every box $\mathcal{C}$ of scale s modelled on $\mathrm{Box}\,(\mathbf{b})$.

Proof: To prove that S is not only linear and isotone but also translation-invariant, let $\mathbf{v} \in \mathcal{V}$ be given. Define $S' : \mathrm{Bbac}\,\mathcal{E} \longrightarrow \mathbf{R}$ by $S'(f) := S(f \circ \mathbf{v})$ for all $f \in \mathrm{Bbac}\,\mathcal{E}$. It is clear that S', as well as S, is linear and isotone. Now let $k \in \mathbb{N}$ and a tile $\mathcal{J} \in \Pi_{2^{-k}}$ be given. Then $\mathcal{J} - \mathbf{v}$ is again a tile of skale 2^{-k} and hence, by (44.10), we have

$$S'(\mathrm{ch}_{\mathcal{J}}) = S(\mathrm{ch}_{\mathcal{J}} \circ \mathbf{v}) = S(\mathrm{ch}_{\mathcal{J}-\mathbf{v}}) = 2^{-kn}.$$

Since $k \in \mathbb{N}$ and $\mathcal{J} \in \Pi_{2^{-k}}$ were arbitrary, it follows from the uniqueness assertion of Prop.2, applied to the case when $\mathfrak{S} := \mathrm{Bbac}\,\mathcal{E}$, that $S' = S$, i.e. that $S(f) := S(f \circ \mathbf{v})$ for all $f \in \mathrm{Bbac}\,\mathcal{E}$. Since $\mathbf{v} \in \mathcal{V}$ was arbitrary, it follows that S is translation-invariant and hence an integral.

Let a box $\mathcal{C}$ of scale s modelled on $\mathrm{Box}\,(\mathbf{b})$ be given. Then there is exactly one tile $\mathcal{K}$ of scale $2s$ such that $\mathcal{C} \subset \mathcal{K}$. Let $\epsilon \in \mathbb{P}^{\times}$ be given. Then there is exactly one tile $\mathcal{K}'$ of scale $2s - \epsilon$ such that $\mathcal{C}$ and $\mathrm{Int}\,\mathcal{K}'$ have the same center. We have $\mathcal{K}' \subset \mathcal{C}$. Applying Prop.1 of Sect.42 to the integral S and using (44.10), we obtain

$$(2s - \epsilon)^n = \mathrm{vol}_S(\mathcal{K}') \leq \mathrm{vol}_S(\mathcal{C}) \leq \mathrm{vol}_S(\mathcal{K}) = (2s)^n.$$

Since $\epsilon \in \mathbb{P}^{\times}$ was arbitrary, the conclusion (44.14) follows. ∎

Proof of the Theorem on Existence and Uniqueness of Integrals: Prop.3 shows that there is an integral, namely S. Given $c \in \mathbb{P}^{\times}$, it is clear than $I := cS$ is also and integral.

Now let an integral I on $\mathcal{E}$ be given. Using the second assertion of Prop.2, applied to the case when $\mathfrak{S} := \mathrm{Bbac}\,\mathcal{E}$, it follows that we can determine $c \in \mathbb{P}^{\times}$ such that $I = cS$.

Now let a norming box $\mathcal{B}$ be given. We may assume that the basis $\mathbf{b}$ is such that $\mathcal{B} = \mathrm{Box}\,(\mathbf{b})$. It follows from (44.14) that

$$\mathrm{vol}_I(\mathcal{C}) = c\mathrm{vol}_S(\mathcal{C}) = (2^n c)s^n,$$

which shows that (42.10) is valid with $\gamma := 2^n c$. ∎

Proof of Prop.3 of Sect.42: Recalling that an integral I is given, we determine $c \in \mathbb{P}^{\times}$ such that $I = cS$, where S is the integral (see Prop.3) defined in Prop.1. Let $\epsilon \in \mathbb{P}^{\times}$ be given and detrmine $\mathcal{A} \in \mathrm{Bnb}\,\mathcal{E}$ such that $\mathcal{S} \subset \mathcal{A}$ and $\mathrm{vol}_I(\mathcal{A}) \leq \frac{c\epsilon}{2}$ and hence $S(\mathrm{ch}_{\mathcal{A}}) = \mathrm{vol}_S(\mathcal{A}) \leq \frac{\epsilon}{2}$. By Prop.1 we may choose $k \in \mathbb{N}$ such that $\overline{S}_k(\mathrm{ch}_{\mathcal{A}}) \leq \epsilon$. By the definition (44.1) of $\overline{S}_k$, this means that

$$2^{-kn}(\#\Gamma_{2^{-k}}(\mathcal{A})) \leq \epsilon$$

where the notation (43.5) has been used. Since $\mathcal{J} \cap \mathcal{S} \neq \emptyset$ implies $\mathcal{J} \cap \mathcal{A} \neq \emptyset$, it follows that (43.8) holds with $s := 2^{-k}$. Since $\epsilon \in \mathbb{P}^{\times}$ was arbitrary, it follows from Prop.3 of Sect.43 that $\mathcal{S}$ is negligible. ∎

45 Extended Integrals.

We assume again that a flat space $\mathcal{E}$ with translation space $\mathcal{V}$ is given.

In Def.1 of Sect. 42, an integral was defined as a linear finctional on the space $\mathrm{Bbac}\,\mathcal{E}$ of all functions that have bounded range, bounded support, and are almost continuous. In this section, we will show how such an integral can be extended to a class of functions that do not necessarily have bounded range and bounded support.

We now assume that a subset $\mathcal{D}$ of $\mathcal{E}$ is given. We use the notations $\mathrm{Nb}\,\mathcal{D}$, $\mathrm{Bnb}\,\mathcal{D}$, $\mathrm{Ac}\,\mathcal{D}$, and $\mathrm{Bbac}\,\mathcal{D}$ introduced in Sect.1.

The following result, analogous to Prop.3 of Sect.41, is an immediate consequence of Prop.2 of Sect.41.

Proposition 1. *The interior of every set belonging to* $\mathrm{Nb}\,\mathcal{D}$ *also belongs to* $\mathrm{Nb}\,\mathcal{D}$. *If* $\mathcal{D}$ *is closed, then the closure of every set belonging to* $\mathrm{Nb}\,\mathcal{D}$ *also belongs to* $\mathrm{Nb}\,\mathcal{D}$. *The union and the intersection of a finite collection of sets in* $\mathrm{Nb}\,\mathcal{D}$ *again belongs to* $\mathrm{Nb}\,\mathcal{D}$. *If a flat isomorphism* $\alpha : \mathcal{E} \longrightarrow \mathcal{E}'$ *from* $\mathcal{E}$ *to a flat space* $\mathcal{E}'$ *and* $\mathcal{A} \in \mathrm{Nb}\,\mathcal{E}$ *are given, then* $\alpha_>(\mathcal{A}) \in \mathrm{Nb}\,\alpha_>(\mathcal{D})$.

Of course, the set $\mathrm{Bnb}\,\mathcal{D}$ is a subset of $\mathrm{Nb}\,\mathcal{D}$.

The set of all functions in $\mathrm{Ac}\,(\mathcal{D})$ that have positive values will be denoted by

$$\mathrm{Pac}\,(\mathcal{D}) := \{f \in \mathrm{Ac}\,(\mathcal{D}) \mid f \geq 0\} \tag{45.1}$$

This set has the structure of a *linear cone* in the sense that it is stable under value-wise addition and multiplication by positive numbers.

We now consider the case when $\mathcal{D}$ is the entire space $\mathcal{E}$.

Definition 1. *Given* $f \in \mathrm{Pac}\,(\mathcal{E})$, *we define, for every* $\gamma \in \mathbb{P}$ *and every subset* $\mathcal{C}$ *of* $\mathcal{E}$, *the* **truncation** $\mathrm{Tr}^{\gamma}_{\mathcal{C}}(f)$ *of* f *at* γ *and* $\mathcal{C}$ *by*

$$\mathrm{Tr}^{\gamma}_{\mathcal{C}}(f)(x) := \begin{cases} f(x) & \textit{if } \ x \in \mathcal{C} \text{ and } f(x) \leq \gamma \ , \\ \gamma & \textit{if } \ x \in \mathcal{C} \text{ and } f(x) \geq \gamma \ , \\ 0 & \textit{if } \ x \in \mathcal{E} \backslash \mathcal{C} \ . \end{cases} \tag{45.2}$$

We abbreviate

$$\mathrm{Tr}^{\gamma}(f) := \mathrm{Tr}^{\gamma}_{\mathcal{E}}(f) \quad \text{and} \quad \mathrm{Tr}_{\mathcal{C}}(f) := \mathrm{Tr}^{\gamma}_{\mathcal{C}}(f) \ \text{ if } \ \gamma \geq \sup f_>(\mathcal{C}) \tag{45.3}$$

Let $f \in \mathrm{Pac}\,(\mathcal{E})$ be given. It is easily seen that, for every $\gamma \in \mathbb{P}$, and every $\mathcal{C} \in \mathrm{Bnb}\,\mathcal{E}$, the truncation $\mathrm{Tr}^{\gamma}_{\mathcal{C}}(f)$ belongs to $\mathrm{Bbac}\,\mathcal{E}$ as defined in Sect.41.

We now assume that an integral Igl on $\mathcal{E}$, as defined in Sect.2, is fixed. Then it makes sense to take the value of the integral at any of the truncations $\mathrm{Tr}^{\gamma}_{\mathcal{C}}(f)$ of f when $\mathcal{C} \in \mathrm{Bnb}\,\mathcal{E}$. Since f and hence all of these truncations have positive values, it follows from (42.5) that the value of Igl at all of these truncations of f is positive.

Definition 2. *The* **extended integral** $\overline{\mathrm{Igl}}^{+} : \mathrm{Pac}\,(\mathcal{E}) \longrightarrow \bar{\mathbb{P}}$ *is defined by*

$$\overline{\mathrm{Igl}}^{+}(f) := \sup \{ \ \mathrm{Igl}\ (\mathrm{Tr}^{\gamma}_{\mathcal{C}}(f)) \mid \gamma \in \mathbb{P}, \ \mathcal{C} \ \in \mathrm{Bnb}\,\mathcal{E} \ \} \ . \tag{45.4}$$

We say that a given $f \in \mathrm{Pac}\,\mathcal{E}$ *is* **integrable** *if* $\overline{\mathrm{Igl}}^{+}(f) < \infty$*.*

Let $\gamma, \gamma' \in \mathbb{P}$ and $\mathcal{C}, \mathcal{C}' \in \mathrm{Bnb}\,\mathcal{E}$ be given. It is clear from Def.1 that

$$\gamma \leq \gamma' \text{ and } \mathcal{C} \subset \mathcal{C}' \implies \mathrm{Tr}^{\gamma}_{\mathcal{C}}(f) \leq \mathrm{Tr}^{\gamma'}_{\mathcal{C}'}(f)$$

and hence, by (42.3), that

$$\gamma \leq \gamma' \text{ and } \mathcal{C} \subset \mathcal{C}' \quad \Longrightarrow \quad \mathrm{Igl}\,(\mathrm{Tr}^{\gamma}_{\mathcal{C}}(f)) \leq \mathrm{Igl}\,(\mathrm{Tr}^{\gamma'}_{\mathcal{C}'}(f)) \tag{45.5}$$

If $f \in \mathrm{Bbac}\,\mathcal{E} \cap \mathrm{Pac}\,\mathcal{E}$ then $\mathrm{Tr}^{\gamma}_{\mathcal{C}}(f) = f$ when $\gamma \geq \sup \mathrm{Rng}\, f$ and when the support of f is included in $\mathcal{C}$. Hence we have $\overline{\mathrm{Igl}}^{+}(f) = \mathrm{Igl}\,(f)$ in this case.

Recall that every cell in $\mathcal{E}$ is bounded and convex and therefore, by Prop.10 of Sect.41, belongs to $\mathrm{Bnb}\,\mathcal{E}$. The following result often leads to practical way of evaluating extended integrals:

Proposition 1. *Choose a point* $q \in \mathcal{E}$ *and a norming cell* $\mathcal{B}$ *(see Def.1 of Sect.51 in Vol.I) arbitrarily. We then have*

$$\overline{\mathrm{Igl}}^{+}(f) = \lim_{s \to \infty} \lim_{\gamma \to \infty} \mathrm{Igl}\,(\mathrm{Tr}^{\gamma}_{q+s\mathcal{B}}(f)) \ . \tag{45.6}$$

Proof: By the Cell-Inclusion Theorem (see Sect.52 of Vol.I) every $\mathcal{C} \in \mathrm{Bnb}\,\mathcal{E}$ is included in a cell of the form $q + s\mathcal{B}$ with $s \in \mathbb{P}$. Therefore, in view of (45.4), it is sufficient to consider only sets in $\mathrm{Bnb}\,\mathcal{E}$ that are of the form $\mathcal{C} := q + s\mathcal{B}$ when evaluating the supremum in (45.3). This fact leads immediately to (45.5). ∎

In the case when f has a bounded support but not necessarily a bounded range, (45.6) reduces to

$$\overline{\mathrm{Igl}}^{+}(f) = \lim_{\gamma \to \infty} \mathrm{Igl}\,(\mathrm{Tr}^{\gamma}(f)) \ . \tag{45.7}$$

In the case when f is continuous but has not nessarily a bounded support, (45.6) reduces to

$$\overline{\mathrm{Igl}}^{+}(f) = \lim_{s \to \infty} \mathrm{Igl}\,(\mathrm{Tr}_{q+s\mathcal{B}}(f)) \ , \tag{45.8}$$

because, by the Compact Image Theorem (see Sect.58 of Vol.I), the restriction of f to every set of the form $q + s\mathcal{B}$ has a bounded range.

Proposition 2. *Let $f, g \in \operatorname{Pac} \mathcal{E}$ be given. Then*

$$f \leq g \Longrightarrow \overline{\operatorname{Igl}}^{+}(f) \leq \overline{\operatorname{Igl}}^{+}(g) \ . \tag{45.9}$$

Also, we have $f + g \in \operatorname{Pac} \mathcal{E}$ and

$$\overline{\operatorname{Igl}}^{+}(f+g) = \overline{\operatorname{Igl}}^{+}(f) + \overline{\operatorname{Igl}}^{+}(g) \ . \tag{45.10}$$

Proof: The implication (45.9) is an immediate consequence of the isotonicity (42.3) if Igl and Def.2.

Let $\gamma, \in \mathbb{P}$ and $\mathcal{C}, \in \operatorname{Bnb} \mathcal{E}$ be given. It is clear from Def.1 that

$$\operatorname{Tr}_{\mathcal{C}}^{\gamma}(f) + \operatorname{Tr}_{\mathcal{C}}^{\gamma}(g) \ \geq \ \operatorname{Tr}_{\mathcal{C}}^{\gamma}(f+g)$$

and hence, by Def.2, that

$$\overline{\operatorname{Igl}}^{+}(f) + \overline{\operatorname{Igl}}^{+}(g) \geq \overline{\operatorname{Igl}}^{+}(f+g) \tag{45.11}$$

On the other hand, let $\gamma, \gamma' \in \mathbb{P}$ and $\mathcal{C}, \mathcal{C}' \in \operatorname{Bnb} \mathcal{E}$ be given. It is clear from Def.1 that

$$\operatorname{Tr}_{\mathcal{C}}^{\gamma}(f) + \operatorname{Tr}_{\mathcal{C}'}^{\gamma'}(g) \ \leq \ \operatorname{Tr}_{\mathcal{C} \cup \mathcal{C}'}^{\max\{\gamma, \gamma'\}}(f+g)$$

and hence, by Def.2, that

$$\operatorname{Igl}(\operatorname{Tr}_{\mathcal{C}}^{\gamma}(f)) + \operatorname{Igl}(\operatorname{Tr}_{\mathcal{C}'}^{\gamma'}(g)) \ \leq \overline{\operatorname{Igl}}^{+}(f+g) \ .$$

Since $\gamma, \gamma' \in \mathbb{P}$ and $\mathcal{C}, \mathcal{C}' \in \operatorname{Bnb} \mathcal{E}$ were arbitrary, it follows from Def.1 that

$$\overline{\operatorname{Igl}}^{+}(f) + \overline{\operatorname{Igl}}^{+}(g) \leq \overline{\operatorname{Igl}}^{+}(f+g) \ . \tag{45.12}$$

(45.11) and (45.12) together give the desired result (45.10). ∎

An immediate consequence of Prop.2 is the following

Corollary. *If $f, g \in \operatorname{Pac} \mathcal{E}$ are both integrable, so is $f + g$. If $f \leq g$ and g is integrable, so is f. If $f \leq g$ and f fails to be integrable, so does f.*

We assume now that a subset $\mathcal{D}$ of $\mathcal{E}$ is given and we consider the space $\operatorname{Ac} \mathcal{D}$ of all almost continuous functions on $\mathcal{D}$. The following facts are easily verified:

Proposition 3. *If f belongs to* $\mathrm{Ac}\,\mathcal{D}$ *then the value-wise absolute value* $|f|$ *of* f *belongs to* $\mathrm{Pac}\,\mathcal{D}$ *and so does* f^+, *defined by*

$$f^+(x) := \begin{cases} f(x) & \text{if } f(x) \geq 0 \\ 0 & \text{if } f(x) < 0 \end{cases} , \tag{45.13}$$

and we have

$$f = f^+ - (-f)^+ , \tag{45.14}$$

$$|f| = f^+ + (-f)^+ . \tag{45.15}$$

Moreover, if f *and* g *belong to* $\mathrm{Pac}\,\mathcal{D}$ *so does* $\inf\{f, g\}$, *defined by*

$$\inf\{f, g\}(x) := \min\{f(x), g(x)\} \quad \text{for all} \quad x \in \mathcal{D} , \tag{45.16}$$

and we have

$$(f - g)^+ = f - \inf\{f, g\} . \tag{45.17}$$

Again, we now consider the case when $\mathcal{D}$ is the entire space $\mathcal{E}$. It is clear from Def.2 and (45.15) that $|f|$ is integrable if and only if both f^+ and $(-f)^+$ are integrable.

Definition 3. *We say that a given* $f \in \mathrm{Ac}\,\mathcal{E}$ *is* **integrable** *if* $|f|$ *is integrable in the sense of Def.2 and we define the* **extended integral** *of* f *by*

$$\overline{\mathrm{Igl}}\,(f) := \overline{\mathrm{Igl}}^{\,+}(f^+) - \overline{\mathrm{Igl}}^{\,+}((-f)^+) \tag{45.18}$$

We use the **notation** $\mathrm{Iac}\,\mathcal{E}$ for the set of all integrable functions in $\mathrm{Ac}\,\mathcal{E}$ and we consider (45.18) as the definition of a mapping $\overline{\mathrm{Igl}} : \mathrm{Iac}\,\mathcal{E} \longrightarrow \mathbf{R}$.

Extended Integral Theorem. *The set* Iac *is a subspace of* Ac *and the mapping* $\overline{\mathrm{Igl}}$ *on the space* $\mathrm{Iac}\,\mathcal{E}$ *is linear in the sense that*

$$\overline{\mathrm{Igl}}\,(f + g) = \overline{\mathrm{Igl}}\,(f) + \overline{\mathrm{Igl}}\,(g) \quad \text{for all} \quad f, g \in \mathrm{Iac}\,\mathcal{E} \tag{45.19}$$

and

$$\overline{\mathrm{Igl}}\,(cf) = c\overline{\mathrm{Igl}}\,(f) \quad \text{for all} \quad c \in \boldsymbol{R} \;,\; f \in \mathrm{Iac}\,\mathcal{E} , \tag{45.20}$$

it is isotone in the sense that

$$f \geq g \implies \overline{\mathrm{Igl}}\,(f) \geq \overline{\mathrm{Igl}}\,(g) \quad \text{for all} \;\; f, g \in \mathrm{Iac}\,\mathcal{E} , \tag{45.21}$$

and translation-invariant in the sense that

$$\overline{\mathrm{Igl}}\,(f \circ \mathbf{v}) = \overline{\mathrm{Igl}}\,(f) \quad \text{for all} \;\; f \in \mathrm{Iac}\,\mathcal{E}, \; \mathbf{v} \in \mathcal{V}. \tag{45.22}$$

Proof: Let $f, g \in \mathrm{Iac}\,\mathcal{E}$ be given. Using (45.14) and (45.16) we find

$$(f + g)^+ = \big(f^+ - (-f)^+ + g^+ - (-g)^+\big)^+ =$$

$$((f^+ + g^+) - ((-f)^+ + (-g)^+))^+ = f^+ + g^+ - h \ ,$$

where

$$h := \inf \ (f^+ + g^+ \ , \ (-f)^+ + (-g)^+) \ , \tag{45.23}$$

amd hence

$$(f+g)^+ + h = f^+ + g^+ \tag{45.24}$$

Replacing f by $-f$ and g by $-g$ in (45.23) and (45.24) and noting that h remains unchanged, we see that

$$(-(f+g))^+ + h = (-f)^+ + (-g)^+ \ . \tag{45.25}$$

Since all the terms in (45.24) and (45.25) belong to $\mathrm{Pac}\,\mathcal{E}$, we can apply (45.10) to both and obtain

$$\overline{\mathrm{Igl}}^+ (f+g)^+ + \overline{\mathrm{Igl}}^+ h = \overline{\mathrm{Igl}}^+ f^+ + \overline{\mathrm{Igl}}^+ g^+ \tag{45.26}$$

and

$$\overline{\mathrm{Igl}}^+ (-(f+g))^+ + \overline{\mathrm{Igl}}^+ h = \overline{\mathrm{Igl}}^+ (-f)^+ + \overline{\mathrm{Igl}}^+ (-g)^+ \ . \tag{45.27}$$

Since both f and g are assumed to be integrable, it follows from the Corallary to Prop.2 that h, as defined by (45.23), is also integrable. Subtracting (45.27) from (45.26) and using Def.2, we see that $f+g$ is integrable and that (45.19) is valid.

The verifications of the remaining assertions of the Theorem are easy and left to the reader. ∎

The theorem just proved states that the extended integral $\overline{\mathrm{Igl}}$ preserves all the properties of the original integral Igl as described in Def.1 of Sect.42. In fact, Igl is the restriction of $\overline{\mathrm{Igl}}$ to the subspace $\mathrm{Bbac}\,\mathcal{E}$ of $\mathrm{Iac}\,\mathcal{E}$.

The concept of *volume* introduced in Def.2 of Sect.42 can be extended to the class $\mathrm{Nb}\,\mathcal{E}$ as defined in the beginning of this section.We note that the characteristic function $\mathrm{ch}_{\mathcal{A}}$ of a set $\mathcal{A} \in \mathrm{Nb}\,\mathcal{E}$, as defined by (41.1), belongs to $\mathrm{Pac}\,\mathcal{E}$.

Definition 4. *Using the extended integral* $\overline{\mathrm{Igl}}^+$ *defined in Def.2 above, the* **volume** *of a given set* $\mathcal{A} \in \mathrm{Nb}\,\mathcal{E}$ *is defined by*

$$\mathrm{vol}\,(\mathcal{A}) := \overline{\mathrm{Igl}}^+ (\mathrm{ch}_{\mathcal{A}}) \ . \tag{45.28}$$

If the set $\mathcal{A}$ is unbounded, then it is likely that $\mathrm{vol}\,(\mathcal{A}) = \infty$. However, it is not difficult to exhibit cases when $\mathrm{vol}\,(\mathcal{A}) < \infty$ even though $\mathcal{A}$ is unbounded. Prop.1 of Sect.42 remains valid if $\mathrm{Bnb}\,\mathcal{E}$ there is replaced by the larger set $\mathrm{Nb}\,\mathcal{E}$.

Let a set $\mathcal{D} \in \mathrm{Nb}\,\mathcal{E}$ be given. We say that a function $f : \mathcal{D} \longrightarrow \mathbf{R}$ is **integrable** if its standard extension $\bar{f}$, as defined by (41.3), belongs to $\mathrm{Iac}\,\mathcal{E}$. We use the **notation** $\mathrm{Iac}\,\mathcal{D}$ for the set of all such functions. It is clear that is a subspace of the space $\mathrm{Ac}\,\mathcal{D}$ introduced in the beginning of this section and that $\mathrm{Bbac}\,\mathcal{D}$ is a subspace of $\mathrm{Iac}\,\mathcal{D}$.

The concept of and *integral over* $\mathcal{D}$ introduced in Def.3 of Sect.42 can be extended to the class $\mathrm{Iac}\,\mathcal{D}$.

Definition 3. *Let a real valued function* f *such that* $\mathcal{D} \subset \operatorname{Dom} f \subset \mathcal{E}$ *and* $f|_{\mathcal{D}} \in \operatorname{Iac} \mathcal{D}$ *be given. Then the* **integral** *of* f **over** $\mathcal{D}$ *is defined by*

$$\int_{\mathcal{D}} f := \overline{\operatorname{Igl}}\,(\overline{f|_{\mathcal{D}}})\ . \tag{45.29}$$

Prop.4 and Prop.8 of Sect.42 remain valid if Bbac there is replaced by Iac.

46. Iterated integrals.

We assume now that flat spaces $\mathcal{E}_1$ and $\mathcal{E}_2$ and integrals Igl_1 and Igl_2 on $\mathcal{E}_1$ and $\mathcal{E}_2$, respectively, are given. We also consider the corresponding extended integrals $\overline{\operatorname{Igl}}_1$ and $\overline{\operatorname{Igl}}_2$, which are linear functionals on the spaces $\operatorname{Iac}\mathcal{E}_1$ and $\operatorname{Iac}\mathcal{E}_2$, respectively, as defined in Def.3 of Sect 54. We denote the corresponding volume functions by vol_1 and vol_2, respectively. We use the notation $\mathcal{E} := \mathcal{E}_1 \times \mathcal{E}_2$ for the product space (see Sect.32 of Vol.I). The translation spaces of $\mathcal{E}_1$ and $\mathcal{E}_2$ will be denoted by $\mathcal{V}_1$ and $\mathcal{V}_2$, respectively. Then $\mathcal{V} := \mathcal{V}_1 \times \mathcal{V}_2$ can be identified with the translation space of $\mathcal{E}$ (see Sect.32 of Vol.I).

Definition 1. *We denote by* $\operatorname{Iit}_{1,2}\mathcal{E}$ *the class of all functions* $f \in \operatorname{Map}(\mathcal{E}, \boldsymbol{R})$ *that satisfy the following two conditions:*

(a) For every $x_1 \in \mathcal{E}_1$ *we have* $f(x_1, \cdot) \in \operatorname{Iac}\mathcal{E}_2$, *so that* $\overline{\operatorname{Igl}}_2(f(x_1, \cdot)) \in \boldsymbol{R}$ *is meaningful.*

(b) The function $(x_1 \mapsto \overline{\operatorname{Igl}}_2(f(x_1, \cdot))) : \mathcal{E}_1 \longrightarrow \boldsymbol{R}$ *belongs to* Iac_1, *so that* $\operatorname{Igl}_1(x_1 \mapsto \operatorname{Igl}_2(f(x_1, \cdot)))$ *is meanigful.*

We define the **iterated integral** $\operatorname{Igl}_{1,2} : \operatorname{Iit}_{1,2}\mathcal{E} \longrightarrow \boldsymbol{R}$ *by*

$$\operatorname{Igl}_{1,2}(f) := \overline{\operatorname{Igl}}_1(x_1 \mapsto \overline{\operatorname{Igl}}_2(f(x_1, \cdot))) \quad \text{for all } f \in \operatorname{Iit}_{1,2}\mathcal{E}. \tag{46.1}$$

In a similar manner, By interchanging the roles of $\mathcal{E}_1$ *and* $\mathcal{E}_2$, *we define the class* $\operatorname{Iit}_{2,1}\mathcal{E}$ *and the iterated integral* $\operatorname{Igl}_{2,1} : \operatorname{Iit}_{2,1}\mathcal{E} \longrightarrow \boldsymbol{R}$ *by*

$$\operatorname{Igl}_{1,2}(f) := \overline{\operatorname{Igl}}_2(x_2 \mapsto \overline{\operatorname{Igl}}_1(f(\cdot, x_2))) \quad \text{for all } f \in \operatorname{Iit}_{2,1}\mathcal{E}. \tag{46.2}$$

It is easily seen that both $\operatorname{Iit}_{1,2}\mathcal{E}$ and $\operatorname{Iit}_{2,1}\mathcal{E}$ are translation-invariant subspaces of $\operatorname{Map}(\mathcal{E}, \mathbf{R})$.

Let $\mathcal{C}_1 \in \operatorname{Nb}\mathcal{E}_1$ and $\mathcal{C}_2 \in \operatorname{Nb}\mathcal{E}_2$ be given. Since

$$\operatorname{ch}_{\mathcal{C}_1 \times \mathcal{C}_2}(x) = \operatorname{ch}_{\mathcal{C}_1}(x_1)\operatorname{ch}_{\mathcal{C}_2}(x_2) \quad \text{for all } x = (x_1, x_2) \in \mathcal{E} \tag{46.3}$$

it is easily seen that $\operatorname{ch}_{\mathcal{C}_1 \times \mathcal{C}_2}$ belongs to both $\operatorname{Iit}_{1,2}\mathcal{E}$ and $\operatorname{Iit}_{2,1}\mathcal{E}$ and that

$$\operatorname{Igl}_{1,2}(\operatorname{ch}_{\mathcal{C}_1 \times \mathcal{C}_2}) = \operatorname{Igl}_{2,1}(\operatorname{ch}_{\mathcal{C}_1 \times \mathcal{C}_2}) = \operatorname{vol}_1(\mathcal{C}_1)\operatorname{vol}_2(\mathcal{C}_2). \tag{46.4}$$

The following result shows that, in many cases, integrals on $\mathcal{E}$ can be evaluated by iterated integrals.

Iterated Integral Theorem. *There is exactly one integral* Igl *on* $\mathcal{E}$ *such that*

$$\overline{\mathrm{Igl}}\,(f) = \mathrm{Igl}_{\,1,2}(f) \qquad \text{for all} \quad f \in \mathrm{Iit}_{1,2}\mathcal{E} \cap \mathrm{Iac}\,\mathcal{E} \tag{46.5}$$

and

$$\overline{\mathrm{Igl}}\,(f) = \mathrm{Igl}_{\,2,1}(f) \qquad \text{for all} \quad f \in \mathrm{Iit}_{2,1}\mathcal{E} \cap \mathrm{Iac}\,\mathcal{E} \tag{46.6}$$

Proof: Of course, $\mathfrak{S} := \mathrm{Iit}_{1,2}\mathcal{E} \ \cap \ \mathrm{Iac}\,\mathcal{E}$ is a translation-invariant subspace of $\mathrm{Iac}\,\mathcal{E}$. We choose basis sets $\mathfrak{b}_1$ and $\mathfrak{b}_2$ of the translation spaces $\mathcal{V}_1$ and $\mathcal{V}_2$ of $\mathcal{E}_1$ and $\mathcal{E}_2$, respectively. Then $\mathfrak{b} := (\mathfrak{b}_1 \times \{\mathbf{0}\}) \ \cup \ (\{\mathbf{0}\} \times \mathfrak{b}_2\})$ is a basis set of $\mathcal{E}$. Every tile in $\mathcal{E}$ corresponding to the basis $\mathfrak{b}$, as defined in Sect.43, is of the form $\mathcal{J}_1 \times \mathcal{J}_2$, where $\mathcal{J}_1$ is a tile in $\mathcal{E}_1$ corresponding to the basis $\mathfrak{b}_1$ and $\mathcal{J}_2$ is a tile in $\mathcal{E}_2$ corresponding to the basis $\mathfrak{b}_2$. By Prop.4 of Sect.43, it follows that the characteristic function of every tile in $\mathcal{E}$ corresponding to $\mathfrak{b}$ belongs to both $\mathrm{Iit}_{1,2}\mathcal{E}$ and $\mathrm{Bbac}\,\mathcal{E}$ and hence to $\mathfrak{S}$.

It is easily verified that $\mathrm{Igl}_{\,1,2}$ is an isotone and translation-invariant linear functional on $\mathfrak{S}$. Therefore, we can apply Prop.4 of Sect.45 to conclude that $\mathrm{Igl}_{\,1,2}$ differs from the restriction to $\mathfrak{S}$ of any given Integral on $\mathcal{E}$ by only a strictly positive factor. By the Theorem on Existence and Uniqueness of Integrals, we can fix the integral on $\mathcal{E}$ such that this factor is 1 and hence that (46.5) holds.

The formula (46.6) follows by interchanging the roles of $\mathcal{E}_1$ and $\mathcal{E}_2$ and noting, using (46.4), that $\mathrm{Igl}_{\,1,2}$ and $\mathrm{Igl}_{\,2,1}$ have the same value when $f := \mathrm{ch}_{\mathcal{T}_1 \times \mathcal{T}_2}$. ∎

Corollary. *For every function* $f \in \mathrm{Iit}_{1,2}\mathcal{E} \cap \mathrm{Iit}_{2,1}\mathcal{E} \cap \mathrm{Iac}\,\mathcal{E}$, *we have* $\mathrm{Igl}_{\,1,2}(f) = \mathrm{Igl}_{\,2,1}(f) = \overline{\mathrm{Igl}}\,(f)$.

Pitfall 1: It is not hard to give examples of functions that belong to $\mathrm{Iac}\,\mathcal{E}$ but not to $\mathrm{Iit}_{1,2}\,\mathcal{E}$ or $\mathrm{Iit}_{2,1}\,\mathcal{E}$. Integrals of such functions cannot be evaluated by iterated integration. ∎

Pitfall 2: It can happen that a function f belongs to both $\mathrm{Iit}_{1,2}\,\mathcal{E}$ and $\mathrm{Iit}_{2,1}\,\mathcal{E}$ but not to $\mathrm{Iac}\,\mathcal{E}$. In this case, the iterated integrals $\mathrm{Igl}_{\,1,2}(f)$ and $\mathrm{Igl}_{\,2,1}(f)$ need not be the same. An Example is the function $f : \mathbf{R} \times \mathbf{R} \longrightarrow \mathbf{R}$ defined, for all $s, t \in \mathbf{R}$, by

$$f(s,t) := \begin{cases} \frac{1}{t^2} & \text{if } 0 < s < t < 1 \\ -\frac{1}{s^2} & \text{if } 0 < t < s < 1 \\ 0 & \text{otherwise.} \end{cases}$$

We take both $\mathrm{Igl}_{\,1}$ and $\mathrm{Igl}_{\,2}$ to be the ordinary real integral. A quick calculation shows that f belongs to $\mathrm{Iit}_{1,2}\,\mathcal{E}$ and that

$$\mathrm{Igl}_{\,1,2}(f) = \int_0^1 (\int_0^1 f(s,t)\mathrm{d}s)\mathrm{d}t = 1.$$

Since $f(s,t) = -f(t,s)$, we also have $f \in \mathrm{Iit}_{2,1}\,\mathcal{E}$ and $\mathrm{Igl}_{\,2,1}(f) = -\mathrm{Igl}_{\,1,2}(f) = -1$. Although the function f is almost continuous and has bounded support, it does not have a bounded range and cannot belong to $\mathrm{Iac}\,\mathbf{R}^{\,2}$. ∎

Proposition 1. *Let* $\mathcal{C}_1 \in \mathrm{Bnb}\,\mathcal{E}_1$ *and* $\mathcal{C}_2 \in \mathrm{Bnb}\,\mathcal{E}_2$ *be given. Then* $\mathcal{C}_1 \times \mathcal{C}_2$ *belongs to* $\mathrm{Bnb}\,\mathcal{E}$ *and and we have*

$$\mathrm{vol}\,(\mathcal{C}_1 \times \mathcal{C}_2) = \mathrm{vol}_1(\mathcal{C}_1)\mathrm{vol}_2(\mathcal{C}_2). \tag{46.7}$$

Proof: In view of Prop.15 of Sect.53 of Vol.I, $\mathcal{C}_1 \times \mathcal{C}_2$ is a bounded subset of $\mathcal{E}$ and we have

$$\mathrm{Bdy}\,(\mathcal{C}_1 \times \mathcal{C}_2) \;=\; (\mathrm{Bdy}\,\mathcal{C}_1 \times \mathrm{Clo}\,\mathcal{C}_2) \;\cup\; (\mathrm{Clo}\,\mathcal{C}_1 \times \mathrm{Bdy}\,\mathcal{C}_2).$$

Thus, $\mathrm{Bdy}\,(\mathcal{C}_1 \times \mathcal{C}_2)$ is the union of two sets that are negligible by Prop.8 of Sect.41, proved above, and hence is itself negligible by (ii) of Prop.2 of Sect.41.

The formula (46.7) follows from (46.4) and (46.5) applied to $f := \mathrm{ch}_{\mathcal{C}_1 \times \mathcal{C}_2}$. ∎

Proof of Prop.8 of Sect.41: We assume, without loss, that $\mathcal{S}_1$ is negligible in $\mathcal{E}_1$ and that $\mathcal{S}_2$ is bounded in $\mathcal{E}_2$. By the Cell-Inclusion Theorem (see Sect.52 of Vol.I) we can choose a box $\mathcal{C}_2$ in $\mathcal{E}_2$ that includes $\mathcal{S}_2$. Also, we choose a norming box $\mathcal{B}_1$ in $\mathcal{V}_1$ and put $n_1 := \dim \mathcal{E}_1$. Using the Theorem on Existence and Uniqueness of Integrals of Sect.42, we see that we can determine $\gamma \in \mathbb{P}^\times$ such that $\mathrm{vol}\,(\mathcal{H}) = \gamma\rho^{n_1}$ for all boxes $\mathcal{H}$ of scale ρ modelled on $\mathcal{B}_1$.

Now let $\epsilon \in \mathbb{P}^\times$ be given. By Prop.1 of Sect.41 we can determine a finite subset $\mathfrak{k}$ of $\mathcal{S}_1$ and, for each $p \in \mathfrak{k}$, a box $\mathcal{H}_p$ centered at p and modelled on $\mathcal{B}_1$, such that

$$\mathcal{S}_1 \;\subset\; \bigcup_{p\in\mathfrak{k}} \mathcal{H}_p \quad \text{and} \quad \sum_{p\in\mathfrak{k}} (\rho_p)^{n_1} \;\le\; \frac{\epsilon}{\gamma\mathrm{vol}_2(\mathcal{C}_2)}\;, \tag{46.8}$$

where, for each $p \in \mathfrak{k}$, ρ_p is the scale of the box $\mathcal{H}_p$. Let $p \in \mathfrak{k}$ be given. It is clear that $\mathcal{H}_p \times \mathcal{C}_2$ is a box in $\mathcal{E} = \mathcal{E}_1 \times \mathcal{E}_2$ and hence, by Prop.5 of Sect.51 (proved in Sect.43), belongs to $\mathrm{Bnb}\,\mathcal{E}$. Hence (46.7) applies and we obtain

$$\mathrm{vol}\,(\mathcal{H}_p \times \mathcal{C}_2) = \mathrm{vol}_1(\mathcal{K}_p)\,\mathrm{vol}_2(\mathcal{C}_2) = \gamma(\rho_p)^{n_1}\mathrm{vol}_2(\mathcal{C}_2).$$

In view of (42.9) and $(46.8)_2$, we conclude that

$$\mathrm{vol}\,(\bigcup_{p\in\mathfrak{k}} \mathcal{H}_p \times \mathcal{C}_2) \;\le\; \sum_{p\in\mathfrak{k}} \mathrm{vol}\,(\mathcal{H}_p \times \mathcal{C}_2) \;\le \epsilon.$$

Since $\epsilon \in \mathbb{P}^\times$ was arbitrary and since $\mathcal{S}_1 \times \mathcal{S}_2 \subset\; (\bigcup_{p\in\mathfrak{k}} \mathcal{H}_p \times \mathcal{C}_2)$ by $(46.8)_1$, it follows from Prop.3 of Sect.41 that $\mathcal{S}_1 \times \mathcal{S}_2$ is negligible. ∎

From now on we also use the notations (42.12) and (42.17), with Igl replaced by Igl_1 or Igl_2 when appropriate. Then (46.5) and (46.6) read

$$\int_{\mathcal{E}} f \;=\; \int_{\mathcal{E}_1} (\int_{\mathcal{E}_2} f(x_1, x_2)\mathrm{d}x_2)\mathrm{d}x_1 \quad \text{for all } f \in \mathrm{Iit}_{1,2}\,\mathcal{E} \cap \mathrm{Iac}\,\mathcal{E} \tag{46.9}$$

and

$$\int_{\mathcal{E}} f \;=\; \int_{\mathcal{E}_2} (\int_{\mathcal{E}_1} f(x_1, x_2)\mathrm{d}x_1)\mathrm{d}x_2 \quad \text{for all } f \in \mathrm{Iit}_{2,1}\,\mathcal{E} \cap \mathrm{Iac}\,\mathcal{E}\;. \tag{46.10}$$

The following result is obtained by applying (46.9) to the characterstic functions of suitable subsets of $\mathcal{E}$ and by observing Def.2 of Sect.42.

Proposition 2. *Let* $\mathcal{A} \in \operatorname{Bnb}\mathcal{E}$ *be given. Assume that, for each* $x_1 \in \mathcal{E}_1$*, the set* $\mathcal{A}_{(x_1,\cdot)} := \{z \in \mathcal{E}_2 \mid (x_1, z) \in \mathcal{A}\}$ *(see Fig.1) belongs to* $\operatorname{Bnb}\mathcal{E}_2$ *and that the function* $(x_1 \mapsto \operatorname{vol}_2(\mathcal{A}_{(x_1,\cdot)})) : \mathcal{E}_1 \longrightarrow \boldsymbol{R}$ *belongs to* $\operatorname{Bbac}\mathcal{E}_1$*. Then*

$$\operatorname{vol}\mathcal{A} = \int_{\mathcal{E}_1} \operatorname{vol}_2(\mathcal{A}_{(x_1,\cdot)})\mathrm{d}x_1. \tag{46.11}$$

Corollary. *Let* $\mathcal{A} \in \operatorname{Bnb}\mathcal{E}$ *and* $\mathcal{A}' \in \operatorname{Bnb}\mathcal{E}$ *be given and assume that they both satisfy the hypotheses of Prop.2. If, for each* $x_1 \in \mathcal{E}_1$*, the set* ${\mathcal{A}'}_{(x_1,\cdot)}$ *can be obtained from* $\mathcal{A}_{(x_1,\cdot)}$*, by a translation in* $\mathcal{V}_2$*, then* $\operatorname{vol}\mathcal{A}' = \operatorname{vol}\mathcal{A}$*.*

Proposition 3. *Let closed sets* $\mathcal{C}_1 \in \operatorname{Bnb}\mathcal{E}_1$ *and* $\mathcal{C}_2 \in \operatorname{Bnb}\mathcal{E}_2$ *and a continuous function* $f : \mathcal{C}_1 \times \mathcal{C}_2 \longrightarrow \boldsymbol{R}$ *be given.Then the standard extension* $\overline{f} : \mathcal{E} \longrightarrow \boldsymbol{R}$ *of* f *(see (41.2)) belongs to* $\operatorname{Iit}_{1,2}\mathcal{E} \cap \operatorname{Iit}_{2,1}\mathcal{E} \cap \operatorname{Iac}\mathcal{E}$ *and we have*

$$\int_{\mathcal{C}_1\times\mathcal{C}_2} = \int_{\mathcal{C}_1}(\int_{\mathcal{C}_2} f(x_1,x_2)\mathrm{d}x_2)\mathrm{d}x_1 = \int_{\mathcal{C}_2}(\int_{\mathcal{C}_1} f(x_1,x_2)\mathrm{d}x_1)\mathrm{d}x_2. \tag{46.12}$$

Proof: The set $\mathcal{C}_1 \times \mathcal{C}_2$ is closed by Prop.15 of Sect.53 of Vol.I and it belongs to $\operatorname{Bnb}\mathcal{E}$ by Prop.1 above. Using the Compactness Theorem and the Compact Image Theorem (see Sect.58 of Vol.I), we find that $\mathcal{C}_1 \times \mathcal{C}_2$ is compact and that f has a bounded range. Therefore, the standard extension $\overline{f}$ belongs to $\operatorname{Bbac}\mathcal{E}$. A similar argument shows that $\overline{f}(x_1,\cdot) \in \operatorname{Bbac}\mathcal{E}_2$ for all $x_1 \in \mathcal{E}_1$.

Now let $\epsilon \in \mathbb{P}^\times$ be given. Since f is uniformly continuous by the Uniform Continuity Theorem of Sect.58 of Vol.I, we can determine $\mathcal{M}_1 \in \operatorname{Nhd}_{\mathbf{0}}(\mathcal{V}_1)$ such that

$$x_1 - y_1 \in \mathcal{M}_1 \implies |f(x_1,\cdot) - f(y_1,\cdot)| < \frac{\epsilon}{\operatorname{vol}_2(\mathcal{C}_2)}$$

and hence, by Prop.5 of Sect.42, such that

$$x_1 - y_1 \in \mathcal{M}_1 \implies |\int_{\mathcal{C}_2} f(x_1,\cdot) - \int_{\mathcal{C}_2} f(y_1,\cdot)| < \epsilon$$

holds for all $x_1, y_1 \in \mathcal{C}_1$. Since $\epsilon \in \mathbb{P}^\times$ was arbitrary, it follows that $(x_1 \mapsto \int_{\mathcal{C}_2} f(x_1,\cdot)) : \mathcal{C}_1 \longrightarrow \mathrm{R}$ is continuous and hence that $(x_1 \mapsto I_2(\overline{f}(x_1,\cdot))$ belongs to $\operatorname{Bbac}\mathcal{E}_1$. Therefore, $\overline{f}$ satisfies the conditions (a) and (b) of Def.1, i.e. we have $\overline{f} \in \operatorname{Iit}_{1,2}\mathcal{E}$. A similar argument shows that $\overline{f} \in \operatorname{Iit}_{2,1}\mathcal{E}$ and hence we have $\overline{f} \in \operatorname{Iit}_{1,2}\mathcal{E} \cap \operatorname{Iit}_{2,1}\mathcal{E} \cap \operatorname{Iac}\mathcal{E}$. The formula (46.12) follows now from the Iterated Integral Theorem. ∎

48. Euclidean integrals and volumes.

In this section, we assume that a genuine Euclidean space $\mathcal{E}$ is given. We denote the translation space of $\mathcal{E}$ by $\mathcal{V}$ and put $n := \dim\mathcal{E} = \dim\mathcal{V}$.

The **norming cube** determined by a given orthonormal basis $\mathbf{e} := (\mathbf{e}_i \mid i \in I)$ is defined to be

$$\operatorname{Cub}(\mathbf{e}) := \operatorname{Box}(\frac{1}{2}\mathbf{e}) = \{\mathbf{v} \in \mathcal{V} \mid |\mathbf{e}_i \cdot \mathbf{v}| < \frac{1}{2} \text{ for all } i \in I\} \tag{48.1}$$

(see Example (A) in Sect.51 o Vol.I). A cell of scale $s \in \mathbb{P}^\times$ modelled on a norming cube will be called a **cube of scale** s.

Characterization of the Euclidean Integral. *There is exactly one integral* Igl , *with corresponding volume function vol on* $\mathcal{E}$ *such that* $\text{vol}\,(\mathcal{C}) = 1$ *for every cube* $\mathcal{C}$ *of scale* 1. *This integral is called the* **Euclidean integral** *on* $\mathcal{E}$ *and the volume-function associated with it is called the* **Euclidean volume**.

The proof of this theorem will be based on the following:

Lemma. *Every integral* Igl *on* $\mathcal{E}$ *is invariant under Euclidean automorphisms of* $\mathcal{E}$. *More precisely, for every* $\alpha \in \text{Eis}\,\mathcal{E}$ *we have*

$$\text{Igl}\,(f \circ \alpha) = \text{Igl}\,(f) \quad \text{for all } f \in \text{Iac}\,\mathcal{E} \tag{48.2}$$

and

$$\text{vol}\,(\alpha_{>}(\mathcal{A})) = \text{vol}\,(\mathcal{A}) \quad \text{for all } \mathcal{A} \in \text{Bnb}\,\mathcal{E}\ . \tag{48.3}$$

Proof: Let an integral Igl on $\mathcal{E}$ and $\alpha \in \text{Eis}\,\mathcal{E}$ be given. Since α is a flat automorphism, Prop.10 of Sect.42 can be applied and (42.23) and (42.24) must hold for a suitable $c \in \mathbb{P}^{\times}$. Now, since balls in $\mathcal{E}$ are bounded and convex (see Example (C) in Sect.51 of Vol.I), they belong to $\text{Bnb}\,\mathcal{E}$ by Prop.10 of Sect.41. In particular, $\text{vol}\,(x + \text{Ubl}\,\mathcal{V})$ is meanigful for all $x \in \mathcal{E}$.

We now choose a point $q \in \mathcal{E}$. Since $\alpha_{>}(q+\text{Ubl}\,\mathcal{V}) = \alpha(q)+\text{Ubl}\,\mathcal{V}$, it follows from (42.24) that

$$\text{vol}\,(q + \text{Ubl}\,\mathcal{V}) = \frac{1}{c}\text{vol}\,(\alpha(q) + \text{Ubl}\,\mathcal{V})\ . \tag{48.4}$$

Since $\alpha(q)+\text{Ubl}\,\mathcal{V}$ can be obtained from $q+\text{Ubl}\,\mathcal{V}$ by the translation $\alpha(q)-q \in \mathcal{V}$, it follows from the translation invariance (42.8) of vol that (48.4) must continue to hold when c is replaced by 1. Since a ball has a non-empty interior and hence non-zero volume by Prop.2 of Sect.42, we conclude that $c = 1$ and hence that (42.23) and (42.24) reduce to (48.2) and (48.3), respectively. ∎

Proof of the Characterization: We choose a cube $\mathcal{C}_0$ of scale 1. By the Theorem on Existence and Uniqueness of Integrals, there is exactly one integral I on $\mathcal{E}$ such that $\text{vol}\,(\mathcal{C}_0) = 1$. Since *every* cube of scale 1 in $\mathcal{E}$ is the image of $\mathcal{C}_0$ under a Euclidean automorphism, it follows from the Lemma that $\text{vol}_I(\mathcal{C}) = 1$ for *all* cubes $\mathcal{C}$ of scale 1 in $\mathcal{E}$. ∎

When dealing with genuine Euclidean spaces, we will omit the adjective "Euclidean" before "integral" and "volume". Also, we will use the notations (42.11) and (42.12) and we will use Igl *only* the Euclidean integral and vol *only* for the corresponding volume. Occasionally, we will write $\text{vol}_{\mathcal{E}}$ instead of just vol when we wish to emphasize the dependence of the volume-function on the space $\mathcal{E}$.

In the case when $\mathcal{E} := \mathbf{R}$, the Euclidean integral reduces to the integral familiar from elementary real analysis and the Euclidean volume reduces to the length le (see Sect.42). In the case when $\dim \mathcal{E} = 0$, $\mathcal{E}$ is a singleton and we have $\text{vol}\,\mathcal{E} = 1$.

Assume, for a moment, that genuine Euclidean spaces $\mathcal{E}_1$ and $\mathcal{E}_2$ are given and let Igl_1 and Igl_2 be the corresponding Euclidean integrals. It is easily seen that the set-product of a cube of scale 1 in $\mathcal{E}_1$ and a cube of scale 1 in $\mathcal{E}_2$ is a cube of scale 1 in $\mathcal{E}_1 \times \mathcal{E}_2$. It follow that (46.7) is valid when vol, vol_1, and vol_2 are interpreted as Euclidean volumes. Also, the Iterated Integral Theorem and Props.1 and 2 of Sect.46 as well as all results of Sect.47 are valid, in particular, when all integrals and volumes are interpreted as Euclidean integrals and volumes.

Proposition 1. *Let Euclidean spaces $\mathcal{E}$ and $\mathcal{E}'$ and a Euclidean isomorphism $\alpha : \mathcal{E} \longrightarrow \mathcal{E}'$ be given. Then*

$$\int_{\mathcal{E}'} f \circ \alpha = \int_{\mathcal{E}} f \quad \text{for all} \;\; f \in \mathrm{Iac}\,\mathcal{E} \; . \tag{48.5}$$

Proof: It follows from Prop.9 of Sect.42 that $(f \mapsto \int_{\mathcal{E}'} f \circ \alpha) : \mathrm{Iac}\,\mathcal{E} \longrightarrow \mathbf{R}$ must be an integral on $\mathcal{E}$. Since the image under α of a cube of scale 1 in $\mathcal{E}'$ is a cube of scale 1 in $\mathcal{E}$, the integral $(f \mapsto \int_{\mathcal{E}'} f \circ \alpha)$ must be the Euclidean integral on $\mathcal{E}$, which means that (48.5) must hold. ∎

From now on we assume that an orthonormal basis $(\mathbf{e}_i \mid i \in I)$ of $\mathcal{V}$, a corresponding family $(a_i \mid i \in I)$ with terms in $\mathbb{P}^\times$, and a point $q \in \mathcal{E}$ are given. The set

$$\mathcal{C} := q + \mathrm{Box}\,(\frac{1}{2} a_i \mathbf{e}_i \mid i \in I) \tag{48.6}$$

is called a **rectangular box centered at** q with $(a_i \mid i \in I)$ as its family of **sidelengths**.

Proposition 2. *The volume of a rectangular box $\mathcal{C}$ with $(a_i \mid i \in I)$ as its family of sidelengths is the product of this family, i.e. we have*

$$\mathrm{vol}\,(\mathcal{C}) = \prod_{i \in I} a_i \; . \tag{48.7}$$

Proof: We proceed by induction over $\dim \mathcal{E}$. If $\dim \mathcal{E} = 0$ then $\mathcal{E} = \mathcal{C}$ is a singleton and its family of sidelenghts is empty. Hence (48.7) is valid because $\mathrm{vol}\,(\mathcal{E}) = 1$ and because the product of the empty family is 1.

Assume now that $\dim \mathcal{E} > 0$ and that the assertion is valid if $\mathcal{E}$ is replaced by a space $\mathcal{E}'$ having one dimension less than $\mathcal{E}$. Let a rectangular box the form (48.6) be given. We choose $j \in I$ and put $I' := I \backslash \{j\}$, $\mathcal{E}' := q + \mathrm{Lsp}\{\mathbf{e}_i \mid i \in I\}$, and

$$\mathcal{C}' := q + \mathrm{Box}\,(\frac{1}{2} a_i \mathbf{e}_i \mid i \in I') \; .$$

Since $\mathcal{C}'$ is a rectangular box in the Euclidean space $\mathcal{E}'$, which has one dimension less than $\mathcal{E}$, we can apply the induction hypothesis and obtain

$$\mathrm{vol}'(\mathcal{C}') = \prod_{i \in I'} a_i \; , \tag{48.8}$$

where vol' is the volume-function for $\mathcal{E}'$. Now, the box $\mathcal{C}$ is the image of $\mathcal{C}'\times\,]-\frac{a_j}{2},\frac{a_j}{2}[$ under the Euclidean isomorphism

$$\big((x',s)\mapsto(x'+s\mathbf{e}_j)\big):\mathcal{E}'\times\mathbf{R}\longrightarrow\mathcal{E}\ .$$

Hence, by Prop.1 above and by (46.7), it follows that

$$\mathrm{vol}\,(\mathcal{C})=\mathrm{vol}'(\mathcal{C}')\,\mathrm{le}\,(]-\frac{a_j}{2},\frac{a_j}{2}[)\ .$$

Since $\mathrm{le}\,(]-\frac{a_j}{2},\frac{a_j}{2}[)=a_j$, it follows from (48.8) that

$$\mathrm{vol}\,(\mathcal{C})=\Big(\prod_{i\in I'}a_i\Big)a_j\ ,$$

which yields (48.7). ∎

Proposition 3. *Let α be a flat automorphism of $\mathcal{E}$ such that*

$$\nabla\alpha=\sum_{i\in I}a_i\,(\mathbf{e}_i\times\mathbf{e}_i)\ . \tag{48.9}$$

Then

$$\int f\circ\alpha=\Big(\prod_{i\in I}a_i\Big)^{-1}\int f\quad\text{for all}\quad f\in\mathrm{Iac}\,\mathcal{E} \tag{48.10}$$

and

$$\mathrm{vol}\,(\alpha_>(\mathcal{A}))=\Big(\prod_{i\in I}a_i\Big)\mathrm{vol}\,\mathcal{A}\quad\text{for all}\quad\mathcal{A}\in\mathrm{Bnb}\,\mathcal{E}\ . \tag{48.11}$$

Proof: In view of Prop.10 of Sect.42 we can determine $c\in\mathbb{P}^\times$ such that

$$\int f\circ\alpha=c\int f\quad\text{for all}\quad f\in\mathrm{Iac}\,\mathcal{E} \tag{48.12}$$

and

$$\mathrm{vol}\,(\alpha_>(\mathcal{A}))=\frac{1}{c}\mathrm{vol}\,\mathcal{A}\quad\text{for all}\quad\mathcal{A}\in\mathrm{Bnb}\,\mathcal{E}\ . \tag{48.13}$$

We now consider the cube of scale 1 given by $\mathcal{C}:=q+\mathrm{Box}\,(\frac{1}{2}\mathbf{e})$. It easily follows from (48.9) that

$$\alpha_>(\mathcal{C}):=\alpha(q)+\mathrm{Box}\,(\frac{1}{2}a_i\mathbf{e}_i\mid i\in I)\ ,$$

which is a rectangular box with $(a_i\mid i\in I)$ as its family of sidelengths. Therefore, by Prop.2, we have $\mathrm{vol}\,(\alpha_>(\mathcal{A}))=\prod_{i\in I}a_i$. We now apply (48.13) to the case when $\mathcal{A}:=\mathcal{C}$ and note that $\mathrm{vol}\,\mathcal{C}=1$. We conclude that $\prod_{i\in I}a_i=\frac{1}{c}$. Thus, (48.12) and (48.13) reduce to (48.10) and (48.11), respectively. ∎

Proposition 4. *The volume*

$$\omega_n := \mathrm{vol}_{\mathcal{V}}(\mathrm{Ubl}\,\mathcal{V}) \ , \quad n := \dim \mathcal{V} \tag{48.14}$$

of the unit ball is given by

$$\omega_n = \prod_{k \in n^]} \alpha_k \ , \tag{48.15}$$

where

$$\alpha_k := \begin{cases} \pi \ \prod_{j \in (\frac{k}{2})^]} \frac{(2j-1)}{2j} & \text{if } \ k \in 2\mathbb{N}, \\ 2 \ \prod_{j \in (\frac{k-1}{2})^]} \frac{(2j)}{(2j+1)} & \text{if } \ k \in 2\mathbb{N}+1 \ . \end{cases} \tag{48.16}$$

Proof: We proceed by induction over the dimension n. If $n := 0$, we trivially have $\omega_0 = 1$, which is in accord with (48.15) because $0^] = \emptyset$.

We assume now that $n > 0$ and that (48.15) becomes valid after n has been replaced by $n-1$. We choose a unit vector $\mathbf{u} \in \mathcal{V}$ and put $\mathcal{U} := \{\mathbf{u}\}^{\perp}$ (see Fig.1). Given $\xi \in \,]-1,1[$, we put

$$\mathcal{B}_\xi := (\mathrm{Ubl}\,\mathcal{V}) \cap (\xi\mathbf{u} + \mathcal{U}) - \xi\mathbf{u} \ . \tag{48.17}$$

It is easily seen that $\mathcal{B}_\xi = \sqrt{1-\xi^2}\,\mathrm{Ubl}\mathcal{U}$ and hence that $\mathcal{B}_\xi = (\sqrt{1-\xi^2}\,\mathbf{1}_{\mathcal{U}})_{>}(\mathrm{Ubl}\mathcal{U})$. By Prop.10 of Sect.42 and (48.14), with $\mathcal{V}$ replaced by $\mathcal{U}$, it follows that $\mathrm{vol}_{\mathcal{U}}(\mathcal{B}_\xi) = \sqrt{1-\xi^2}^{\,n-1}\omega_{n-1}$. Using Prop.2 of Sect.46 and (48.17), we find that

$$\mathrm{vol}_{\mathcal{V}}(\mathrm{Ubl}\,\mathcal{V}) = \int_{-1}^{1} \mathrm{vol}_{\mathcal{U}}(\mathcal{B}_\xi)\mathrm{d}\xi = \omega_{n-1}\int_{-1}^{1} \sqrt{1-\iota^2}^{\,n-1} \ . \tag{48.18}$$

It is an exercise in elemantary calculus to show that

$$\alpha_k = \int_{-1}^{1} \sqrt{1-\iota^2}^{\,k-1} \qquad \text{for all} \ \ k \in \mathbb{N}^{\times}$$

when α_k is defined by (48.16). Therefore, in view of (48.14), (48.18) reduces to $\omega_n = \omega_{n-1}\alpha_{n-1}$, from which (48.15) follows by the induction hypothesis. ▮

Prop.4 yields, in particular,

$$\omega_1 = 2, \ \ \omega_2 = \pi, \ \ \omega_3 = \frac{4}{3}\pi, \ \ \omega_4 = \frac{1}{2}\pi^2, \ \ \omega_5 = \frac{8}{15}\pi^2 \ .$$

Of course , the values for ω_1, ω_2, and ω_3 are familiar from elementary mathematics.

The following result is an immediate consequence of Prop.4 above and Prop.10 of Sect.42.

Proposition 5. *Every ball in $\mathcal{E}$ or $\mathcal{V}$ of a given radius $r \in \mathbb{P}^\times$ has volume $r^n \omega_n$.*

The set

$$\mathcal{G} := q + \Big(\sum_{i \in I} a_i\,(\mathbf{e}_i \times \mathbf{e}_i)\Big)_{>}(\mathrm{Ubl}\,\mathcal{V}) \tag{48.19}$$

is called an **ellipsoid centered at** q with $(a_i \mid i \in I)$ as its family of **semi-axes**.

The following result is an immediate consequence of Props.3 and 4 above.

Proposition 6. *The volume of an ellipsoid $\mathcal{G}$ with $(a_i \mid i \in I)$ as its family of semi-axes is given by*

$$\mathrm{vol}\,\mathcal{G} = \omega_n \prod_{i \in I} a_i\ , \tag{48.20}$$

where ω_n is given by (48.15) and (48.16).

49. Set functions

In this section, we assume that a flat space $\mathcal{E}$ is given. We denote its translation space by $\mathcal{V}$ and we put $n := \dim \mathcal{E} = \dim \mathcal{V}$. We also assume that an integral on $\mathcal{E}$ and an *open* subset $\mathcal{D}$ of $\mathcal{E}$ are given, and we use the **notation**

$$\mathrm{Bnb}\,\mathcal{D} := \{\mathcal{A} \in \mathrm{Bnb}\,\mathcal{E} \mid \mathrm{Clo}\,\mathcal{A} \subset \mathcal{D}\}\ . \tag{49.1}$$

In view of Prop.5 of Sect.41, we have

Proposition 1. *The interior and the closure of every set in $\mathrm{Bnb}\,\mathcal{D}$ also belongs to $\mathrm{Bnb}\,\mathcal{D}$. The union and the intersection of a finite collection of sets in $\mathrm{Bnb}\,\mathcal{D}$ again belong to $\mathrm{Bnb}\,\mathcal{D}$. The set-difference of two sets in $\mathrm{Bnb}\,\mathcal{D}$ again belongs to $\mathrm{Bnb}\,\mathcal{D}$.*

The term **set-function** is used for a function whose codomain is $\mathbf{R}$ and whose domain is a collection of subsets of a given set. Here we consider set functions from $\mathrm{Bnb}\,\mathcal{D}$ to $\mathbf{R}$.

Definition 1. *We say that a given set-function $F : \mathrm{Bnb}\,\mathcal{D} \longrightarrow \mathbf{R}$ is*

(a) **additive** *if*

$$F(\mathcal{A}_1 \cup \mathcal{A}_2) = F(\mathcal{A}_1) + F(\mathcal{A}_2) \tag{49.2}$$

for all disjoint pairs $(\mathcal{A}_1, \mathcal{A}_1)$ with terms in $\mathrm{Bnb}\,\mathcal{D}$,

(b) **isotone** *if*

$$\mathcal{A}_1 \subset \mathcal{A}_2 \quad \Longrightarrow \quad F(\mathcal{A}_1) \leq F(\mathcal{A}_2) \tag{49.3}$$

for all $\mathcal{A}_1, \mathcal{A}_2 \in \mathrm{Bnb}\,\mathcal{D}$,

(c) **positive** *if $\mathrm{Rng}\,F \subset \mathbb{P}$, and*

(d) **compactly differentiable** *if one can find a continuous function $f : \mathcal{D} \longrightarrow \mathbf{R}$ with the following property: for every norming cell $\mathcal{B}$ of $\mathcal{V}$, every compact subset $\mathcal{K}$ of $\mathcal{D}$ and every $\epsilon \in \mathbb{P}^\times$ there is a $\delta \in \mathbb{P}^\times$ such that*

$$\Big|\frac{F(x + s\mathcal{B})}{\mathrm{vol}\,(x + s\mathcal{B})} - f(x)\Big| < \epsilon \tag{49.4}$$

for all $x \in \mathcal{K}$ and all $s \in]0, \delta]$ such that $x + s\mathcal{B} \in \operatorname{Bnb}\mathcal{D}$.

It is evident that if F is compactly differentiable, then there is exactly one function f with the property described in (d) of Def.1; it is called the **density** of F.

Proposition 2. *If the given set-function $F : \operatorname{Bnb}\mathcal{D} \longrightarrow \boldsymbol{R}$ is additive and positive, it is also isotone.*

Proof: Let $\mathcal{A}_1, \mathcal{A}_2 \in \operatorname{Bnb}\mathcal{D}$ be given such that $\mathcal{A}_1 \subset \mathcal{A}_2$. The pair $(\mathcal{A}_1, \mathcal{A}_2 \backslash \mathcal{A}_1)$ is disjoint and its union is $\mathcal{A}_2$. Moreover, $\mathcal{A}_2 \backslash \mathcal{A}_1$ belongs to $\operatorname{Bnb}\mathcal{D}$ by Prop.1. Hence, if F is additive, we have $F(\mathcal{A}_2) = F(\mathcal{A}_1) + F(\mathcal{A}_2 \backslash \mathcal{A}_1)$. If F is also positive, it follows that $F(\mathcal{A}_1) \leq F(\mathcal{A}_2)$, and (49.3) is proved. ∎

The following result states that *every* continuous function $f : \mathcal{D} \longrightarrow \mathbf{R}$ is the density of an additive and compactly differentable set function.

Proposition 3. *Let a continuous function $f : \mathcal{D} \longrightarrow \boldsymbol{R}$ be given. Then $\int_{\mathcal{A}} f$ is meaningful for every $\mathcal{A} \in \operatorname{Bnb}\mathcal{D}$, so that we can define the set-function $F : \operatorname{Bnb}\mathcal{D} \longrightarrow \boldsymbol{R}$ by*

$$F(\mathcal{A}) := \int_{\mathcal{A}} f \quad \text{for all} \ \ \mathcal{A} \in \operatorname{Bnb}\mathcal{D} \,. \tag{49.5}$$

This set function F is additive and compactly differentiable, and its density is f.

Proof: Let $\mathcal{A} \in \operatorname{Bnb}\mathcal{D}$ be given. Since $\operatorname{Clo}\mathcal{A}$ is a compact subset of $\mathcal{D}$, we can apply the Compact Image Theorem (Sect.58 of Vol.I) to conclude that $f_{>}(\operatorname{Clo}\mathcal{A})$ is bounded and hence that $\overline{f|_{\mathcal{A}}} \in \operatorname{Bbac}\mathcal{E}$, which shows that $\int_{\mathcal{A}} f$ is meaningful by Def.3 of Sect.42.

The additivity of F follows from Prop.8 of Sect.42.

Now let a norming cell $\mathcal{B}$, a compact subset $\mathcal{K}$ of $\mathcal{D}$ and $\epsilon \in \mathbb{P}^{\times}$ be given. By Prop.6 of Sect.58 of Vol.I we can determine $\sigma \in \mathbb{P}^{\times}$ such that $\mathcal{K} + \sigma\overline{\mathcal{B}}$ is compact and included in $\mathcal{D}$. Hence we can apply the Uniform Continuity Theorem (Sect.58 of Vol.I) to conclude that $f|_{\mathcal{K}+\sigma\overline{\mathcal{B}}}$ is uniformly continuous. Therefore, we can determine $\delta \in]0, \sigma]$ such that

$$0 \ \leq \ \sup f_{>}(x + s\mathcal{B}) - \inf f_{>}(x + s\mathcal{B}) \ \leq \ \epsilon \tag{49.6}$$

for all $x \in \mathcal{K}$ and all $s \in]0, \delta[$. On the other hand, by Prop.5 of Sect.42, we have

$$\inf f_{>}(x + s\mathcal{B}) \ \leq \ \frac{F(x + s\mathcal{B})}{\operatorname{vol}(x + s\mathcal{B})} \ \leq \ \sup f_{>}(x + s\mathcal{B}) \tag{49.7}$$

for all $x \in \mathcal{K}$ and all $s \in \mathbb{P}^{\times}$ such that $x + s\mathcal{B} \in \operatorname{Bnb}\mathcal{D}$. Since $x + s\mathcal{B} \in \operatorname{Bnb}\mathcal{D}$ for every $x \in \mathcal{K}$ and $s \in]0, \delta]$, it follows from (49.6) and (49.7) that (49.4) holds for all $x \in \mathcal{K}$ and $s \in]0, \delta]$. ∎

The following is a converse of Prop.3 for the case when $f \geq 0$.

Density Theorem. *Let $F : \text{Bnb}\,\mathcal{D} \longrightarrow \boldsymbol{R}$ be a set function that is additive, compactly differentiable, and positive, and denote its density by f. Then $f \geq 0$ and*

$$F(\mathcal{A}) = \int_{\mathcal{A}} f \quad \text{for all} \quad \mathcal{A} \in \text{Bnb}\,\mathcal{D}\ . \tag{49.8}$$

The proof will be based on several lemmas. We choose a point $q \in \mathcal{E}$ and a basis $\mathbf{b}$ of $\mathcal{V}$ and we consider tiles and pavings as in Sect.43. We may assume that tiles of scale 1 have volume 1, and hence that tiles of scale $s \in \mathbb{P}^{\times}$ have volume s^n. We use the abbreviation $\mathcal{B} := \text{Box}(\mathbf{b})$, so that

$$\text{vol}\,(x + \frac{t}{2}\mathcal{B}) = t^n \quad \text{for all} \quad x \in \mathcal{E},\ t \in \mathbb{P}^{\times}\ . \tag{49.9}$$

We assume now that $\mathcal{A} \in \text{Bnb}\,\mathcal{D}$ is given.

Lemma 1. *There is a $\sigma \in \mathbb{P}^{\times}$ such that*

$$\mathcal{K} := \text{Clo}\,\mathcal{A} + \sigma\overline{\mathcal{B}} \tag{49.10}$$

is included in $\mathcal{D}$ and compact. Moreover, given a tile $\mathcal{J}$ of a scale $s \in\ ]0, \sigma]$ such that $\text{Clo}\,\mathcal{J} \cup \text{Clo}\,\mathcal{A} \neq \emptyset$, we have $\text{Clo}\,\mathcal{J} \subset \mathcal{K}$ and hence $\mathcal{J} \in \text{Bnb}\,\mathcal{D}$.

Proof: Since $\text{Clo}\,\mathcal{A}$ is compact, the existence of a $\sigma \in \mathbb{P}^{\times}$ such that $\mathcal{K}$, as defined by (49.10), is included in $\mathcal{D}$ and compact follows from Prop.6 of Sect.58 of Vol.I. Now, if $\mathcal{J}$ is a tile of scale $s \in]0, \sigma[$ such that $\text{Clo}\,\mathcal{J} \cap \text{Clo}\,\mathcal{A} \neq \emptyset$ we may choose $x \in \text{Clo}\,\mathcal{J} \cap \text{Clo}\,\mathcal{A}$. It is easily seen that $\text{Clo}\,\mathcal{J} \subset x + s\overline{\mathcal{B}}$ and hence, by (49.10), that $\text{Clo}\,\mathcal{J} \subset \mathcal{K} \subset \mathcal{D}$, which shows that $\mathcal{J} \in \text{Bnb}\,\mathcal{D}$. ∎

We now choose $\sigma \in \mathbb{P}^{\times}$ according to Lemma 1 and define $\mathcal{K} \in \text{Bnb}\,\mathcal{D}$ by (49.10). Moreover we assume that a positive, isotone, and compactly differentiable set function $H : \text{Bnb}\,\mathcal{D} \longrightarrow \mathbf{R}$ is given. We denote the density of H by $h : \mathcal{D} \longrightarrow \mathbf{R}$. Since H is positIve, we have $h \geq 0$.

Lemma 2. *For every $\epsilon \in \mathbb{P}^{\times}$ there is a $\delta \in\ ]0, \sigma]$ with the following property: for each tile $\mathcal{J}$ whose scale s is in the interval $]0, \delta[$ and which satisfies $\text{Clo}\,\mathcal{J} \cap \text{Clo}\,\mathcal{A} \neq \emptyset$, we have $\mathcal{J} \in \text{Bnb}\,\mathcal{D}$ and*

$$\Big|\frac{H(\mathcal{J})}{s^n} - h(x)\Big| \leq \epsilon \quad \text{when} \quad x \quad \text{is the center of} \quad \text{Int}\mathcal{J}\ . \tag{49.11}$$

Proof: Since $\mathcal{K}$ is compact by Lemma 1, it follows from (d) of Def.1 that we may choose $\delta \in]0, \sigma[$ such that

$$\Big|\frac{H(x + \frac{t}{2}\mathcal{B})}{\text{vol}\,(x + \frac{t}{2}\mathcal{B})} - h(x)\Big| < \epsilon \tag{49.12}$$

for all $x \in \mathcal{K}$ and all $t \in\ ]0, \delta]$ such that $(x + \frac{t}{2}\mathcal{B}) \in \text{Bnb}\,\mathcal{D}$. Let a tile $\mathcal{J}$ of a scale $s \in\ ]0, \sigma]$ such that $\text{Clo}\,\mathcal{J} \cap \text{Clo}\,\mathcal{A} \neq \emptyset$ be given and denote the center of the box $\text{Int}\mathcal{J}$ by x. Then $\text{Int}\mathcal{J} = x + \frac{s}{2}\mathcal{B}$ and $\text{Clo}\,\mathcal{J} = x + \frac{s}{2}\overline{\mathcal{B}}$ and hence

$$x + \frac{s}{2}\mathcal{B} \subset x + \frac{s}{2}\overline{\mathcal{B}} \subset x + \frac{t}{2}\mathcal{B} \quad \text{for all} \quad t \in\ ]s, \delta]\ . \tag{49.13}$$

Now let $t \in]s,\delta]$ be given. Then we can determine exactly one tile $\mathcal{J}'$ of scale t such that $\mathrm{Int}\mathcal{J}' = x + \frac{t}{2}\mathcal{B}$. Since $\mathcal{J} \subset \mathcal{J}'$, we have $\mathrm{Clo}\,\mathcal{J}' \cap \mathrm{Clo}\,\mathcal{A} \neq \emptyset$. It follows from Lemma 1 that $\mathrm{Clo}\,\mathcal{J}' \subset \mathcal{K}$. We conclude that $x \in \mathcal{K}$ and $x + \frac{t}{2}\mathcal{B} \in \mathrm{Bnb}\,\mathcal{D}$ and hence that (49.12) can be applied. Since H is isotone, it follows from (49.13) that

$$H(x + \frac{s}{2}\mathcal{B}) \leq H(\mathcal{J}) \leq H(x + \frac{t}{2}\mathcal{B})$$

and hence

$$H(x + \frac{s}{2}\mathcal{B}) - h(x)t^n \leq H(\mathcal{J}) - h(x)t^n \leq H(x + \frac{t}{2}\mathcal{B}) - h(x)t^n \ .$$

Using (49.9) and (49.12), we conclude that

$$H(x + \frac{s}{2}\mathcal{B}) - h(x)t^n \leq H(\mathcal{J}) - h(x)t^n < \epsilon t^n \ .$$

Since $t \in]s,\delta[$ was arbitrary and since ι^n is continuous, it follows that

$$H(x + \frac{s}{2}\mathcal{B}) - h(x)s^n \leq H(\mathcal{J}) - h(x)s^n \leq \epsilon s^n \ .$$

Using (49.9) and (49.12) once more, with t replaced by s, we conclude that

$$-\epsilon s^n \leq H(\mathcal{J}) - h(x)s^n < \epsilon s^n \ ,$$

which proves (49.11). ∎

The final Lemma refers to pavings Π_s as defined by Def.1 of Sect.43.

Lemma 3. *For every $\epsilon \in \mathbb{P}^\times$, there is a $\delta' \in \mathbb{P}^\times$ such that*

$$\sum_{\mathcal{J} \in \Gamma_s^{\mathrm{Clo}}(\mathrm{Bdy}\,\mathcal{A})} H(\mathcal{A} \cap \mathcal{J}) \leq \epsilon \quad \text{for all } s \in]0,\delta'] \ , \tag{49.14}$$

where $\Gamma_s^{\mathrm{Clo}}(\mathrm{Bdy}\,\mathcal{A})$ is defined in accord with (43.6).

Proof: Since the density h of H is continuous by (d) of Def.1 and since $\mathcal{K}$, as defined by (49.10), is compact by Lemma 1, we can apply the Compact Image Theorem (see Sect.58 of Vol.I) to conclude that $h_>(\mathcal{K})$ is bounded. Hence we may choose $b \in 1 + \mathbb{P}^\times$ such that $0 \leq h(x) \leq b - 1$ for all $x \in \mathcal{K}$. We now determine $\delta \in]0,\sigma]$ according to Lemma 2 with the choice $\epsilon := 1$. Let $s \in]0,\delta[$ be given. In view of the definition (43.6), it follows from Lemma 2 that

$$H(\mathcal{J}) \leq (h(x) + 1)s^n \leq bs^n$$

holds for every tile $\mathcal{J} \in \Gamma_s^{\mathrm{Clo}}(\mathrm{Bdy}\,\mathcal{A})$ when x denotes the center of the box $\mathrm{Int}\mathcal{J}$, which is included in $\mathcal{K}$ by Lemma 1. Since H is isotone, we conclude that $H(\mathcal{J} \cap \mathcal{A}) \leq H(\mathcal{J}) \leq bs^n$ for all $\mathcal{J} \in \Gamma_s^{\mathrm{Clo}}(\mathrm{Bdy}\,\mathcal{A})$ and hence that

$$\sum_{\mathcal{J} \in \Gamma_s^{\mathrm{Clo}}(\mathrm{Bdy}\,\mathcal{A})} H(\mathcal{A} \cap \mathcal{J}) \leq bs^n(\#\Gamma_s^{\mathrm{Clo}}(\mathrm{Bdy}\,\mathcal{A})) \ . \tag{49.15}$$

Now let $\epsilon \in \mathbb{P}^\times$ be given. Since $\text{Bdy}\,\mathcal{A}$ is negligible, we can apply Prop.3 of Sect.43 with $\mathcal{S} := \text{Bdy}\,\mathcal{A}$ and determine $\delta' \in]0, \delta]$ such that $\#\Gamma_s^{\text{Clo}}(\text{Bdy}\,\mathcal{A}) \leq \frac{\epsilon}{bs^n}$ when $s \in]0, \delta']$. This result and (49.15) yield the desired conclusion (49.14). ∎

Proof of the Density Theorem: We define $F' : \text{Bnb}\,\mathcal{D} \longrightarrow \mathbb{R}$ by

$$F'(\mathcal{A}) := \int_{\mathcal{A}} f \qquad \text{for all } \mathcal{A} \in \text{Bnb}\,\mathcal{D} \ . \tag{49.16}$$

It follows from Prop.3 that F' is additive and compactly differentiable and that f is the density of F'. Since $f \geq 0$, it is clear that F' is positive and hence, by Prop.2, isotone.

Now let $\mathcal{A} \in \text{Bnb}\,\mathcal{D}$ and $s \in \mathbb{P}^\times$ be given. We note that $\Gamma_s^{\text{Clo}}(\text{Bdy}\,\mathcal{A})$ and $\Delta_s^{\text{Clo}}(\text{Int}\mathcal{A})$, as defined according to (43.6) and (43.4), are disjoint finite subsets of Π_s and hence that $\Delta_s^{\text{Clo}}(\text{Int}\mathcal{A}) \cup \{\mathcal{A} \cap \mathcal{J} \mid \mathcal{J} \in \Gamma_s^{\text{Clo}}(\text{Bdy}\,\mathcal{A})\}$ is a finite partition of $\mathcal{A}$. Since both F and F' are additive, it follows that

$$\begin{aligned} F'(\mathcal{A}) - F(\mathcal{A}) = &\sum_{\mathcal{J} \in \Delta_s^{\text{Clo}}(\text{Int}\mathcal{A})} (F'(\mathcal{J}) - F(\mathcal{J})) \ + \\ &\sum_{\mathcal{J} \in \Gamma_s^{\text{Clo}}(\text{Bdy}\,\mathcal{A})} (F'(\mathcal{A} \cap \mathcal{J}) - F(\mathcal{A} \cap \mathcal{J})) \end{aligned} \ . \tag{49.18}$$

Now let $\epsilon \in \mathbb{P}^\times$ be given. We determine $\sigma \in \mathbb{P}^\times$ according to Lemma 1. By Lemmas 2 and 3, we can then determine $\delta \in]0, \sigma]$ such that

(i) (49.11) holds with $h := f$ for $H := F$ and also for $H := F'$ whenever $s \in]0, \delta]$ and $\mathcal{J} \in \Delta_s^{\text{Clo}}(\text{Int}\mathcal{A})$.

(ii) (49.14) holds for $H := F$ and also for $H := F'$ whenever $s \in]0, \delta]$.

Now let $s \in]0, \delta]$ be given. Using (49.11) as indicated in (i), we find that

$$|F'(\mathcal{J}) - F(\mathcal{J})| \leq 2s^n\epsilon \quad \text{for all } \mathcal{J} \in \Delta_s^{\text{Clo}}(\text{Int}\mathcal{A}) \ . \tag{49.19}$$

Using (49.14) as indicated in (ii), we obtain

$$\Bigg| \sum_{\mathcal{J} \in \Gamma_s^{\text{Clo}}(\text{Bdy}\,\mathcal{A})} (F'(\mathcal{A} \cap \mathcal{J}) - F(\mathcal{A} \cap \mathcal{J})) \Bigg| \leq 2\epsilon \ . \tag{49.20}$$

Using (49.19) and (49.20), it follows from (49.18) that

$$|F'(\mathcal{A}) - F(\mathcal{A})| \leq 2s^n\epsilon\ (\#\Delta_s^{\text{Clo}}(\text{Int}\mathcal{A})) + 2\epsilon \ . \tag{49.21}$$

Since $\text{vol}\,\mathcal{J} = s^n$ for all $\mathcal{J} \in \Pi_s$, since $\text{vol} : \text{Bnb}\,\mathcal{E} \longrightarrow \mathbb{R}$ is additive and isotone, and since $\bigcup \Delta_s^{\text{Clo}}(\text{Int}\mathcal{A}) \subset \text{Int}\mathcal{A} \subset \mathcal{A}$ by (43.4), we have

$$s^n(\#\Delta_s^{\text{Clo}}(\text{Int}\mathcal{A})) = \text{vol}\left(\bigcup \Delta_s^{\text{Clo}}(\text{Int}\mathcal{A})\right) \leq \text{vol}\,(\mathcal{A})$$

and hence, by (49.21)

$$|F'(\mathcal{A}) - F(\mathcal{A})| \leq \epsilon \,(\mathrm{vol}\,(\mathcal{A}) + 1) \ .$$

Since $\epsilon \in \mathbb{P}^\times$ and $\mathcal{A} \in \mathrm{Bnb}\,\mathcal{D}$ were arbitrary, we conclude that $F' = F$ and hence, by (49.16), that (49.8) is valid. ∎

410. Transformation of Integrals.

In this section, we assume again that a flat space $\mathcal{E}$ is given. We denote its translation space by $\mathcal{V}$ and we put $n := \dim \mathcal{E} = \dim \mathcal{V}$. We also assume that an integral on $\mathcal{E}$ is given and we use the notation (42.11) for it. We also make use of the integral induced on $\mathcal{V}$ and the corresponding volume-function $\mathrm{vol}_{\mathcal{V}}$.

The following result shows how integrals transform under flat automorphisms of $\mathcal{E}$. It states how the factor c of Prop.10 of Sect.42 can be computed from the automorphism.

Proposition 1. *Let a flat automorphism α of $\mathcal{E}$ be given. Then*

$$\int_{\mathcal{E}} f \circ \alpha = \frac{1}{|\det \nabla\alpha|} \int_{\mathcal{E}} f \quad \text{for all} \quad f \in \mathrm{Bbac}\mathcal{E} \tag{410.1}$$

and

$$\mathrm{vol}\,(\alpha_{>}(\mathcal{A})) = |\det \nabla\alpha| \mathrm{vol}\,(\mathcal{A}) \quad \text{for all} \quad \mathcal{A} \in \mathrm{Bnb}\,\mathcal{E} \ . \tag{410.2}$$

The proof will be based on the following:

Lemma 1. *Let $\boldsymbol{\lambda} \in \mathcal{V}^*$ be given and put $\mathcal{W} := \{\boldsymbol{\lambda}\}^\perp$. Also, let $\mathcal{C} \in \mathrm{Bnb}\,\mathcal{W}$ be given and define $\boldsymbol{\sigma} : \mathcal{V} \longrightarrow \boldsymbol{R}$ by*

$$\boldsymbol{\sigma}(\mathbf{v}) := \mathrm{sgn}(\boldsymbol{\lambda}\mathbf{v}) \mathrm{vol}_{\mathcal{V}}(\mathcal{C} + [-1,1]\mathbf{v}) \quad \text{for all} \quad \mathbf{v} \in \mathcal{V} \ . \tag{410.3}$$

Then $\boldsymbol{\sigma}$ is linear and proportional to $\boldsymbol{\lambda}$, i.e., we have $\boldsymbol{\sigma} \in \mathcal{V}^*$ and $\boldsymbol{\sigma} = \rho\boldsymbol{\lambda}$ for some $\rho \in \boldsymbol{R}$.

Proof: The assertion is trivially valid when $\boldsymbol{\lambda} := \mathbf{0}$. Hence we may assume that $\boldsymbol{\lambda} \neq \mathbf{0}$ and choose $\mathbf{e} \in \mathcal{E}$ such that $\boldsymbol{\lambda}\mathbf{e} = 1$. By the Theorem on Existence and Uniqueness of Integral of Sect.42, there is exectly one integral on $\mathcal{W}$ such that the corresponding volume-function $\mathrm{vol}_{\mathcal{W}}$ satisfies $2\mathrm{vol}_{\mathcal{W}}(\mathcal{C}) = \mathrm{vol}_{\mathcal{V}}(\mathcal{C}{+}[-1,1]\mathbf{e})$. Using the isomorphism $\gamma : \mathbf{R} \times \mathcal{W} \longrightarrow \mathcal{V}$ defined by

$$\gamma(\xi, \mathbf{w}) := \mathbf{w} + \xi\mathbf{e} \quad \text{for all} \quad \mathbf{w} \in \mathcal{W}, \quad \xi \in \mathbf{R} \ , \tag{410.4}$$

we easily conclude from (46.7) that

$$2s \ \mathrm{vol}_{\mathcal{W}}(\mathcal{C}) = \mathrm{vol}_{\mathcal{V}}(\mathcal{C} + [-s,s]\mathbf{e}) \quad \text{for all} \quad s \in \mathbb{P} \ . \tag{410.5}$$

We caim that

$$\boldsymbol{\sigma}(\mathbf{v}) = 2\mathrm{vol}_{\mathcal{W}}(\mathcal{C})(\boldsymbol{\lambda}\mathbf{v}) \quad \text{for all} \quad \mathbf{v} \in \mathcal{V} \tag{410.6}$$

and hence that $\boldsymbol{\sigma} = \rho\boldsymbol{\lambda}$ holds when $\rho := 2\mathrm{vol}_{\mathcal{W}}(\mathcal{C})$. If $\mathbf{v} := \mathbf{0}$ then (410.6) is a trivial consequence of (410.3) because $\mathrm{sgn}(0) = 0$. Let $\mathbf{v} \in \mathcal{V}^{\times}$ be given and put $\mathbf{w} := \mathbf{v} - (\boldsymbol{\lambda}\mathbf{v})\mathbf{e}$. It is easilily seen that $\mathbf{w}$ belongs to $\mathcal{W}$, so that (410.4) gives $\gamma((\boldsymbol{\lambda}\mathbf{v}), \mathbf{w}) = \mathbf{v}$. We then apply the Corollary to Prop.2 of Sect.46 to the case when $\mathcal{E}_1 := \mathbf{R}$, $\mathcal{E}_2 := \mathcal{W}$, $\mathcal{A} := [-|\boldsymbol{\lambda}\mathbf{v}|,\ |\boldsymbol{\lambda}\mathbf{v}|] \times \mathcal{C}$, and $\mathcal{A}' := \bigcup\Big\{\{\xi\} \times (\mathcal{C} + \frac{\xi}{|\boldsymbol{\lambda}\mathbf{v}|}\mathbf{w}) \ \Big|\ \xi \in [-|\boldsymbol{\lambda}\mathbf{v}|,\ |\boldsymbol{\lambda}\mathbf{v}|]\Big\}$ (see Fig.1).

Since $\mathcal{A}'_{(\xi,\cdot)} = \mathcal{C} + \frac{\xi}{|\boldsymbol{\lambda}\mathbf{v}|}\mathbf{w}$ differs from $\mathcal{A}_{(\xi,\cdot)} = \mathcal{C}$ only by a translation in $\mathcal{W}$, we conclude that $\mathrm{vol}_{\mathbf{R}\times\mathcal{W}}(\mathcal{A}) = \mathrm{vol}_{\mathbf{R}\times\mathcal{W}}(\mathcal{A}')$ and hence that $\mathrm{vol}_{\mathcal{V}}(\gamma_{>}(\mathcal{A})) = \mathrm{vol}_{\mathcal{V}}(\gamma_{>}(\mathcal{A}'))$, i.e.,

$$\mathrm{vol}_{\mathcal{V}}(\mathcal{C} + [-|\boldsymbol{\lambda}\mathbf{v}|,\ |\boldsymbol{\lambda}\mathbf{v}|]\ \mathbf{e}) = \mathrm{vol}_{\mathcal{C}}(\mathcal{C} + [-1,1]\ \mathbf{v})\ . \tag{410.7}$$

Using (410.5) with $s := |\boldsymbol{\lambda}\mathbf{v}|$ and the fact that $\boldsymbol{\lambda}\mathbf{v} = \mathrm{sgn}\,(\boldsymbol{\lambda}\mathbf{v})|\boldsymbol{\lambda}\mathbf{v}|$, it follows from (410.7) and (410.3) that (410.6) is valid. ∎

Proof of Prop.1: In view of Prop.10 of Sect.42, it suffices to show that (410.2) holds for a particular set $\mathcal{A} \in \mathrm{Bnb}\,\mathcal{E}$ of non-zero volume. We choose a point $q \in \mathcal{E}$ and a basis $\mathbf{b} := (\mathbf{b}_i \mid i \in n^{]})$ of $\mathcal{V}$ and claim that (410.2) holds when $\mathcal{A} := q + \overline{\mathrm{Box}}\,(\mathbf{b})$. We put $\boldsymbol{\epsilon} := \wedge\mathbf{b}^* \in \mathrm{Skew}_n(\mathcal{V}^n, \mathbf{R})$ and define $\boldsymbol{\omega} : \mathcal{V}^n \longrightarrow \mathbf{R}$ by

$$\boldsymbol{\omega}(\mathbf{w}) := \mathrm{sgn}(\boldsymbol{\epsilon}(\mathbf{w}))\ \mathrm{vol}_{\mathcal{V}}\Big(\sum_{i\in n^{]}}[-1,1]\mathbf{w}_i\Big) \quad \text{for all}\ \ \mathbf{w} \in \mathcal{V}^n\ . \tag{410.8}$$

Let $j \in n^{]}$ and $\mathbf{w} \in \mathcal{W}^n$ be given. Applying Lemma 1 to the case when $\boldsymbol{\lambda} := \boldsymbol{\epsilon}(\mathbf{w}.j)$ and $\mathcal{C} := \sum([-1,1]\mathbf{w}_i \mid i \in n^{]}\backslash\{j\})$, we infer from (410.8) that $\boldsymbol{\omega}(\mathbf{w}.j)$ is linear. Since $j \in n^{]}$ and $\mathbf{w} \in \mathcal{W}^n$ were arbitrary, it follows that $\boldsymbol{\omega}$ is multilinear. Since $\boldsymbol{\epsilon}$ is skew, it is evident from (410.8) that $\boldsymbol{\omega}$ is also skew, so that $\boldsymbol{\omega} \in \mathrm{Skew}_n(\mathcal{V}^n, \mathbf{R})$.

Since $\boldsymbol{\epsilon}(\mathbf{b}) = 1$ and $\overline{\mathrm{Box}}\,(\mathbf{b}) = \sum([-1,1]\mathbf{b}_i \mid i \in n^{]})$, it follows from (410.8) that

$$\mathrm{vol}\,(q + \overline{\mathrm{Box}}\,(\mathbf{b})) = \mathrm{vol}_{\mathcal{V}}(\overline{\mathrm{Box}}\,(\mathbf{b})) = \boldsymbol{\omega}(\mathbf{b})\ . \tag{410.9}$$

Since $\alpha : \mathcal{E} \longrightarrow \mathcal{E}$ is flat, we have

$$\alpha_{>}(q + \overline{\mathrm{Box}}\,(\mathbf{b})) = \alpha(q) + (\nabla\alpha)_{>}(\overline{\mathrm{Box}}\,(\mathbf{b})) = \alpha(q) + \sum_{i\in n^{]}}[-1,1](\nabla\alpha)\mathbf{b}_i$$

and hence, using (410.8) again,

$$\mathrm{vol}\,(\alpha_{>}(q + \overline{\mathrm{Box}}\,(\mathbf{b}))) = \mathrm{vol}_{\mathcal{V}}\Big(\sum_{i\in n^{]}}[-1,1](\nabla\alpha)\mathbf{b}_i\Big) = |\boldsymbol{\omega}((\nabla\alpha)^{\times n}\mathbf{b})|\ . \tag{410.10}$$

Since $\boldsymbol{\omega}((\nabla\alpha)^{\times n}\mathbf{b}) = (\det(\nabla\alpha))\boldsymbol{\omega}(\mathbf{b})$ by the Theorem on Characterization of Determinants of Sect.14, the desired conclusion, namely

$$\mathrm{vol}\,(\alpha_{>}(q + \overline{\mathrm{Box}}\,(\mathbf{b}))) = |\det(\nabla\alpha)|\mathrm{vol}\,(q + \overline{\mathrm{Box}}\,(\mathbf{b})))\ ,$$

follows from (410.9) and (410.10). ▮

From now on we assume that, in addition to the flat space $\mathcal{E}$, a flat space $\mathcal{E}'$ with translation space $\mathcal{V}'$, and a C^1-diffeomorphism ϕ from an open subset $\mathcal{D}$ of $\mathcal{E}$ onto an open subset $\mathcal{D}'$ of $\mathcal{E}'$ are given. The following Lemma is the basis for the remaining results of this section.

Lemma 2. *For every norming cell $\mathcal{B}$ of $\mathcal{V}$, for every compact subset $\mathcal{K}$ of $\mathcal{D}$, and for every $\eta \in \mathbb{P}^\times$ there is a $\delta \in \mathbb{P}^\times$ such that $x + s\mathcal{B} \in \mathrm{Bnb}\,\mathcal{D}$ and*

$$\phi(x) + \frac{s}{1+\eta}(\nabla_x\phi)_>(\mathcal{B}) \subset \phi_>(x+s\mathcal{B}) \subset \phi(x) + (1+\eta)s(\nabla_x\phi)_>(\mathcal{B}) \quad (410.11)$$

for all $x \in \mathcal{K}$ and all $s \in\,]0, \delta]$.

Proof: Let a norming cell $\mathcal{B}$ of $\mathcal{V}$, a compact subset $\mathcal{K}$ of $\mathcal{D}$, and $\eta \in \mathbb{P}^\times$ be given. We put $\nu := \mathrm{no}_{\mathcal{B}}$ and choose a norm ν' on $\mathcal{V}'$. Since $\mathrm{Rng}\,\nabla\phi \subset \mathrm{Lis}\,(\mathcal{V}, \mathcal{V}')$ and since $\nabla\phi$ and ν' are continuous (see Prop.7 of Sect.56 of Vol.I), the function $\big((x, \mathbf{u}) \mapsto \nu'((\nabla_x\phi)\mathbf{u})\big) : \mathcal{K} \times \mathrm{Bdy}\,\mathcal{B} \longrightarrow \mathbb{P}^\times$ is continuous. Since $\mathcal{K} \times \mathrm{Bdy}\,\mathcal{B}$ is compact by Prop.4 of Sect.58 of Vol.I, it follows from the Theorem on Attainment of Extrema of Sect.58 of Vol.I that we can determine $\alpha, \beta \in \mathbb{P}^\times$ such that

$$\alpha \le \nu'((\nabla_x\phi)\mathbf{u}) \le \beta \quad \text{for all} \;\; x \in \mathcal{K}\;,\; \mathbf{u} \in \mathrm{Bdy}\,\mathcal{B}$$

and hence that

$$\alpha\nu(\mathbf{u}) \le \nu'((\nabla_x\phi)\mathbf{u}) \le \beta\nu(\mathbf{u}) \quad \text{for all} \;\; x \in \mathcal{K}\;,\; \mathbf{u} \in \mathcal{V}\;. \quad (410.12)$$

We now use Prop.4 of Sect 64 of Vol.I and determine $\delta_1 \in \mathbb{P}^\times$ such that $\mathcal{K} + \delta_1\overline{\mathcal{B}} \subset \mathcal{D}$ and such that, for each $x \in \mathcal{K}$, the mapping $\mathbf{n}_x : \delta_1\mathcal{B} \longrightarrow \mathcal{V}'$ defined by

$$\mathbf{n}_x(\mathbf{u}) := \phi(x + \mathbf{u}) - \phi(x) - (\nabla_x\phi)\mathbf{u} \quad \text{for all} \;\; \mathbf{u} \in \delta_1\mathcal{B} \quad (410.13)$$

satifies $\mathrm{str}(\mathbf{n}_x; \nu, \nu') \le \eta\alpha$. By (64.2) of Sect.64 of Vol.I, we then have

$$\nu'(\mathbf{n}_x(\mathbf{u})) \le \eta\alpha\nu(\mathbf{u}) \quad \text{for all} \;\; x \in \mathcal{K}\;,\; \mathbf{u} \in \delta_1\mathcal{B}\;. \quad (410.14)$$

Now let $s \in\,]0, \delta_1]$, $\mathbf{v} \in \mathcal{B}$, and $x \in \mathcal{K}$ be given. Using (410.14) with $\mathbf{u} := s\mathbf{v}$ and $(410.12)_1$ with $\mathbf{u} := (\nabla_x\phi)^{-1}\mathbf{n}_x(s\mathbf{v})$, we find that

$$\alpha\nu((\nabla_x\phi)^{-1}\mathbf{n}_x(s\mathbf{v})) \;\le\; \nu'(\mathbf{n}_x(s\mathbf{v})) \;\le\; \eta\alpha\nu(s\mathbf{v}) \;\le\; \eta\alpha s\nu(\mathbf{v})$$

and hence that $\nu((\nabla_x\phi)^{-1}\mathbf{n}_x(s\mathbf{v})) \le \eta s\nu(\mathbf{v})$, which is equivalent to $(\nabla_x\phi)^{-1}\mathbf{n}_x(s\mathbf{v}) \in \eta s\mathcal{B}$ and hence to

$$\mathbf{n}_x(s\mathbf{v}) \;\in\; \eta s(\nabla_x\phi)_>(\mathcal{B})\;. \quad (410.15)$$

In view of (410.13), we conclude that

$$\begin{aligned} \phi(x + s\mathbf{v}) \;&=\; \phi(x) + s(\nabla_x\phi)\mathbf{v} + \mathbf{n}_x(s\mathbf{v}) \\ &\in\; \phi(x) + s(\nabla_x\phi)_>(\mathcal{B}) + \eta s(\nabla_x\phi)_>(\mathcal{B})\;. \end{aligned} \quad (410.16)$$

Since $(\nabla_x \phi)_>(\mathcal{B})$ is a norming cell of $\mathcal{V}'$, we can use Prop.1 of Sect.51 of Vol.I to conclude from (410.16) that

$$\phi_>(x+s\mathcal{B}) \subset \phi(x)+s(1+\eta)(\nabla_x\phi)_>(\mathcal{B}) \quad \text{for all } x\in\mathcal{K}\ ,\ s\in\,]0,\delta_1]\ . \tag{410.17}$$

Noting that $\phi_>(\mathcal{K})$ is a compact subset of $\mathcal{D}'$ by the Compact Image Theorem of Sect.58 of Vol.I, we now apply Prop.4 of Sect.64 of Vol.I to $\phi^{\leftarrow}$ instead of ϕ and thus determine $\delta_2 \in \mathbb{P}^\times$ such that

$$\phi_>(\mathcal{B}) + \beta\delta_2\overline{\mathrm{Ce}}\,(\nu') \ \in\ \mathrm{Bnb}\,\mathcal{D}' \tag{410.18}$$

and such that, for each $y\in\phi_>(\mathcal{B})$, the mapping $\mathbf{m}_y : \beta\delta_2\mathrm{Ce}\,(\nu') \longrightarrow \mathcal{V}$ defined by

$$\mathbf{m}_y(\mathbf{w}) := \phi^{\leftarrow}(y+\mathbf{w}) - \phi^{\leftarrow}(y) - (\nabla_y\phi^{\leftarrow})\mathbf{w} \quad \text{for all } \mathbf{w}\in\beta\delta_2\mathrm{Ce}\,(\nu') \tag{410.19}$$

satisfies

$$\nu(\mathbf{m}_y(\mathbf{w})) \le \frac{\eta}{\beta}\nu'(\mathbf{w}) \quad \text{for all } y\in\phi_>(\mathcal{K})\ ,\ \mathbf{w}\in\beta\delta_2\mathrm{Ce}\,(\nu')\ . \tag{410.20}$$

Now let $t\in\,]0,\delta_2]$, $x\in\mathcal{K}$, and $\mathbf{v}\in\mathcal{B}$ be given. Using $(410.12)_2$ with the choice $\mathbf{u} := t\mathbf{v}$ and noting that $\mathbf{v}\in\mathcal{B}$ is equivalent to $\nu(\mathbf{v}) < 1$, we see that $\nu'(t(\nabla_x\phi)\mathbf{v}) \le \beta t\nu(\mathbf{v}) < \beta t$ and hence that $t(\nabla_x\phi)\mathbf{v} \in \beta t\mathrm{Ce}\,(\nu') \subset \beta\delta_2\mathrm{Ce}\,(\nu')$. Therefore, we may use (410.20) with the choices $\mathbf{w} := t(\nabla_x)\mathbf{v}$ and $y := \phi(x)$ to obtain

$$\nu\big(\mathbf{m}_{\phi(x)}(t(\nabla_x\phi)\mathbf{v})\big) \le \frac{\eta}{\beta}\nu'(t(\nabla_x\phi)\mathbf{v}) < \frac{\eta}{\beta}\beta t = \eta t\ ,$$

which is equivalent to

$$\mathbf{m}_{\phi(x)}(t(\nabla_x\phi)\mathbf{v}) \ \in\ \eta t\mathcal{B}\ . \tag{410.21}$$

In view of (410.19), it follows from (410.21) that

$$\begin{aligned}\phi^{\leftarrow}(\phi(x)+t(\nabla_x\phi)\mathbf{v}) \ &=\ \phi^{\leftarrow}(\phi(x)) + t(\nabla_{\phi(x)}\phi^{\leftarrow})(\nabla_x\phi)\mathbf{v} + \mathbf{m}_{\phi(x)}(t(\nabla_x\phi)\mathbf{v}) \\ &\in\ x+t\mathbf{v}+\eta t\mathcal{B} \ \subset\ x+t\mathcal{B}+\eta t\mathcal{B}\ .\end{aligned}$$

Using Prop.1 of Sect.51 of Vol.I again, we conclude that

$$\phi(x)+t(\nabla_x\phi)_>(\mathcal{B}) \subset \phi_>(x+(1+\eta)t\mathcal{B}) \quad \text{for all } x\in\mathcal{K}\ ,\ t\in\,]0,\delta_2]\ . \tag{410.22}$$

Using (410.22) with the choice $t := \frac{s}{1+\eta}$ and then (410.17) we see that the assertion of the Lemma is valid with $\delta := \min\{\delta_1, \frac{\delta_2}{1+\eta}\}$. ∎

Proposition 2. *The image under ϕ of every negligible set in* $\mathrm{Bnb}\,\mathcal{D}$ *is a negligible set in* $\mathrm{Bnb}\,\mathcal{D}'$.

Proof : We first choose norms ν and ν' on $\mathcal{V}$ and $\mathcal{V}'$, repectively and we put $\mathcal{B} := \mathrm{Ce}\,(\nu)$, $\mathcal{B}' := \mathrm{Ce}\,(\nu')$.

Now let $\mathcal{S} \in \mathrm{Bnb}\,\mathcal{D}$ be given and assume that it is negligible. By Prop.2 of Sect.41, the closure $\mathrm{Clo}\,\mathcal{S}$ is also negligible and by (49.1) we have $\mathrm{Clo}\,\mathcal{S} \subset \mathcal{D}$. Since $\mathrm{Clo}\,\mathcal{S}$ is closed and bounded and hence compact by the Compactness Theorem, it follows from Prop.4 of Sect.58 of Vol.I that $\mathrm{Clo}\,\mathcal{S} \times \overline{\mathcal{B}}$ is also compact. Since the mapping

$$\big((x, \mathbf{u}) \mapsto \nu'((\nabla_x\phi)\mathbf{u})\big) : \mathrm{Clo}\,\mathcal{S} \times \overline{\mathcal{B}} \longrightarrow \mathbf{R}$$

is continuous, we can apply the Theorem on Attainment of Extrema and determine $\beta \in \mathbb{P}^\times$ such that $\nu'((\nabla_x\phi)\mathbf{u}) < \beta$ for all $x \in \mathcal{S}$ and all $\mathbf{u} \in \mathcal{B}$, which is equivalent to

$$(\nabla_x\phi)_>(\mathcal{B}) \ \subset \ \beta\mathcal{B}' \quad \text{for all} \ \ x \in \mathcal{S} \ . \tag{410.23}$$

We now apply Lemma 2 with the choices $\mathcal{K} := \mathrm{Clo}\,\mathcal{S}$ and $\eta := 1$ and determine $\delta \in \mathbb{P}^\times$ such that $x + s\mathcal{B} \in \mathrm{Bnb}\,\mathcal{D}$ and

$$\phi_>(x + s\mathcal{B}) \ \subset \ \phi(x) + 2s(\nabla_x\phi)_>(\mathcal{B}) \quad \text{for all} \ \ x \in \mathcal{S} \ , \ \ s \in \,]0, \delta] \ . \tag{410.24}$$

Now let $\epsilon \in \mathbb{P}^\times$ be given. Since $\mathcal{S}$ is negligible we can apply Prop.1 of Sect.41 and determine a finite subset $\mathfrak{k}$ of $\mathcal{S}$ and a family $(\rho_p \mid p \in \mathfrak{k})$ in $\mathbb{P}^\times$ such that

$$\mathcal{S} \ \subset \ \bigcup_{p\in\mathfrak{k}}(p + \rho_p\mathcal{B}) \tag{410.25}$$

and

$$\sum_{p\in\mathfrak{k}} {\rho_p}^n \ \leq \ \min\{\delta^n, \frac{\epsilon}{(2\beta)^n}\} \ . \tag{410.26}$$

It follows from (410.26) that $\rho_p \in \,]0, \delta]$ for all $p \in \mathfrak{k}$. Hence, by (410.25), (410.24), and (410.23), we have

$$\begin{aligned} \phi_>(\mathcal{S}) \ \subset \ \bigcup_{p\in\mathfrak{k}} \phi_>(p + \rho_p\mathcal{B}) \ &\subset \ \bigcup_{p\in\mathfrak{k}} \big(\phi(p) + 2\rho_p(\nabla_p\phi)_>(\mathcal{B})\big) \\ &\subset \ \bigcup_{p\in\mathfrak{k}} (\phi(p) + 2\rho_p\beta\mathcal{B}') \ . \end{aligned} \tag{410.27}$$

Since (410.26) implies that

$$\sum_{p\in\mathfrak{k}} (2\rho_p\beta)^n = (2\beta)^n \sum_{p\in\mathfrak{k}} {\rho_p}^n \leq \epsilon \ ,$$

we conclude from (410.27) and Def.1 of Sect.41 that $\phi_>(\mathcal{S})$ is indeed negligible. ∎

Theorem on Transformation of Integrals. *Let a flat space $\mathcal{E}$ and a C^1-diffeomorphism ϕ from an open subset $\mathcal{D}$ of $\mathcal{E}$ onto an open subset $\mathcal{D}'$ of $\mathcal{E}$ be given. Then*

$$\phi_{>>}(\operatorname{Bnb}\mathcal{D}) \;\subset\; \operatorname{Bnb}\mathcal{D}' \tag{410.28}$$

and

$$\int_{\phi_>(\mathcal{A})} f \;=\; \int_{\mathcal{A}} (f\circ\phi)|\det\circ\nabla\phi| \quad \text{for all } \;\mathcal{A}\in\operatorname{Bnb}\mathcal{D} \tag{410.29}$$

and all continuous functions $f:\mathcal{D}'\longrightarrow \mathbf{R}$.

Proof: Let $\mathcal{A}\in\operatorname{Bnb}\mathcal{D}$ be given. Then $\operatorname{Clo}\mathcal{A}$ is a closed and bounded, and hence compact, subset of $\mathcal{D}$. By the Compact Image Theorem, $\operatorname{Clo}\phi_>(\mathcal{A}) = \phi_>(\operatorname{Clo}\mathcal{A})$ is a bounded subset of $\mathcal{D}'$. On the other hand, since $\operatorname{Bdy}\mathcal{A}$ is a negligible set belonging to $\operatorname{Bnb}\mathcal{D}$, it follows from Prop.2 that $\operatorname{Bdy}\phi_>(\mathcal{A}) = \phi_>(\operatorname{Bdy}\mathcal{A})$ is negligible. Therefore $\phi_>(\mathcal{A})$ belongs to $\operatorname{Bnb}\mathcal{D}'$. Since $\mathcal{A}\in\operatorname{Bnb}\mathcal{D}$ was arbitrary, (410.28) follows.

We note that it is sufficient to prove (410.29) when f is positive. Indeed, every continuous function f is the value-wise difference of two positive continuous functions, namely $f^+ := \frac{1}{2}(|f|+f)$ and $f^- := \frac{1}{2}(|f|-f)$. Hence, if (410.29) is known to be valid when f is replaced by f^+ and by f^-, it is clearly valid also for $f = f^+ - f^-$.

Now let a positive continuous function $f:\mathcal{D}'\longrightarrow\mathbf{R}$ be given and define the set function $G:\operatorname{Bnb}\mathcal{D}\longrightarrow\mathbf{R}$ by

$$G(\mathcal{A}) := \int_{\phi_>(\mathcal{A})} f \quad \text{for all } \;\mathcal{A}\in\operatorname{Bnb}\mathcal{D}\;. \tag{410.30}$$

It is evident that G is both additive and positive. We will show that G is compactly differentiable and that its density is

$$g := (f\circ\phi)|\det\circ\nabla\phi|\;. \tag{410.31}$$

By the Density Theorem of Sect.49 and by (410.30), it will then follow that (410.29) holds.

We assume that a compact subset $\mathcal{K}$ of $\mathcal{D}$ and a norming cell $\mathcal{B}$ of $\mathcal{V}$ are given. We may assume, without loss, that the given integral on $\mathcal{E}$ is such that $\operatorname{vol}_{\mathcal{A}}(\mathcal{B}) = 1$ and hence that

$$\operatorname{vol}(x+s\mathcal{B}) = s^n \quad \text{for all } \; s\in\mathbb{P}^\times\;,\; x\in\mathcal{E}\;. \tag{410.32}$$

We now assume that $\eta\in\mathbb{P}^\times$ is given and we determine $\delta\in\mathbb{P}^\times$ according to Lemma 2. Since volume is an isotone set-function, it then follows from (410.11), Prop.1, and (410.32) that

$$\left(\frac{1}{1+\eta}\right)^n|\det(\nabla_x\phi)| \;\le\; \frac{\operatorname{vol}\phi_>(x+s\mathcal{B})}{\operatorname{vol}(x+s\mathcal{B})} \;\le\; (1+\eta)^n|\det(\nabla_x\phi)| \tag{410.33}$$

for all $x \in \mathcal{K}$ and all $s \in]0, \delta]$. On the other hand, it follows from (410.30) and Prop.5 of Sect.42 that

$$\inf f_{>}(\phi_{>}(x+s\mathcal{B})) \;\leq\; \frac{G(x+s\mathcal{B})}{\operatorname{vol}\phi_{>}(x+s\mathcal{B})} \;\leq \sup f_{>}(\phi_{>}(x+s\mathcal{B})) \qquad (410.34)$$

for all $x \in \mathcal{K}$ and all $s \in]0, \delta]$. Combining (410.33) and (410.34), we find that

$$\begin{aligned}(\frac{1}{1+\eta})^n |\det(\nabla_x\phi)| \inf(f\circ\phi)_{>}(x+s\mathcal{B}) \;&\leq\; \frac{G(x+s\mathcal{B})}{\operatorname{vol}(x+s\mathcal{B})} \\ &\leq\; (1+\eta)^n |\det(\nabla_x\phi)| \sup(f\circ\phi)_{>}(x+s\mathcal{B})\ .\end{aligned} \qquad (410.35)$$

is valid for all $x \in \mathcal{K}$ and all $s \in]0, \delta]$. Now, since $x + s\mathcal{B} \subset \mathcal{K} + \delta\overline{\mathcal{B}}$ for all $x \in \mathcal{K}$ and all $s \in]0, \delta]$ and since $\mathcal{K} + \delta\overline{\mathcal{B}}$ is a compact subset of $\mathcal{D}$, it follows from the Uniform Continuity Theorem that we can determine $\delta' \in]0, \delta]$ such that

$$(f\circ\phi)(x) - \eta \leq \inf(f\circ\phi)_{>}(x+s\mathcal{B}) \leq \sup(f\circ\phi)_{>}(x+s\mathcal{B}) \leq (f\circ\phi)(x) + \eta$$

for all $x \in \mathcal{K}$ and all $s \in]0, \delta']$. Hence, by (410.35), we conclude that

$$\begin{aligned}(\frac{1}{1+\eta})^n |\det(\nabla_x\phi)|((f\circ\phi)(x)-\eta)) \;&\leq\; \frac{G(x+s\mathcal{B})}{\operatorname{vol}(x+s\mathcal{B})} \\ &\leq\; (1+\eta)^n |\det(\nabla_x\phi)|((f\circ\phi)(x)+\eta)\end{aligned} \qquad (410.36)$$

for all $x \in \mathcal{K}$ and all $s \in]0, \delta']$.

Now let $\epsilon \in \mathbb{P}^\times$ be given. The function

$$\big((\eta, x) \mapsto (1+\eta)^n |\det(\nabla_x\phi)|(f\circ\phi)(x)+\eta)\big) : [0,1]\times\mathcal{K} \longrightarrow \mathbb{P}^\times$$

is continuous and has the value $g(x)$, as defined by (410.31), at $(0, x)$ when $x \in \mathcal{K}$. Since $[0,1]\times\mathcal{K}$ is compact by Prop.4 of Sect.58 of Vol.I, it follows from the Uniform Continuity Theorem that, if η is sufficiently small, we have

$$(1+\eta)^n |\det(\nabla_x\phi)|(f\circ\phi)(x)+\eta) \;\leq g(x)+\epsilon \quad \text{for all} \;\; x \in \mathcal{K}\ . \qquad (410.37)$$

Similarly, if η is sufficiently small, we have

$$g(x)-\epsilon \leq \;(\frac{1}{1+\eta})^n |\det(\nabla_x\phi)|(f\circ\phi)(x)-\eta) \quad \text{for all} \;\; x \in \mathcal{K}\ . \qquad (410.38)$$

Therefore, if we choose $\eta \in \mathbb{P}^\times$ appropriately and determine δ' such that (410.36) holds for all $x \in \mathcal{K}$ and all $s \in]0, \delta']$, it follows from (410.37) and (410.38) that

$$g(x)-\epsilon \;\leq\; \frac{G(x+s\mathcal{B})}{\operatorname{vol}(x+s\mathcal{B})} \;\leq g(x)+\epsilon \quad \text{for all} \;\; x \in \mathcal{K}\ ,\;\; s \in]0, \delta']\ .$$

Since $\mathcal{K}$, $\mathcal{B}$, and ϵ were arbitrary, it follows that G is indeed compactly differentiable in the sense of Def.1 (d) of Sect.49 and that its density is the function g defined by (410.31). ∎

411. Transformation of Euclidean Integrals.

From now on we assume that Euclidean spaces $\mathcal{E}$ and $\mathcal{E}'$ with translation spaces $\mathcal{V}$ and $\mathcal{V}'$ are given. We assume that the spaces all have the same dimension, so that the set of Euclidean isomorphisms from $\mathcal{E}$ to $\mathcal{E}'$ is not empty. Let Igl$'$ be the Euclidean Integral on $\mathcal{E}'$ and let $\alpha : \mathcal{E}' \to \mathcal{E}$ be a Euclidean isomorphism. Define Igl: $\text{Bbac}\,(\mathcal{E}) \to \mathbf{R}$ by

$$\text{Igl}(f) := \text{Igl}'(f \circ \alpha) \qquad \text{for all} \;\; f \in \text{Bbac}\,(\mathcal{E}) \tag{411.1}$$

It follows very easily from the Characterization of the Euclidean Integral in Sect.48 that Igl is in fact the Euclidean integral on $\mathcal{E}$. In view of (411.1), (410.28) and Def.3 of Sect.42, we then have

$$\int_{\mathcal{A}} f = \int_{\alpha^{<}(\mathcal{A})} f \circ \alpha \quad \text{for all} \;\; \mathcal{A} \in \text{Bnb}\,\mathcal{E},\;\; f \in \text{Bbac}\,(\mathcal{E}). \tag{411.2}$$

Theorem on Transformation of Euclidean Integrals. *Let Euclidean spaces $\mathcal{E}$ and $\mathcal{E}'$ be given and let $\psi : \mathcal{D}' \to \mathcal{D}$ be a C^1- diffeomorphism from an open subset $\mathcal{D}'$ of $\mathcal{E}'$ to an open subset $\mathcal{D}$ of $\mathcal{E}$. We then have*

$$\int_{\mathcal{A}} f = \int_{\psi^{<}(\mathcal{A})} (f \circ \psi)\, \sqrt{\det \circ\, ((\nabla\psi)^{\top}(\nabla\psi)} \quad \text{for all} \;\; \mathcal{A} \in \text{Bnb}\,\mathcal{D} \tag{411.3}$$

and all continuous functions $f : \mathcal{D} \longrightarrow \mathbf{R}$.

Proof. We chose an isomorphism $\alpha : \mathcal{E} \to \mathcal{E}'$. Then $\bar{\mathcal{D}} := \alpha^{<}(\mathcal{D}')$ is an open subset of $\mathcal{E}$ and the adjustment $\alpha|_{\bar{\mathcal{D}}}^{\mathcal{D}'} : \bar{\mathcal{D}} \to \mathcal{D}'$ is invertible. From now on, we simply write α for this adjustment and put $\phi := \psi \circ \alpha : \bar{\mathcal{D}} \to \mathcal{D}$, which is is a C^1- diffeomorphism. Let a continuous functions $f : \mathcal{D} \to \mathbf{R}$ and $\mathcal{A} \in \text{Bnb}\,\mathcal{D}$ be given. We can apply the Theorem on Transformation of Integrals (410.20), with $\mathcal{D}$ replaced by $\bar{\mathcal{D}}$, $\mathcal{D}'$ replaced by $\mathcal{D}$, and $\mathcal{A}$ replaced by $\phi^{<}(\mathcal{A})$, obtaining

$$\int_{\mathcal{A}} f = \int_{\phi^{<}(\mathcal{A})} (f \circ \phi)\; |\det \circ \nabla\phi|\;. \tag{411.4}$$

Now let $\xi \in \mathcal{D}'$ and be given and put $x := \alpha^{\leftarrow}(\xi)$. Then, by the Chain Rule, we have

$$\nabla_x \phi = \nabla_x(\psi \circ \alpha) = \nabla_\xi \psi \nabla\alpha\;. \tag{411.5}$$

The gradient $\nabla\alpha$ of the Euclidean isomorphism α belongs to $\text{Orth}(\mathcal{V}, \mathcal{V}')$ and hence satisfies $\nabla\alpha\,(\nabla\alpha)^{\top} = 1_{\mathcal{V}'}$ (see Sect.43 of Vol.I). Thus, it follows from (411.5) that

$$\nabla_x\phi\,(\nabla_x\phi)^\top = \nabla_\xi\psi\nabla\alpha\,(\nabla_\xi\psi\nabla\alpha)^\top = \nabla_\xi\psi\,\nabla\alpha\,(\nabla\alpha)^\top(\nabla_\xi\psi)^\top = \nabla_\xi\psi\,(\nabla_\xi\psi)^\top\ .$$

Using the Basic Rules for the determinant in Sect.14, it follows that $|\det\nabla_x\phi\,|^2 = \det((\nabla_\xi\psi)^\top\nabla_\xi\psi)$ and hence, since $\xi\in\mathcal{D}'$ was arbitrary, that

$$|\det\nabla\phi\,| = \sqrt{\det\circ((\nabla\psi)^\top\nabla\psi)}\circ\alpha \qquad (411.6)$$

Using (411.2), the fact $\phi^< = \alpha^<\circ\psi^<$, and (411.6), we see that the desired result (411.3) follows from (411.4). ∎

The Theorem just proved can be used to evaluate specific integrals by using an appropriate coordinate system. Let a Euclidean space $\mathcal{E}$ and an open subset $\mathcal{D}$ of $\mathcal{E}$ be given. Choose a coordinate system Γ on $\mathcal{D}$ as defined and discussed in Sect.71 of Vol.I. According to Prop.1 of Sect.71 in Vol.I, $\bar{\mathcal{D}} := \operatorname{Rng}\Gamma$ is then an open subset of $\mathbf{R}^\Gamma$ and $\psi := (\Gamma^{\mathrm{Rng}})^{\leftarrow} : \bar{\mathcal{D}}\to\mathcal{D}$ is a C^2-diffeomorphism.

Corollary. *Let G be the matrix of the inner-product components of the coordinate system Γ and put*

$$g := \sqrt{|\det\circ G|} : \mathcal{D}\to\boldsymbol{R} \qquad (411.7)$$

Then

$$\int_{\mathcal{A}} f = \int_{\Gamma_>(\mathcal{A})}(fg)\circ\psi \quad \text{for all } \ \mathcal{A}\in\operatorname{Bnb}\mathcal{D} \qquad (411.8)$$

and all continuous functions $f:\mathcal{D}\longrightarrow\mathbf{R}$.

Proof: Let $x\in\mathcal{D}$ be given and put $\xi := \Gamma(x) = (c(x)\mid c\in\Gamma)$, so that $x=\psi(\xi)$. By (71.2) and (71.4) if Vol.I, we have

$$(\nabla_\xi\psi)\delta_c^\Gamma = \psi^{,c}(\xi) = \mathbf{b}_c(x) \quad \text{for all } \ c\in\Gamma \qquad (411.9)$$

and hence

$$\delta_c^\Gamma\cdot(\nabla_\xi\psi)^\top(\nabla_\xi\psi)\delta_d^\Gamma = (\nabla_\xi\psi)\delta_c^\Gamma\cdot(\nabla_\xi\psi)\delta_d^\Gamma = \mathbf{b}_c(x)\cdot\mathbf{b}_d(x)$$

for all $c,d\in\Gamma$. Using the identification $\mathbf{R}^{\Gamma\times\Gamma}\cong\operatorname{Lin}(\mathbf{R}^\Gamma)$ (see (16.5) of Vol.I), we conclude, by (73.1) of Vol.I, that

$$(\nabla_\xi\psi)^\top(\nabla_\xi\psi) = G(x)\ . \qquad (411.10)$$

Since $x\in\mathcal{D}$ was arbitrary and $x=\psi(\xi)$ it follows from (411.7), (411.10), and (411.3) with $\mathcal{D}$ replaced by $\bar{\mathcal{D}}$, that the desired result (411.8) is valid. ∎

Example. We assume that $\mathcal{E}$ is a genuite Euclidean space of dimension 2 and use polar coordinates as in (B) of Sect.74 of Vol.I. To do so, we must prescribe a point $q\in\mathcal{E}$ and a unit vector $\mathbf{e}\in\mathcal{V}$. Define $r:\mathcal{E}\ddot{\mathbb{P}}$ by

$$r(x) := |x-q| \quad \text{for all } \ x\in\mathcal{E}\ , \qquad (411.11)$$

which means that $r(x)$ is the distance from x to q.

Consider a circular disc $\mathcal{C}_R$ of radius $R \in \mathbb{P}^\times$ centered at q and let a continuous function $h : [0, R] \to \mathbb{R}$ be given. By (74.14) of Vol.I, we have $g = r$. Since $\mathcal{C}_R$ differs from $\mathcal{A} := \mathcal{C}_R \backslash (q + \mathbb{P}\,\mathbf{e})$ only by a negligible set, we can apply (411.8) and obtain

$$\int_{\mathcal{C}_R} h \circ r = \int_{\Gamma(\mathcal{A})} r\,(h \circ r) \circ \psi, \qquad (411.12)$$

Where ψ is defined as in Sect.74 of Vol.I. We have $\Gamma(\mathcal{A}) =]0, R[\times]0, 2\pi[$. Therefore, using the Iterated Integral Theorem in the form (46.12) of Vol.I, it follows from (411.12) that

$$\int_{\mathcal{C}_R} h \circ r = \int_0^R \Big(\int_0^{2\pi} \mathrm{d}t\Big) s h(s) \mathrm{d}s = 2\pi \int_0^R s h(s) \mathrm{d}s\ . \qquad (411.13)$$

In the particular case when $h := \exp \circ\, \iota^{-2}$, using the fact that $h^\bullet = -2\iota h$ and the Fundamental Theorem of Calculus in the form (08.44) of Vol.I, we obtain

$$\int_{\mathcal{C}_R} exp \circ (-r^2) = \pi(1 - e^{-R^2})\ . \qquad (411.14)$$

In the limit $R \to \infty$ the right side of (411.14) becomes and extended integral in the sense of Def.3 of Sect.45 and we have

$$\int_{\mathcal{E}} exp \circ (-r^2) = \pi\ . \qquad (411.15)$$

Problems for Chapter 4

(1) Let a flat space $\mathcal{E}$ with translation space $\mathcal{V}$, an integral Igl on $\mathcal{E}$, and a linear space $\mathcal{W}$ be given. We say that a mapping $\mathbf{h} : \mathcal{E} \to \mathcal{W}$ belongs to $\mathrm{Bbac}\,(\mathcal{E}, \mathcal{W})$, if , for every $\boldsymbol{\mu} \in \mathcal{W}^*$, the function $\boldsymbol{\mu}\mathbf{h} : \mathcal{E} \to \mathbb{R}$ belongs to $\mathrm{Bbac}\,\mathcal{E}$.

(a) Show that there is exactly one mapping $\mathrm{Igl}^{\,\mathcal{W}} : \mathrm{Bbac}\,(\mathcal{E}.\mathcal{W}) \to \mathcal{W}$ such that

$$\mathrm{Igl}\,(\boldsymbol{\mu}\mathbf{h}) = \boldsymbol{\mu}\mathrm{Igl}^{\,\mathcal{W}}(\mathbf{h}) \quad \text{for all} \ \ \boldsymbol{\mu} \in \mathcal{W}^* \text{and all} \ \ \mathbf{h} \in \mathrm{Bbac}\,(\mathcal{E}.\mathcal{W})\ . \qquad (P4.1)$$

We generalize Def.3 of Sect.42 as follows: We assume that a subset $\mathcal{D} \in \mathrm{Nb}\,(\mathcal{E})$ and a mapping $\mathbf{h}$ with codomain $\mathcal{W}$ are given such that $\mathcal{D} \subset \mathrm{Dom}\,\mathbf{h} \subset \mathcal{E}$ and $\mathbf{h}|_{\mathcal{D}} \in \mathrm{Bbac}\,(\mathcal{E}, \mathcal{W})$. The **integral** of $\mathbf{h}$ over $\mathcal{D}$ is then defined by

$$\int_{\mathcal{D}} \mathbf{h}(x) \mathrm{d}x = \int_{\mathcal{D}} \mathbf{h} := \mathrm{Igl}^{\,\mathcal{W}} \overline{(\mathbf{h}|_{\mathcal{D}})}\ , \qquad (P4.2)$$

where

$$\overline{(\mathbf{h}|_{\mathcal{D}})}(x) := \begin{cases} \mathbf{h}(x) & \text{if } x \in \mathcal{D} \\ 0 & \text{if } x \in \mathcal{E}\backslash\mathcal{D} \end{cases} . \qquad (P4.3)$$

(b) Let ν be a norm on $\mathcal{W}$ according to Def.1 of Sect.5 of Vol.I. Prove that

$$\nu\Big(\int_{\mathcal{D}} \mathbf{h}\Big) \leq \int_{\mathcal{D}} (\nu \circ \mathbf{h}) \qquad (P4.4)$$

(This is a generalization of Prop.2 of Sect.610 of Vol.I.)

(2) Let a flat space $\mathcal{E}$ with translation space $\mathcal{V}$ and an integral Igl on $\mathcal{E}$ be given.

(a) Prove: For every $f \in \operatorname{Bbac}\mathcal{E}$ such that $\sigma := \int_{\mathcal{E}} f \neq 0$, there is exactly one point $b \in \mathcal{E}$ such that

$$\int_{\mathcal{E}} f(x)(x-b)\mathrm{d}x = \mathbf{0} \ , \qquad (P4.5)$$

where the integral is understood in the sense of (P4.2) with $\mathcal{W}$ replaced by $\mathcal{V}$.

(b) Show that b satifies

$$b = q + \frac{1}{\sigma}\int_{\mathcal{E}} f(x)(x-q)\mathrm{d}x \quad \text{for all} \ \ q \in \mathcal{E} \ . \qquad (P4.6)$$

The point b is called the **barycenter** of the function f.
In the case when $f := \mathrm{ch}_{\mathcal{A}}$ for a given $\mathcal{A} \in \operatorname{Bnb}\mathcal{E}$, we have $\int_{\mathcal{E}}(x-b)\mathrm{d}x = \mathbf{0}$ and b is then called the **centroid** of $\mathcal{A}$.

Note: The results and terminology of this problem are analogous to those of Sect.34 of Vol.I.

(3) Let a flat space $\mathcal{E}$ with translation space $\mathcal{V}$ and an integral Igl on $\mathcal{E}$ be given. Prove: There is an integral on $\mathcal{E}' := \mathcal{E} \times \mathbf{R}$ such that the corresponding volume function $\mathrm{vol}_{\mathcal{E}'}$ satisfies

$$\mathrm{vol}_{\mathcal{E}'}\{(x,s) \mid x \in \mathcal{E}, s \in [0, f(x)]\} = \overline{\mathrm{Igl}}\,(f) \qquad (P4.7)$$

for all integrable functions $f \in \operatorname{Pac}\mathcal{E}$. (See Def.2 in Sect.45.)

(4) Let a flat space $\mathcal{E}$ with translation space $\mathcal{V}$ and a hyperplane $\mathcal{F}$ in $\mathcal{E}$ be given. (See Sect.32 of Vol.I.) Let $\mathbf{e} \in \mathcal{V}$ be a non-zero vector that does not belong to the translation space of $\mathcal{F}$. Let $\mathcal{D} \in \operatorname{Bnb}\mathcal{E}$ be given and define, for each $z \in \mathbf{R}$, the *cross-section* $\mathcal{S}_z$ of $\mathcal{D}$ by

$$\mathcal{S}_z := (\mathcal{D} \cap (\mathcal{F} + z\mathbf{e})) - z\mathbf{e} \subset \mathcal{F} \ . \qquad (P4.8)$$

(a) Let $\mathrm{vol}_{\mathcal{F}}$ be the volume function associated with some integral on $\mathcal{F}$. (See Def.2 of Sect.42.) Show that there is a volume function $\mathrm{vol}_{\mathcal{E}}$ associated with a suitable integral on $\mathcal{E}$ such that

$$\mathrm{vol}_{\mathcal{E}}(\mathcal{D}) = \int_{\mathbf{R}} \mathrm{vol}_{\mathcal{F}}(\mathcal{S}_z)\mathrm{d}z \ , \qquad (P4.9)$$

provided that $\mathcal{D}$ satisfies suitable conditions. State these conditions precisely.

(b) Assume now the space $\mathcal{E}$ is a genuine Euclidean space and assume that $\mathbf{e}$ is a unit vector perpendicular to the hyperplane $\mathcal{F}$. Let $q \in \mathcal{F}$, $r \in \mathbb{P}^{\times}$, and $h \in \mathbb{P}^{\times}$ be given. Consider the *hyperboloidal solid*

$$\mathcal{D} := \{p + z\mathbf{e} \mid p \in \mathcal{F} \ , \ |p-q| \leq r \ , \ z \in [0, \, h\big(1 - \frac{|p-q|^2}{r^2}\big)\} \ , \qquad (P4.10)$$

whose *height* is h and whose *base* $\mathcal{S}_0$ is a ball of radius r centered at q in the hyperplace $\mathcal{F}$. (See Figure P1). Determine the Euclidean volume of $\mathcal{D}$.

(5) Let a genuine Euclidean space $\mathcal{E}$ with translation space $\mathcal{V}$ and put $n := \dim \mathcal{E}$. Also, let a line $\mathcal{L}$ be given. Choosing a point $q \in \mathcal{L}$ and a unit vector $\mathbf{e}$ in the direction space of $\mathcal{L}$, we have

$$\mathcal{L} := \{q + \xi\mathbf{e} \mid \xi \in \mathbf{R}\} \ . \qquad (P4.11)$$

Let $\mathcal{G} \in \mathrm{Bnb}\,(\mathbf{R} \times \mathbb{P})$ be given such that

$$\mathcal{G}_\xi := \{\eta \in \mathbb{P} \mid (\xi, \eta) \in \mathcal{G}\} \qquad (P4.12)$$

is a finite disjoint union of bounded intervals for every $\xi \in \mathbf{R}$. Define

$$\mathcal{D} := \{q + \xi\mathbf{e} + \mathbf{u} \mid \xi \in \mathbf{R}, \ \mathbf{u} \in \mathcal{V}, \ \mathbf{e} \cdot \mathbf{u} = 0, \ (\xi, |\mathbf{u}|) \in \mathcal{G}_\xi\} \ . \qquad (P4.13)$$

One can interpret $\mathcal{D}$ to be a *body of revolution* about the line $\mathcal{L}$ when $n = 3$.

(a) Prove that $\mathcal{D} \in \mathrm{Bnb}\,\mathcal{E}$.

(b) Let (ξ_c, η_c) be the centroid of $\mathcal{G}$ as defined in Problem (2) and let ω_{n-1} be defined according to Prop.4 of Sect.48. Show that

$$\mathrm{vol}\,(\mathcal{D}) = (n-1)\,\eta_c\,\omega_{n-1} \ . \qquad (P4.14)$$

(This result is known as the *Theorem of Pappus* when n=3)

(c) Evaluate the volume of a torus whose center-circle radius is r and whose cross-section radius is C.

(6) Let a genuine inner-product space $\mathcal{V}$, an orthonormal pair $(\mathbf{e}, \mathbf{f})$ in $\mathcal{V}$ and $r \in \mathbb{P}^{\times}$ be given. Put

$$\mathcal{A} := r\,(\mathrm{Ubl}\mathcal{V}) \ \cap \ (\frac{r}{2}\mathbf{f} + \mathbb{R}\,\mathbf{e} + \frac{r}{2}\mathrm{Ubl}\,(\{\mathbf{e}\}^{\perp})\ . \qquad (P4.15)$$

This is the intersection of a Ball of radius r with a cylinder whose boundary passes through the center of the ball and is tangent to it. See Figure P2.

Evaluate $\mathrm{vol}\,(\mathcal{A})$.

(7) Using (411.15), prove that

$$\int_{o}^{\infty} e^{-s^2}\mathrm{d}s = \int_{\mathbb{P}} exp \circ (-\iota^2) = \frac{\sqrt{\pi}}{2}\ .$$

(Hint: Use Cartesian coordinates.)