Chapter 7

Coordinate Systems

In this chapter we assume again that all spaces (linear, flat, or Euclidean) under consideration are finite-dimensional.

71 Coordinates in Flat Spaces

We assume that a flat space $\mathcal{E}$ with translation space $\mathcal{V}$ is given. Roughly speaking, a coordinate system is a method for specifying points in $\mathcal{E}$ by means of families of numbers. The coordinates are the functions that assign the specifying numbers to the points. It is useful to use the coordinate functions themselves as indices when describing the family of numbers that specifies a given point. For most "curvilinear" coordinate systems, the coordinate functions can be defined unambiguously only on a subset $\mathcal{D}$ of $\mathcal{E}$ obtained from $\mathcal{E}$ by removing a suitable set of "exceptional" points. Examples will be considered in Sect.74.

Definition: *A* **coordinate system** *on a given open subset $\mathcal{D}$ of $\mathcal{E}$ is a finite set $\Gamma \subset \mathrm{Map}\,(\mathcal{D}, \mathbb{R})$ of functions $c : \mathcal{D} \to \mathbb{R}$, which are called* **coordinates***, subject to the following conditions:*

(C1) *Every coordinate $c \in \Gamma$ is of class C^2.*

(C2) *For every $x \in \mathcal{D}$, the family $(\nabla_x c \mid c \in \Gamma)$ of the gradients of the coordinates at x is a basis of $\mathcal{V}^*$.*

(C3) *The mapping $\Gamma : \mathcal{D} \to \mathbb{R}^\Gamma$, identified with the set Γ by self-indexing* (see Sect.02) *and term-wise evaluation* (see Sect.04), *i.e. by $\Gamma(x) := (c(x) \mid c \in \Gamma)$ for all $x \in \mathcal{D}$, is injective.*

For the remainder of this section we assume that an open subset $\mathcal{D}$ of $\mathcal{E}$ and a coordinate system Γ on $\mathcal{D}$ are given. Using the identification $\mathcal{V}^{*\Gamma} = \mathrm{Lin}(\mathcal{V}, \mathbb{R})^\Gamma \cong \mathrm{Lin}(\mathcal{V}, \mathbb{R}^\Gamma)$ (see Sect.14), we obtain

$$\nabla_x \Gamma = (\nabla_x c \mid c \in \Gamma). \tag{71.1}$$

The condition (C2) expresses the requirement that the gradient of $\Gamma : \mathcal{D} \to \mathbb{R}^\Gamma$ have invertible values. Using the Local Inversion Theorem of Sect.68 and (C3) we obtain the following result.

Proposition 1: *The range $\overline{\mathcal{D}} := \operatorname{Rng}\Gamma$ of Γ is an open subset of $\mathbb{R}^\Gamma$, the mapping $\gamma := \Gamma|^{\overline{\mathcal{D}}} : \mathcal{D} \to \overline{\mathcal{D}}$ is invertible and has an inverse $\psi := \gamma^{\leftarrow} : \overline{\mathcal{D}} \to \mathcal{D}$ that is of class C^2.*

By (65.11), the family

$$(\psi,_c \mid c \in \Gamma) \in \mathrm{Map}\,(\overline{\mathcal{D}}, \mathcal{V})^\Gamma \cong \mathrm{Map}\,(\overline{\mathcal{D}}, \mathcal{V}^\Gamma)$$

of partial derivatives of ψ with respect to the coordinates is related to the gradient $\nabla\psi : \overline{\mathcal{D}} \to \mathrm{Lin}(\mathbb{R}^\Gamma, \mathcal{V})$ of ψ by

$$\nabla\psi = \mathrm{lnc}_{(\psi,_c \mid c\in\Gamma)}, \tag{71.2}$$

where the value at $\xi \in \overline{\mathcal{D}}$ of the right side is understood to be the linear combination mapping of the family $(\psi,_c(\xi) \mid c \in \Gamma)$. Since $\psi = \gamma^{\leftarrow}$, the Chain Rule shows that $\nabla_\xi\psi$ is invertible with inverse $(\nabla_\xi\psi)^{-1} = (\nabla_{\psi(\xi)}\Gamma) = \nabla_{\psi(\xi)}\Gamma$ for all $\xi \in \overline{\mathcal{D}}$. In view of (71.2), it follows that $(\psi,_c(\xi) \mid c \in \Gamma)$ is a basis of $\mathcal{V}$ and, in view of (71.1), that

$$(\nabla_{\psi(\xi)} c)\psi,_d(\xi) = \delta^c_d := \left\{ \begin{array}{lll} 0 & \text{if} & c \neq d \\ 1 & \text{if} & c = d \end{array} \right\} \tag{71.3}$$

for all $c, d \in \Gamma$ and all $\xi \in \overline{\mathcal{D}}$. We use the notation

$$\mathbf{b}_c := \psi,_c \circ \gamma, \quad \boldsymbol{\beta}^c := \nabla c \quad \text{for all} \quad c \in \Gamma. \tag{71.4}$$

By (C1) and Prop.1, $\mathbf{b}_c : \mathcal{D} \to \mathcal{V}$ and $\boldsymbol{\beta}^c : \mathcal{D} \to \mathcal{V}^*$ are of class C^1 for all $c \in \Gamma$. It follows from (71.3) that

$$\boldsymbol{\beta}^c \mathbf{b}_d = \delta^c_d \quad \text{for all} \quad c, d \in \Gamma. \tag{71.5}$$

By Prop.4 of Sect.23, we conclude that for each $x \in \mathcal{D}$, the family $\boldsymbol{\beta}(x) := (\boldsymbol{\beta}^c(x) \mid c \in \Gamma)$ in $\mathcal{V}^*$ is the dual basis of the basis $\mathbf{b}(x) := (\mathbf{b}_c(x) \mid c \in \Gamma)$ of $\mathcal{V}$. We call $\mathbf{b} : \mathcal{D} \to \mathcal{V}^\Gamma$ the **basis field** and $\boldsymbol{\beta} : \mathcal{D} \to \mathcal{V}^{*\Gamma}$ the **dual basis field** of the given coordinate system Γ.

Remark: The reason for using superscripts rather than subscripts as indices to denote the terms of certain families will be explained in Sect.73. The placing of the indices is designed in such a way that in most of the summations that occur, the summation dummy is used exactly twice, once as a superscript and once as a subscript. ∎

We often use the word "field" for a mapping whose domain is $\mathcal{D}$ or an open subset of $\mathcal{D}$. If the codomain is $\mathbb{R}$, we call it a **scalar field**; if the codomain is $\mathcal{V}$, we call it a **vector field**; if the codomain is $\mathcal{V}^*$, we call it a **covector field**; and if the codomain is Lin$\mathcal{V}$, we call it a **lineon field**. If $\mathbf{h}$ is a vector field, we define the **component-family**

$$[\mathbf{h}] := ([\mathbf{h}]^c \mid c \in \Gamma) \in (\operatorname{Map}(\operatorname{Dom}\mathbf{h}, \mathbb{R}))^\Gamma$$

of $\mathbf{h}$ relative to the given coordinate system by $[\mathbf{h}](x) := \operatorname{lnc}_{\mathbf{b}(x)}^{-1}\mathbf{h}(x)$ for all $x \in \operatorname{Dom}\mathbf{h}$, so that

$$\mathbf{h} = \sum_{d\in\Gamma} [\mathbf{h}]^d \mathbf{b}_d, \quad [\mathbf{h}]^c = \boldsymbol{\beta}^c \mathbf{h} \quad \text{for all} \quad c \in \Gamma. \tag{71.6}$$

If $\boldsymbol{\eta}$ is a covector field, we define the component family $[\boldsymbol{\eta}] := ([\boldsymbol{\eta}]_c \mid c \in \Gamma)$ of $\boldsymbol{\eta}$ by $[\boldsymbol{\eta}](x) := \operatorname{lnc}_{\boldsymbol{\beta}(x)}^{-1}\boldsymbol{\eta}(x)$ for all $x \in \operatorname{Dom}\boldsymbol{\eta}$, so that

$$\boldsymbol{\eta} = \sum_{d\in\Gamma} [\boldsymbol{\eta}]_d \boldsymbol{\beta}^d, \quad [\boldsymbol{\eta}]_c = \boldsymbol{\eta}\mathbf{b}_c \quad \text{for all} \quad c \in \Gamma. \tag{71.7}$$

If $\mathbf{T}$ is a lineon field, we define the component-matrix $[\mathbf{T}] := ([\mathbf{T}]^c{}_d \mid (c,d) \in \Gamma^2)$ of $\mathbf{T}$ by

$$[\mathbf{T}](x) := (\operatorname{lnc}_{(\mathbf{b}_c \otimes \boldsymbol{\beta}^d \mid (c,d)\in\Gamma^2)})^{-1}\mathbf{T}(x)$$

for all $x \in \operatorname{Dom}\mathbf{T}$, so that

$$\mathbf{T} = \sum_{(c,d)\in\Gamma^2} [\mathbf{T}]^c{}_d (\mathbf{b}_c \otimes \boldsymbol{\beta}^d). \tag{71.8}$$

The matrix $[\mathbf{T}]$ is given by

$$[\mathbf{T}]^c{}_d = \boldsymbol{\beta}^c \mathbf{T} \mathbf{b}_d \quad \text{for all} \quad c, d \in \Gamma \tag{71.9}$$

and characterized by

$$\mathbf{T}\mathbf{b}_c = \sum_{d\in\Gamma} [\mathbf{T}]^d{}_c \mathbf{b}_d \quad \text{for all} \quad c \in \Gamma. \tag{71.10}$$

In general, if $\mathbf{F}$ is a field whose codomain is a linear space $\mathcal{W}$ constructed from $\mathcal{V}$ by some natural construction, we define the component family $[\mathbf{F}]$ of $\mathbf{F}$ as follows: For each $x \in \operatorname{Dom}\mathbf{F}$, $[\mathbf{F}](x)$ is the family of components of $\mathbf{F}$ relative to the basis of $\mathcal{W}$ induced by the basis $\mathbf{b}(x)$ of $\mathcal{V}$ by the construction of $\mathcal{W}$. For example, if $\mathbf{F}$ is a field with codomain $\operatorname{Lin}(\mathcal{V}^*, \mathcal{V})$ then the component-matrix $[\mathbf{F}] := ([\mathbf{F}]^{cd} \mid (c,d) \in \Gamma^2)$ of $\mathbf{F}$ is determined by

$$\mathbf{F} = \sum_{(c,d)\in\Gamma^2} [\mathbf{F}]^{cd}(\mathbf{b}_c \otimes \mathbf{b}_d). \tag{71.11}$$

It is given by

$$[\mathbf{F}]^{cd} = \boldsymbol{\beta}^c \mathbf{F} \boldsymbol{\beta}^d \quad \text{for all} \quad c, d \in \Gamma \tag{71.12}$$

and characterized by

$$\mathbf{F}\boldsymbol{\beta}^c = \sum_{d\in\Gamma} [\mathbf{F}]^{dc}\mathbf{b}_d \quad \text{for all} \quad c \in \Gamma. \tag{71.13}$$

A field of any type is continuous, differentiable, or of class C^1 if and only if all of its components have the corresponding property.

We say that Γ is a **flat coordinate system** if the members of Γ are all flat functions or restrictions thereof. If also $\mathcal{D} = \mathcal{E}$, then $\gamma = \Gamma \ : \ \mathcal{E} \to \mathbb{R}^\Gamma$ is a flat isomorphism. The point $q := \psi(0) = \gamma^{\leftarrow}(0) \in \mathcal{E}$ is then called the **origin** of the given flat coordinate system. A flat coordinate system can be specified by prescribing the origin $q \in \mathcal{E}$ and a set basis $\mathfrak{b}$ of $\mathcal{V}$. Then each member $\boldsymbol{\lambda}$ of the dual basis $\mathfrak{b}^*$ can be used to specify a coordinate $c \ : \ \mathcal{E} \to \mathbb{R}$ by $c(x) := \boldsymbol{\lambda}(x - q)$ for all $x \in \mathcal{E}$.

The basis field $\mathbf{b}$ (or its dual $\boldsymbol{\beta}$) is constant if and only if the coordinate system is flat.

Let $\xi \in \overline{\mathcal{D}} \subset \mathbb{R}^\Gamma$ and $c \in \Gamma$ be given and let $\psi(\xi.c)$ be defined according to (04.25). It is easily seen that $\operatorname{Dom}\psi(\xi.c)$ is an open subset of $\mathbb{R}$. We call $\operatorname{Rng}\psi(\xi.c)$ the *coordinate curve* through the point $\psi(\xi)$ cooresponding to the coordinate c. If Γ is a flat coordinate system, then the coordinate curves are straight lines.

Pitfall: It is easy to give examples of non-flat coordinate systems which are "rectilinear" in the sense that all coordinate curves are straight lines. Thus, there is a distinction between non-flat coordinate systems and "curvilinear coordinate systems", i.e. coordinate systems having some coordinate curves that are not straight. ∎

Notes 71

(1) In all discussions of the theory of coordinate systems I have seen in the literature, the coordinates are assumed to be enumerated so as to form a list $(c^i \mid i \in n^]),\ n := \dim\mathcal{E}$. The numbers in $n^]$ are then used as indices in all formulas involving coordinates and components. However, when dealing with a *specific* coordinate system, most authors do not follow their theory but instead use the coordinates themselves as indices. I believe that one should do in theory what one does in practice and always use the set of coordinates itself as the index set. Enumeration of the coordinates is artificial and serves no useful purpose.

(2) Many textbooks contain a lengthy discussion about the transformation from one coordinate system to another. I believe that such "coordinate transformations" serve no useful purpose and may lead to enormous unnecessary calculations. In a specific situation, one should choose from the start a coordinate system that fits the situation and then stick with it. Even if a second coordinate system is considered, as is useful on rare occasions, one does not need "coordinate transformations".

72 Connection Components Components of Gradients

Let a coordinate system Γ on an open subset $\mathcal{D}$ of a flat space $\mathcal{E}$ with translation space $\mathcal{V}$ be given. As we have seen in the previous section, the basis field $\mathbf{b}$ and the dual basis field $\boldsymbol{\beta}$ of the system Γ are of class C^1.

We use the notations

$$C_c{}^d{}_e := \boldsymbol{\beta}^d(\nabla \mathbf{b}_c)\mathbf{b}_e \quad \text{for all} \quad c, d, e \in \Gamma \tag{72.1}$$

and

$$D_c := \operatorname{div} \mathbf{b}_c \quad \text{for all} \quad c \in \Gamma. \tag{72.2}$$

The functions $C_c{}^d{}_e : \mathcal{D} \to \mathbb{R}$ and $D_c : \mathcal{D} \to \mathbb{R}$ are called the **connection components** and the **deviation components**, respectively, of the system Γ.

The connection components give the component matrices of the gradients not only of the terms of the basis field $\mathbf{b}$ but also of its dual $\boldsymbol{\beta}$:

Proposition 1: *For each coordinate $c \in \Gamma$, the component matrices of $\nabla \mathbf{b}_c$ and $\nabla \boldsymbol{\beta}^c$ are given by*

$$[\nabla \mathbf{b}_c] = (C_c{}^d{}_e \mid (d, e) \in \Gamma^2) \tag{72.3}$$

and

$$[\nabla \boldsymbol{\beta}^c] = (-C_d{}^c{}_e \mid (d, e) \in \Gamma^2). \tag{72.4}$$

Proof: We obtain (72.3) simply by comparing (72.1) with (71.9). To derive (72.4), we use the product rule (66.6) when taking the gradient of (71.5) and obtain

$$(\nabla \boldsymbol{\beta}^c)^\top \mathbf{b}_d + (\nabla \mathbf{b}_d)^\top \boldsymbol{\beta}^c = \mathbf{0} \quad \text{for all} \quad d \in \Gamma.$$

Value-wise operation on $\mathbf{b}_e$ gives

$$\mathbf{b}_d(\nabla \boldsymbol{\beta}^c)\mathbf{b}_e + \boldsymbol{\beta}^c(\nabla \mathbf{b}_d)\mathbf{b}_e = 0 \quad \text{for all} \quad d, e \in \Gamma.$$

The result (72.4) follows now from (72.1) and the fact that the matrix $[\nabla \boldsymbol{\beta}^c]$ is given by

$$[\nabla \boldsymbol{\beta}^c] = (\mathbf{b}_d(\nabla \boldsymbol{\beta}^c)\mathbf{b}_e \mid (d, e) \in \Gamma^2).$$

■

Proposition 2: *The connection components satisfy the symmetry relations*

$$C_c{}^d{}_e = C_e{}^d{}_c \quad \text{for all} \quad c, d, e \in \Gamma. \tag{72.5}$$

Proof: In view of the definition $\boldsymbol{\beta}^c := \nabla c$ we have $\nabla \boldsymbol{\beta}^c = \nabla\nabla c$. The assertion (72.5) follows from (72.4) and the Theorem on Symmetry of Second Gradients of Sect.611, applied to c. ■

Proposition 3: *The deviation components are obtained from the connection components by*

$$D_c = \sum_{d \in \Gamma} C_c{}^d{}_d \quad \text{for all} \quad c \in \Gamma. \tag{72.6}$$

Proof: By (72.2), (67.1), and (67.2), we have $D_c(x) = \operatorname{tr}(\nabla_x \mathbf{b}_c)$ for all $x \in \mathcal{D}$. The desired result (72.6) then follows from (72.3) and (26.8). ■

Let $\mathcal{W}$ be a linear space and let $\mathbf{F}$ be a differentiable field with domain $\mathcal{D}$ and with codomain $\mathcal{W}$. We use the notation

$$\mathbf{F}_{;c} := (\mathbf{F} \circ \psi)_{,c} \circ \gamma \quad \text{for all} \quad c \in \Gamma, \tag{72.7}$$

which says that $\mathbf{F}_{;c}$ is obtained from $\mathbf{F}$ by first looking at the dependence of the values of $\mathbf{F}$ on the coordinates, then taking the partial c-derivative, and then looking at the result as a function of the point. It is easily seen that

$$(\nabla \mathbf{F})\mathbf{b}_c = \mathbf{F}_{;c} \quad \text{for all} \quad c \in \Gamma. \tag{72.8}$$

If $\mathbf{F} := f$ is a differentiable scalar field, then (72.8) states that the component family $[\nabla f]$ of the covector field $\nabla f : \mathcal{D} \to \mathcal{V}^*$ is given by

$$[\nabla f] = (f_{;c} \mid c \in \Gamma). \tag{72.9}$$

The connection components are all zero if the coordinate system is flat. If it is not flat, one must know what the connection components are in order to calculate the components of gradients of vector and covector fields.

Proposition 4: *Let* $\mathbf{h}$ *be a differentiable vector field and* $\boldsymbol{\eta}$ *a differentiable covector field with component-families* $[\mathbf{h}]$ *and* $[\boldsymbol{\eta}]$*, respectively. Then the component-matrices of* $\nabla\mathbf{h}$ *and* $\nabla\boldsymbol{\eta}$ *have the terms*

$$[\nabla\mathbf{h}]^c{}_d = [\mathbf{h}]^c{}_{;d} + \sum_{e\in\Gamma}[\mathbf{h}]^e C_e{}^c{}_d, \tag{72.10}$$

and

$$[\nabla\boldsymbol{\eta}]_{cd} = [\boldsymbol{\eta}]_{c;d} - \sum_{e\in\Gamma}[\boldsymbol{\eta}]_e\, C_c{}^e{}_d \tag{72.11}$$

for all $c, d \in \Gamma$*.*

Proof: To obtain (72.10), one takes the gradient of $(71.6)_1$, uses the product rule (66.5), the formula (71.9) with $\mathbf{T} := \nabla\mathbf{h}$, the formula (72.9) with $f := [\mathbf{h}]^c$ and (72.3). The same procedure, starting with $(71.7)_1$ and using (72.4), yields (72.11). ∎

It is easy to derive formulas analogous to (72.10) and (72.11) for the components of gradients of other kinds of fields.

Proposition 5: *Let* $\boldsymbol{\eta}$ *be a differentiable covector field. Then the component-matrix of* $\operatorname{Curl}\boldsymbol{\eta} := \nabla\boldsymbol{\eta} - (\nabla\boldsymbol{\eta})^\top : \operatorname{Dom}\boldsymbol{\eta} \to \operatorname{Lin}(\mathcal{V}, \mathcal{V}^*)$ (see Sect.611) *is given by*

$$[\operatorname{Curl}\boldsymbol{\eta}]_{cd} = [\boldsymbol{\eta}]_{c;d} - [\boldsymbol{\eta}]_{d;c} \quad \text{for all} \quad c, d \in \Gamma. \tag{72.12}$$

Proof: By Def.1 of Sect.611, the components of $\operatorname{Curl}\boldsymbol{\eta}$ are obtained from the components of $\nabla\boldsymbol{\eta}$ by

$$[\operatorname{Curl}\boldsymbol{\eta}]_{cd} = [\nabla\boldsymbol{\eta}]_{cd} - [\nabla\boldsymbol{\eta}]_{dc} \quad \text{for all} \quad c, d \in \Gamma.$$

After substituting (72.11), we see from (72.5) that the terms involving the connection components cancel and hence that (72.12) holds. ∎

Note that the connection components do not appear in (72.12).

Proposition 6: *Let* $\mathbf{h}$ *be a differentiable vector field. The divergence of* $\mathbf{h}$ *is given, in terms of the component family* $[\mathbf{h}]$ *of* $\mathbf{h}$ *and the deviation components, by*

$$\operatorname{div}\mathbf{h} = \sum_{c\in\Gamma}([\mathbf{h}]^c{}_{;c} + [\mathbf{h}]^c D_c). \tag{72.13}$$

Proof: Using $(71.6)_1$ and Prop.1 of Sect.67, we obtain

$$\operatorname{div}\mathbf{h} = \sum_{c\in\Gamma}(\nabla([\mathbf{h}]^c)\mathbf{b}_c + [\mathbf{h}]^c \operatorname{div}\mathbf{b}_c).$$

Using (72.8) with $\mathbf{F} := [\mathbf{h}]^c$ and the definition (72.2) of D_c, we obtain the desired result (72.13). ∎

Proposition 7: *Let* $\mathbf{F}$ *be a differentiable field with codomain* $\mathrm{Lin}(\mathcal{V}^*, \mathcal{V})$. *Then the component family of* $\mathrm{div}\, \mathbf{F} : \mathrm{Dom}\, \mathbf{F} \to \mathcal{V}$ (see Def.1 of Sect.67) *is given by*

$$[\mathrm{div}\, \mathbf{F}]^c = \sum_{d\in\Gamma} ([\mathbf{F}]^{cd}{}_{;d} + [\mathbf{F}]^{cd} D_d) + \sum_{(e,d)\in\Gamma^2} [\mathbf{F}]^{ed} C_e{}^c{}_d \tag{72.14}$$

for all $c \in \Gamma$.

Proof: Let $c \in \Gamma$ be given. Then $\boldsymbol{\beta}^c\mathbf{F}$ is a differentiable vector field. If we apply Prop.2 of Sect.67 with $\mathcal{W} := \mathcal{V}$, $\mathbf{H} := \mathbf{F}$, and $\boldsymbol{\rho} := \boldsymbol{\beta}^c$, we obtain

$$\mathrm{div}(\boldsymbol{\beta}^c\mathbf{F}) = \boldsymbol{\beta}^c \mathrm{div}\, \mathbf{F} + \mathrm{tr}(\mathbf{F}^\top \nabla \boldsymbol{\beta}^c). \tag{72.15}$$

It follows from (72.4), Prop.5 of Sect.16, (23.9), and (26.8) that

$$\mathrm{tr}(\mathbf{F}^\top \nabla \boldsymbol{\beta}^c) = - \sum_{(e,d)\in\Gamma^2} [\mathbf{F}]^{ed} C_e{}^c{}_d,$$

and from Prop.6 that

$$\mathrm{div}(\boldsymbol{\beta}^c\mathbf{F}) = \sum_{d\in\Gamma} ([\boldsymbol{\beta}^c\mathbf{F}]^d{}_{;d} + [\boldsymbol{\beta}^c\mathbf{F}]^d D_d).$$

Substituting these two results into (72.15) and observing that $[\mathrm{div}\, \mathbf{F}]^c = \boldsymbol{\beta}^c \mathrm{div}\, \mathbf{F}$ and $[\boldsymbol{\beta}^c\mathbf{F}]^d = [\mathbf{F}]^{cd}$ for all $d \in \Gamma$, we obtain the desired result (72.14). ∎

Assume now that the coordinate system Γ is flat and hence that its basis field $\mathbf{b}$ and the dual basis field $\boldsymbol{\beta}$ are constant. By (72.1) and (72.2), the connection components and deviation components are zero. The formulas (72.10) and (72.11) for the components of the gradients of a differentiable vector field $\mathbf{h}$ and a differentiable covector field $\boldsymbol{\eta}$ reduce to

$$[\nabla\mathbf{h}]^c{}_d = [\mathbf{h}]^c{}_{;d}, \quad [\nabla\boldsymbol{\eta}]_{cd} = [\boldsymbol{\eta}]_{c;d}, \tag{72.16}$$

valid for all $c, d \in \Gamma$. The formula (72.13) for the divergence of a differentiable vector field $\mathbf{h}$ reduces to

$$\mathrm{div}\, \mathbf{h} = \sum_{c\in\Gamma} [\mathbf{h}]^c{}_{;c}, \tag{72.17}$$

and the formula (72.14) for the divergence of a differentiable field $\mathbf{F}$ with values in $\mathrm{Lin}(\mathcal{V}^*, \mathcal{V})$ becomes

$$[\mathrm{div}\, \mathbf{F}]^c = \sum_{d\in\Gamma} [\mathbf{F}]^{cd}{}_{;d} \quad \text{for all} \quad c \in \Gamma. \tag{72.18}$$

Notes 72

(1) The term "connection components" is used here for the first time. More traditional terms are "Christoffel components", "Christoffel symbols", "three-index symbols", or "Gamma symbols". The last three terms are absurd because it is not the symbols that matter but what they stand for. The notations $\{{}_i{}^j{}_k\}$ or $\Gamma_i{}^j{}_k$ instead of our $C_c{}^d{}_e$ are often found in the literature.

(2) The term "deviation components" is used here for the first time. I am not aware of any other term used in the literature.

(3) The components of gradients as described, for example, in Prop.4 are often called "covariant derivatives".

73 Coordinates in Euclidean Spaces

We assume that a coordinate system Γ on an open subset of a (not necessarily genuine) Euclidean space $\mathcal{E}$ with translation space $\mathcal{V}$ is given. Since $\mathcal{V}$ is an inner-product space, we identify $\mathcal{V}^* \cong \mathcal{V}$, so that the basis field $\mathbf{b}$ as well as the dual basis field $\boldsymbol{\beta}$ of Γ have terms that are vector fields of class C^1. We use the notations

$$G_{cd} := \mathbf{b}_c \cdot \mathbf{b}_d \quad \text{for all} \quad c, d \in \Gamma \tag{73.1}$$

and

$$\overline{G}^{cd} := \boldsymbol{\beta}^c \cdot \boldsymbol{\beta}^d \quad \text{for all} \quad c, d \in \Gamma, \tag{73.2}$$

so that $G := (G_{cd} \mid (c,d) \in \Gamma^2)$ and $\overline{G} := (\overline{G}^{cd} \mid (c,d) \in \Gamma^2)$ are symmetric matrices whose terms are scalar fields of class C^1. These scalar fields are called the **inner-product components** of the system Γ. In terms of the notation (41.13), G and $\overline{G}$ are given by

$$G(x) = G_{\mathbf{b}(x)}, \quad \overline{G}(x) = G_{\boldsymbol{\beta}(x)} \tag{73.3}$$

for all $x \in \mathcal{D}$. Since $\boldsymbol{\beta}(x)$ is the dual basis of $\mathbf{b}(x)$ for each $x \in \mathcal{D}$, we obtain the following result from (41.16) and (41.15).

Proposition 1: *For each $x \in \mathcal{D}$, the matrix $\overline{G}(x)$ is the inverse of $G(x)$ and hence we have*

$$\sum_{e \in \Gamma} G_{ce}\overline{G}^{ed} = \delta_c^d \quad \text{for all} \quad c, d \in \Gamma. \tag{73.4}$$

The basis fields $\mathbf{b}$ *and* $\boldsymbol{\beta}$ *are related by*

$$\mathbf{b}_c = \sum_{d\in\Gamma} G_{cd}\boldsymbol{\beta}^d, \quad \boldsymbol{\beta}^c = \sum_{d\in\Gamma} \overline{G}^{cd}\mathbf{b}_d \tag{73.5}$$

for all $c \in \Gamma$*.*

The following result shows that the connection components can be obtained directly from G and $\overline{G}$.

Theorem on Connection Components: *The connection components of a coordinate system on an open subset of a Euclidean space can be obtained from the inner-product components of the system by*

$$C_c{}^d{}_e = \frac{1}{2}\sum_{f\in\Gamma} \overline{G}^{df}(G_{fc;e} + G_{fe;c} - G_{ce;f}) \tag{73.6}$$

for all $c, d, e \in \Gamma$*.*

Proof: We use the abbreviation

$$B_{cde} := \mathbf{b}_d \cdot (\nabla \mathbf{b}_c)\mathbf{b}_e \quad \text{for all} \quad c, d, e \in \Gamma. \tag{73.7}$$

By (72.1) and Prop.1 we have

$$C_c{}^d{}_e = \sum_{f\in\Gamma} \overline{G}^{df} B_{cfe}, \quad B_{cde} = \sum_{f\in\Gamma} G_{df} C_c{}^f{}_e \tag{73.8}$$

for all $c, d, e \in \Gamma$. We note that, in view of $(73.8)_2$ and (72.5),

$$B_{cde} = B_{edc} \quad \text{for all} \quad c, d, e \in \Gamma. \tag{73.9}$$

Let $c, d, e \in \Gamma$ be given. Applying the Product Rule (66.9) to (73.1), we find that

$$\nabla G_{cd} = (\nabla \mathbf{b}_c)^\top \mathbf{b}_d + (\nabla \mathbf{b}_d)^\top \mathbf{b}_c.$$

Hence, by (72.8), (41.10), and (73.7) we obtain

$$G_{cd;e} = (\nabla G_{cd}) \cdot \mathbf{b}_e = B_{cde} + B_{dce}.$$

We can view this as a system of equations which can be solved for B_{cde} as follows: Since $c, d, e \in \Gamma$ were arbitrary, we can rewrite the system with c, d, e cyclically permuted and find that

$$G_{cd;e} = B_{cde} + B_{dce},$$

$$G_{de;c} = B_{dec} + B_{edc},$$

$$G_{ec;d} = B_{ecd} + B_{ced}$$

are valid for all $c, d, e \in \Gamma$. If we subtract the last of these equations from the sum of the first two and observe (73.9), we obtain

$$G_{cd;e} + G_{de;c} - G_{ec;d} = 2B_{cde},$$

valid for all $c, d, e \in \Gamma$. Using $(73.8)_1$, we conclude that (73.6) holds. ■

For the following results, we need the concept of *determinant*, which will be explained only in Vol.II of this treatise. However, most readers undoubtedly know how to compute determinants of square matrices of small size, and this is all that is needed for the application of the results to special coordinate systems. We are concerned here with the determinant for $\mathbb{R}^\Gamma$. In Vol.II we will see that it is a mapping $\det : \operatorname{Lin} \mathbb{R}^\Gamma \to \mathbb{R}$ of class C^1 whose gradient satisfies

$$(\nabla_M \det)N = \det(M)\operatorname{tr}(M^{-1}N) \tag{73.10}$$

for all $M \in \operatorname{Lis} \mathbb{R}^\Gamma$ and all $N \in \operatorname{Lin} \mathbb{R}^\Gamma$. If $M \in \operatorname{Lin} \mathbb{R}^\Gamma \cong \mathbb{R}^{\Gamma\times\Gamma}$ is a diagonal matrix (see Sect.02), then $\det(M)$ is the product of the diagonal of M, i.e. we have

$$\det(M) = \prod_{c\in\Gamma} M_{cc}. \tag{73.11}$$

We have $\det(M) \neq 0$ if and only if $M \in \operatorname{Lin} \mathbb{R}^\Gamma$ is invertible. Hence, since G has invertible values by Prop.1, $\det \circ G : \mathcal{D} \to \mathbb{R}$ is nowhere zero.

Theorem on Deviation Components: *The deviation components of a coordinate system on an open subset of a Euclidean space can be obtained from the determinant of the inner-product matrix of the system by*

$$D_c = 2\frac{(\det\circ G);c}{\det\circ G} \quad \text{for all} \quad c \in \Gamma. \tag{73.12}$$

Proof: By (73.10), the Chain Rule, and Prop.1 we obtain

$$(\det\circ G)_{;c} = (\det\circ G)\operatorname{tr}(\overline{G}G_{;c}). \tag{73.13}$$

On the other hand, by (73.6) and (72.6), we have

$$D_c = \frac{1}{2}\sum_{(d,f)\in\Gamma^2} \overline{G}^{df}(G_{fc;d} + G_{fd;c} - G_{cd;f}). \tag{73.14}$$

Since the matrices G and $\overline{G}$ are symmetric we have

$$\sum_{(d,f)\in\Gamma^2} \overline{G}^{df} G_{fc;d} = \sum_{(f,d)\in\Gamma^2} \overline{G}^{fd} G_{dc;f} = \sum_{(d,f)\in\Gamma^2} \overline{G}^{df} G_{cd;f},$$

and hence, by (73.14), Prop.5 of Sect.16, and (26.8),

$$D_c = \frac{1}{2} \sum_{(d,f)\in\Gamma^2} \overline{G}^{df} G_{fd;c} = \frac{1}{2}\mathrm{tr}(\overline{G}G_{;c}).$$

Comparing this result with (73.13), we obtain (73.12). ■

In practice, it is useful to introduce the function

$$g := \sqrt{|\det \circ G|}, \tag{73.15}$$

which is of class C^1 because $\det \circ G$ is nowhere zero. Since $\det \circ G = \pm g^2$, (73.12) is equivalent to

$$D_c = \frac{g_{;c}}{g} \quad \text{for all} \quad c \in \Gamma. \tag{73.16}$$

Remark: One can show that the sign of $\det \circ G$ depends only on the signature of the inner product space $\mathcal{V}$ (see Sect.47). In fact $\det \circ G$ is strictly positive if $\mathrm{sig}^- \mathcal{V}$ is even, strictly negative if $\mathrm{sig}^- \mathcal{V}$ is odd. In particular, if $\mathcal{E}$ is a genuine Euclidean space, $\det \circ G$ is strictly positive and the absolute value symbols can be omitted in (73.15). ■

The identification $\mathcal{V} \cong \mathcal{V}^*$ makes the distinction between vector fields and covector fields disappear. This creates ambiguities because a vector field $\mathbf{h}$ now has two component families, one with respect to the basis field $\mathbf{b}$ and the other with respect to the dual basis field $\boldsymbol{\beta}$. Thus, the symbol $[\mathbf{h}]$ becomes ambiguous. For the terms of $[\mathbf{h}]$, the ambiguity is avoided by careful attention to the placing of indices as superscripts or subscripts. Thus, we use $([\mathbf{h}]^c \mid c \in \Gamma) := \boldsymbol{\beta} \cdot \mathbf{h}$ for the component family of $\mathbf{h}$ relative to $\mathbf{b}$ and $([\mathbf{h}]_c \mid c \in \Gamma) := \mathbf{b} \cdot \mathbf{h}$ for the component family of $\mathbf{h}$ relative to $\boldsymbol{\beta}$ (see also (41.17)). The symbol $[\mathbf{h}]$ by itself can no longer be used. It follows from (73.5) that the two types of components of $\mathbf{h}$ are related by the formulas

$$[\mathbf{h}]^c = \sum_{d\in\Gamma} \overline{G}^{cd} [\mathbf{h}]_d, \qquad [\mathbf{h}]_c = \sum_{d\in\Gamma} G_{cd} [\mathbf{h}]^d, \tag{73.17}$$

valid for all $c \in \Gamma$. To avoid clutter, we often omit the brackets and write $\mathbf{h}^c$ for $[\mathbf{h}]^c$ and $\mathbf{h}_c$ for $[\mathbf{h}]_c$ if no confusion can arise.

A lineon field $\mathbf{T}$ has four component matrices because of the identifications $\mathrm{Lin}\mathcal{V} \cong \mathrm{Lin}\mathcal{V}^* \cong \mathrm{Lin}(\mathcal{V}, \mathcal{V}^*) \cong \mathrm{Lin}(\mathcal{V}^*, \mathcal{V})$. The resulting ambiguity is avoided again by careful attention to the placing of indices as superscripts and subscripts. The four types of components are given by the formulas

$$\begin{array}{rclcrcl} [\mathbf{T}]^c{}_d & = & \boldsymbol{\beta}^c \cdot \mathbf{T}\mathbf{b}_d, & & [\mathbf{T}]_c{}^d & = & \mathbf{b}_c \cdot \mathbf{T}\boldsymbol{\beta}^d, \\ [\mathbf{T}]_{cd} & = & \mathbf{b}_c \cdot \mathbf{T}\mathbf{b}_d, & & [\mathbf{T}]^{cd} & = & \boldsymbol{\beta}^c \cdot \mathbf{T}\boldsymbol{\beta}^d, \end{array} \tag{73.18}$$

valid for all $c, d \in \Gamma$. The various types of components are related to each other by formulas such as

$$[\mathbf{T}]^{cd} = \sum_{e \in \Gamma} \overline{G}^{de} [\mathbf{T}]^c{}_e = \sum_{(e,f) \in \Gamma^2} \overline{G}^{ce} \overline{G}^{df} [\mathbf{T}]_{ef}, \tag{73.19}$$

valid for all $c, d \in \Gamma$. Again, we often omit the brackets to avoid clutter.

Using (73.16) and the product rule, we see that Props.6 and 7 of Sect.72 have the following corollaries.

Proposition 2: *The divergence of a differentiable vector field* $\mathbf{h}$ *is given by*

$$\operatorname{div} \mathbf{h} = \frac{1}{g} \sum_{c \in \Gamma} (g[\mathbf{h}]^c)_{;c}. \tag{73.20}$$

Proposition 3: *The components of the divergence of a differentiable lineon field* $\mathbf{T}$ *are given by*

$$[\operatorname{div} \mathbf{T}]^c = \frac{1}{g} \sum_{d \in \Gamma} (g[\mathbf{T}]^{cd})_{;d} + \sum_{(e,d) \in \Gamma^2} [\mathbf{T}]^{ed} C_e{}^c{}_d \tag{73.21}$$

for all $c \in \Gamma$.

Using Def.2 of Sect.67 and observing (72.9) and $(73.17)_1$, we obtain the following immediate consequence of Prop.2.

Proposition 4: *The Laplacian of a twice differentiable scalar field* f *is given by*

$$\Delta f = \frac{1}{g} \sum_{(c,d) \in \Gamma^2} (g \overline{G}^{cd} f_{;d})_{;c}. \tag{73.22}$$

Notes 73

(1) If $\mathbf{h}$ is a vector field, the components $[\mathbf{h}]^c$ of $\mathbf{h}$ relative to the basis field $\mathbf{b}$ are often called the "contravariant components" of $\mathbf{h}$, and the components $[\mathbf{h}]_c$ of $\mathbf{h}$ relative to the dual basis field $\boldsymbol{\beta}$ are then called the "covariant components" of $\mathbf{h}$. (See also Note (5) to Sect.41.)

(2) If $\mathbf{T}$ is a lineon field, the components $[\mathbf{T}]_{cd}$ and $\mathbf{T}^{cd}$ are often called the "covariant components" and "contravariant components", respectively, while $[\mathbf{T}]_c{}^d$ and $[\mathbf{T}]^c{}_d$ are called "mixed components" of $\mathbf{T}$.

74 Special Coordinate Systems

In this section a *genuine* Euclidean space $\mathcal{E}$ with translation space $\mathcal{V}$ is assumed given.

(A) Cartesian Coordinates: A flat coordinate system Γ is called a *Cartesian coordinate system* if $(\nabla c \mid c \in \Gamma)$ is a (genuine) orthonormal basis of $\mathcal{V}$. To specify a Cartesian coordinate system, we may prescribe a point $q \in \mathcal{E}$ and an orthonormal basis set $\mathfrak{e}$ of $\mathcal{V}$. For each $\mathbf{e} \in \mathfrak{e}$, we define a function $c : \mathcal{E} \to \mathbb{R}$ by

$$c(x) := \mathbf{e} \cdot (x - q) \quad \text{for all} \quad x \in \mathcal{E}. \tag{74.1}$$

The set Γ of all functions c defined in this way is then a Cartesian coordinate system on $\mathcal{E}$ with origin q. Since $\mathfrak{e}$ is orthonormal, we have $\boldsymbol{\beta}^c = \mathbf{b}_c$ for all $c \in \Gamma$. The mapping $\Gamma : \mathcal{E} \to \mathbb{R}^\Gamma$ defined by $\Gamma(x) := (c(x) \mid c \in \Gamma)$ for all $x \in \mathcal{E}$ is invertible and its inverse $\psi : \mathbb{R}^\Gamma \to \mathcal{E}$ is given by

$$\psi(\xi) = q + \sum_{c \in \Gamma} \xi_c \mathbf{b}_c \quad \text{for all} \quad \xi \in \mathbb{R}^\Gamma. \tag{74.2}$$

The matrices of the inner-product components of Γ are constant and given by $G = \overline{G} = 1_{\mathbb{R}^\Gamma}$. If $\mathbf{h}$ is a vector field, then $[\mathbf{h}]^c = [\mathbf{h}]_c$ for all $c \in \Gamma$ and if $\mathbf{T}$ is a lineon field, then $[\mathbf{T}]^c{}_d = [\mathbf{T}]^{cd} = [\mathbf{T}]_c{}^d = [\mathbf{T}]_{cd}$ for all $c, d \in \Gamma$. Thus, all indices can be written as subscripts without creating ambiguity. Since $\det \circ G = 1$ and hence $g = 1$, the formulas (73.20), (73.22), and (73.21) reduce to

$$\begin{aligned} \operatorname{div} \mathbf{h} &= \sum_{c \in \Gamma} [\mathbf{h}]_{c;c}, \\ \Delta f &= \sum_{c \in \Gamma} f_{;c;c}, \\ [\operatorname{div} \mathbf{T}]_c &= \sum_{d \in \Gamma} [\mathbf{T}]_{cd;d}, \end{aligned} \tag{74.3}$$

respectively.

(B) Polar Coordinates: We assume that $\dim \mathcal{E} = 2$. We prescribe a point $q \in \mathcal{E}$, a lineon $\mathbf{J} \in \operatorname{Skew}\mathcal{V} \cap \operatorname{Orth}\mathcal{V}$ (see Problem 2 of Chap.4) and a unit vector $\mathbf{e} \in \mathcal{V}$. ($\mathbf{J}$ is a *perpendicular turn* as defined in Sect.87.) We define $\mathbf{h} : \mathbb{R} \to \mathcal{V}$ by

$$\mathbf{h} := \exp_{\mathcal{V}} \circ (\iota \mathbf{J})\mathbf{e} = \cos \mathbf{e} + \sin \mathbf{J}\mathbf{e} \tag{74.4}$$

(see Problem 9 of Chap.6). Roughly, $\mathbf{h}(t)$ is obtained from $\mathbf{e}$, by a rotation with angle t. We have

$$\mathbf{h}^{\bullet}(t) = \mathbf{J}\mathbf{h}(t) \quad \text{for all} \quad t \in \mathbb{R}, \tag{74.5}$$

and hence

$$\mathbf{h} \cdot \mathbf{h}^{\bullet} = 0, \quad |\mathbf{h}| = |\mathbf{h}^{\bullet}| = 1. \tag{74.6}$$

We now consider the mapping $\Psi : \mathbb{P}^{\times} \times \mathbb{R} \to \mathcal{E}$ defined by

$$\Psi(s,t) := q + s\mathbf{h}(t) \quad \text{for all} \quad (s,t) \in \mathbb{P}^{\times} \times \mathbb{R}. \tag{74.7}$$

It is clear that Ψ is of class C^1, and, in view of (65.11), its gradient is given by

$$\nabla_{(s,t)}\Psi = \mathrm{lnc}_{(\mathbf{h}(t),s\mathbf{h}^{\bullet}(t))} \quad \text{for all} \quad (s,t) \in \mathbb{P}^{\times} \times \mathbb{R}.$$

It is clear from (74.6) that $(\mathbf{h}(t), s\mathbf{h}^{\bullet}(t))$ is a basis of $\mathcal{V}$ for every $(s,t) \in \mathbb{P}^{\times} \times \mathbb{R}$ and hence that $\nabla\Psi$ has only invertible values. It follows from the Local Inversion Theorem of Sect.68 that Ψ is locally invertible. It is not injective because $\Psi(s,t) = \Psi(s, t+2\pi)$ for all $(s,t) \in \mathbb{P}^{\times} \times \mathbb{R}$. An invertible adjustment of Ψ with open domain is $\psi := \Psi|_{\overline{\mathcal{D}}}^{\mathcal{D}}$, where

$$\overline{\mathcal{D}} := \mathbb{P}^{\times} \times \,]0, 2\pi[, \quad \mathcal{D} := \mathcal{E} \setminus (q + \mathbb{P}\mathbf{e}). \tag{74.8}$$

We put $\gamma := \psi^{\leftarrow}$ and define $r : \mathcal{D} \to \mathbb{R}$ and $\theta : \mathcal{D} \to \mathbb{R}$ by $(r, \theta) := \gamma|^{\mathbb{R}^2}$. Then $\Gamma := \{r, \theta\}$ is called a **polar coordinate system**. The function r is given by

$$r(x) = |x - q| \quad \text{for all} \quad x \in \mathcal{D}. \tag{74.9}$$

and the function θ is characterized by $\mathrm{Rng}\,\theta = \,]0, 2\pi[$ and

$$\mathbf{h}(\theta(x)) = \frac{x - q}{r(x)} \quad \text{for all} \quad x \in \mathcal{D}. \tag{74.10}$$

The values of the coordinates of a point x are indicated in Fig.1; the argument x is omitted to avoid clutter.

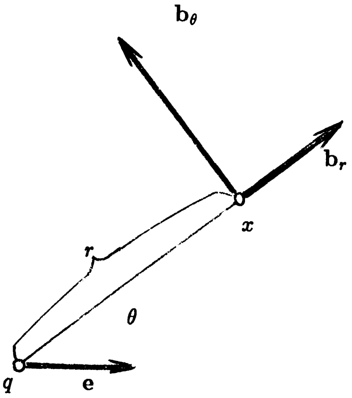

Figure 1.

If we interpret the first term in a pair as the r-term and the second as the θ-term of a family indexed on $\Gamma = \{r, \theta\}$, then the mappings γ and ψ above coincide with the mappings denoted by the same symbols in Sect.71. Since ψ is an adjustment of the mapping Ψ defined by (74.7), it follows from $(71.4)_1$ that

$$\mathbf{b}_r = \mathbf{h} \circ \theta, \qquad \mathbf{b}_\theta = r(\mathbf{h}^\bullet \circ \theta). \tag{74.11}$$

By (74.6) we have $\mathbf{b}_r \cdot \mathbf{b}_\theta = 0$, $|\mathbf{b}_r| = 1$, and $|\mathbf{b}_\theta| = r$ (see Fig.1). Hence the inner-product components (73.1) are

$$G_{r\theta} = 0, \;\; G_{rr} = 1, \;\; G_{\theta\theta} = r^2. \tag{74.12}$$

The value-wise inverse $\overline{G}$ of the matrix G is given by

$$\overline{G}^{r\theta} = 0, \;\; \overline{G}^{rr} = 1, \;\; \overline{G}^{\theta\theta} = \frac{1}{r^2}. \tag{74.13}$$

By (73.11) we have $\det \circ G = r^2$, and hence $g := \sqrt{|\det \circ G|}$ becomes

$$g = r. \tag{74.14}$$

Using the Theorem on Connection Components of Sect.73, we find that the only non-zero connection components are

$$C_\theta{}^\theta{}_r = C_r{}^\theta{}_\theta = \frac{1}{r}, \qquad C_\theta{}^r{}_\theta = -r. \tag{74.15}$$

The formula (73.20) for the divergence of a differentiable vector field $\mathbf{h}$ becomes

$$\operatorname{div} \mathbf{h} = \frac{1}{r}(r[\mathbf{h}]^r)_{;r} + [\mathbf{h}]^\theta{}_{;\theta}. \tag{74.16}$$

The formula (73.22) for Laplacian of a twice differentiable scalar field f becomes

$$\Delta f = \frac{1}{r}(rf_{;r})_{;r} + \frac{1}{r^2} f_{;\theta;\theta}. \tag{74.17}$$

The formula (73.21) for the components of the divergence of a differentiable lineon field $\mathbf{T}$ yields

$$\begin{aligned} [\text{div}\,\mathbf{T}]^r &= \tfrac{1}{r}(r\mathbf{T}^{rr})_{;r} + \mathbf{T}^{r\theta}{}_{;\theta} - r\mathbf{T}^{\theta\theta}, \\ [\text{div}\,\mathbf{T}]^\theta &= \tfrac{1}{r}(r\mathbf{T}^{\theta r})_{;r} + \mathbf{T}^{\theta\theta}{}_{;\theta} + \tfrac{1}{r}(\mathbf{T}^{r\theta} + \mathbf{T}^{\theta r}). \end{aligned} \tag{74.18}$$

(On the right sides, brackets are omitted to avoid clutter.)

Remark: The adjustment of Ψ described above is only one of several that yield a suitable definition of a polar coordinate system. Another would be obtained by replacing (74.8) by

$$\overline{\mathcal{D}} := \mathbb{P}^\times \times\ ]-\pi, \pi[, \qquad \mathcal{D} := \mathcal{E} \setminus (q - \mathbb{P}\mathbf{e}). \tag{74.19}$$

In this case θ would be characterized by (74.10) and $\text{Rng}\,\theta =\ ]-\pi, \pi[$. ■

(C) Cylindrical Coordinates: We assume that $\dim \mathcal{E} = 3$. We first prescribe a point $q \in \mathcal{E}$ and a unit vector $\mathbf{f} \in \mathcal{V}$. We then put $\mathcal{U} := \{\mathbf{f}\}^\perp$ and prescribe a lineon $\mathbf{J} \in \text{Skew}\mathcal{U} \cap \text{Orth}\mathcal{U}$ and a unit vector $\mathbf{e} \in \mathcal{U}$. We define $\mathbf{h} : \mathbb{R} \to \mathcal{V}$ by (74.4) and consider the mapping $\Psi : \mathbb{P}^\times \times \mathbb{R} \times \mathbb{R} \to \mathcal{E}$ defined by

$$\Psi(s, t, u) := q + s\mathbf{h}(t) + u\mathbf{f}. \tag{74.20}$$

It is easily seen that Ψ is of class C^1 and locally invertible but not injective. An invertible adjustment of Ψ with open domain is $\psi := \Psi|_{\overline{\mathcal{D}}}^{\mathcal{D}}$, where

$$\overline{\mathcal{D}} := \mathbb{P}^\times \times]0, 2\pi[\ \times \mathbb{R}, \qquad \mathcal{D} := \mathcal{E} \setminus (q + \mathbb{P}\mathbf{e} + \mathbb{R}\mathbf{f}). \tag{74.21}$$

We put $\gamma := \psi^{\leftarrow}$ and define the functions r, θ, z, all with domain $\mathcal{D}$, by $(r, \theta, z) := \gamma|^{\mathbb{R}^3}$. Then $\Gamma := \{r, \theta, z\}$ is called a **cylindrical coordinate system**. let $\mathbf{E}$ be the symmetric idempotent for which $\text{Rng}\,\mathbf{E} = \mathcal{U}$ (see Prop.4 of Sect.41). Then the functions r and z are given by

$$r(x) = |\mathbf{E}(x - q)|, \quad z(x) = \mathbf{f} \cdot (x - q) \qquad \text{for all} \qquad x \in \mathcal{D}, \tag{74.22}$$

and θ is characterized by $\text{Rng}\,\theta =]0, 2\pi[$ and

$$\mathbf{h}(\theta(x)) = \frac{1}{r(x)}\mathbf{E}(x - q) \quad \text{for all} \quad x \in \mathcal{D}. \tag{74.23}$$

The values of the coordinates of a point x are indicated in Fig.2; the argument x is omitted to avoid clutter.

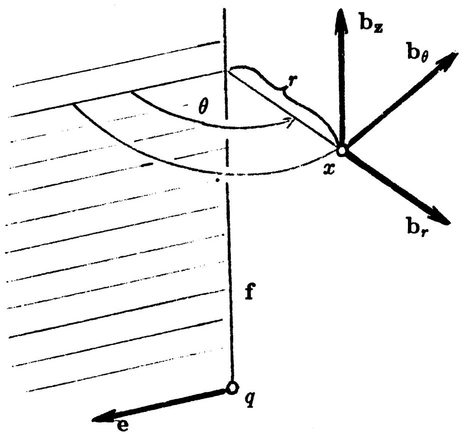

Figure 2.

If we interpret the first term in a triple as the r-term, the second as the θ-term, and the third as the z-term of a family indexed on $\Gamma = \{r, \theta, z\}$, then the notation γ and ψ above is in accord with the one used in Sect.1. By the same reasoning as used in Example (B), we infer from (74.20) that the basis field $\mathbf{b}$ of Γ is given by

$$\mathbf{b}_r = \mathbf{h} \circ \theta, \quad \mathbf{b}_\theta = r(\mathbf{h}^\bullet \circ \theta), \quad \mathbf{b}_z = \mathbf{f}. \tag{74.24}$$

The matrices G and $\overline{G}$ of the inner-product components of Γ are diagonal matrices and their diagonals are given by

$$G_{rr} = 1,\ G_{\theta\theta} = r^2,\ G_{zz} = 1, \tag{74.25}$$

$$\overline{G}^{rr} = 1,\ \overline{G}^{\theta\theta} = \frac{1}{r^2},\ \overline{G}^{zz} = 1. \tag{74.26}$$

The relation (74.14) remains valid and the only non-zero connection components of Γ are again given by (74.15). The formulas (74.16), (74.17), and (74.18) must be replaced by

$$\operatorname{div} \mathbf{h} = \frac{1}{r}(r\mathbf{h}^r)_{;r} + \mathbf{h}^\theta{}_{;\theta} + \mathbf{h}^z{}_{;z}, \tag{74.27}$$

$$\Delta f = \frac{1}{r}(rf_{;r})_{;r} + \frac{1}{r^2} f_{;\theta;\theta} + f_{;z;z}, \tag{74.28}$$

and

$$\begin{aligned}
[\mathrm{div}\,\mathbf{T}]^r &= \frac{1}{r}(r\mathbf{T}^{rr})_{;r} + \mathbf{T}^{r\theta}{}_{;\theta} + \mathbf{T}^{rz}{}_{;z} - r\mathbf{T}^{\theta\theta}, \\
[\mathrm{div}\,\mathbf{T}]^\theta &= \frac{1}{r}(r\mathbf{T}^{\theta r})_{;r} + \mathbf{T}^{\theta\theta}{}_{;\theta} + \mathbf{T}^{\theta z}{}_{;z} + \frac{1}{r}(\mathbf{T}^{r\theta} + \mathbf{T}^{\theta r}), \\
[\mathrm{div}\,\mathbf{T}]^z &= \frac{1}{r}(r\mathbf{T}^{zr})_{;r} + \mathbf{T}^{z\theta}{}_{;\theta} + \mathbf{T}^{zz}{}_{;z}.
\end{aligned} \tag{74.29}$$

(Brackets are omitted on the right sides.)

(D) Spherical Coordinates: We assume $\dim \mathcal{E} = 3$ and prescribe q, $\mathbf{f}$, and $\mathbf{J}$ as in Example (C). We replace the formula (74.20) by

$$\Psi(s,t,u) := q + s(\cos(t)\mathbf{f} + \sin(t)\mathbf{h}(u)). \tag{74.30}$$

The definition of $\overline{\mathcal{D}}$ in (74.21) must be replaced by

$$\overline{\mathcal{D}} := \mathbb{P}^\times \times]0,\pi[\times]0,2\pi[, \tag{74.31}$$

but $\mathcal{D}$ remains the same. Then $\psi := \Psi|_{\overline{\mathcal{D}}}^{\mathcal{D}}$ is invertible, and the functions r, θ, φ, all with domain $\mathcal{D}$, are defined by $\gamma := \psi^{\leftarrow}$ and $(r,\theta,\varphi) := \gamma|^{\mathbb{R}^3}$. Then $\Gamma := \{r,\theta,\varphi\}$ is called a **spherical coordinate system**. The functions r and θ are given by

$$r(x) = |x-q|, \quad \theta(x) = \arccos\left(\frac{\mathbf{f}\cdot(x-q)}{r(x)}\right) \quad \text{for all} \quad x \in \mathcal{D} \tag{74.32}$$

and φ is characterized by $\mathrm{Rng}\,\varphi =]0,2\pi[$ and

$$\mathbf{h}(\varphi(x)) = \frac{1}{r(x)\sin(\theta(x))}\mathbf{E}(x-q) \quad \text{for all} \quad x \in \mathcal{D} \tag{74.33}$$

where $\mathbf{E}$ is defined as in Example (C). The values of the coordinates of a point x are indicated in Fig.3, again with arguments omitted.

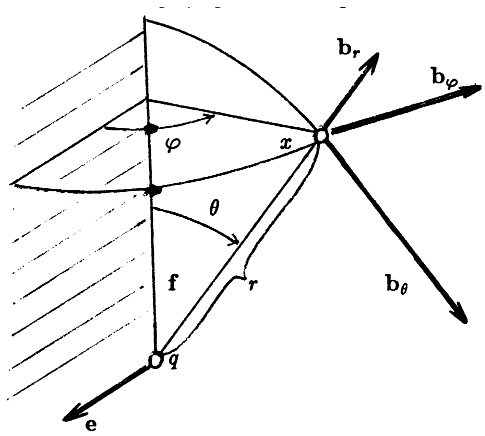

Figure 3.

By the same procedure as used in the previous examples, we find that the basis field $\mathbf{b}$ of Γ is given by

$$\begin{aligned} \mathbf{b}_r &= (\cos\circ\theta)\mathbf{f} + (\sin\circ\theta)(\mathbf{h}\circ\varphi), \\ \mathbf{b}_\theta &= r(-(\sin\circ\theta)\mathbf{f} + (\cos\circ\theta)(\mathbf{h}\circ\varphi)), \\ \mathbf{b}_\varphi &= r(\sin\circ\theta)(\mathbf{h}^\bullet\circ\varphi). \end{aligned} \tag{74.34}$$

The matrices G and $\overline{G}$ of the inner-product components are again diagonal matrices and their diagonals are given by

$$G_{rr} = 1,\ G_{\theta\theta} = r^2,\ G_{\varphi\varphi} = r^2(\sin\circ\theta)^2, \tag{74.35}$$

$$\overline{G}^{rr} = 1,\ \overline{G}^{\theta\theta} = \frac{1}{r^2},\ \overline{G}^{\varphi\varphi} = \frac{1}{r^2(\sin\circ\theta)^2}. \tag{74.36}$$

By (73.11) we have $\det\circ G = r^4(\sin\circ\theta)^2$ and hence $g := \sqrt{|\det\circ G|}$ becomes

$$g = r^2(\sin\circ\theta). \tag{74.37}$$

Using (73.6), we find that the only non-zero connection components are

$$C_\theta{}^\theta{}_r = C_r{}^\theta{}_\theta = C_\varphi{}^\varphi{}_r = C_r{}^\varphi{}_\varphi = \frac{1}{r}, \qquad C_\theta{}^r{}_\theta = -r, \tag{74.38}$$

$$C_\varphi{}^\varphi{}_\theta = C_\theta{}^\varphi{}_\varphi = \frac{1}{\tan\circ\theta},\ C_\varphi{}^r{}_\varphi = -r(\sin\circ\theta)^2,\ C_\varphi{}^\theta{}_\varphi = -(\sin\cos)\circ\theta.$$

The formulas (73.20) and (73.22) for the divergence and for the Laplacian specialize to

$$\text{div}\,\mathbf{h} = \frac{1}{r^2}(r^2\mathbf{h}^r)_{;r} + \frac{1}{\sin\circ\theta}((\sin\circ\theta)\mathbf{h}^\theta)_{;\theta} + \mathbf{h}^\varphi{}_{;\varphi} \tag{74.39}$$

and

$$\Delta f = \frac{1}{r^2}(r^2 f_{;r})_{;r} + \frac{1}{r^2(\sin\circ\theta)}((\sin\circ\theta)f_{;\theta})_{;\theta} + \frac{1}{r^2(\sin\circ\theta)^2}f_{;\varphi;\varphi}. \tag{74.40}$$

the formula (73.21) for the divergence of a lineon field gives

$$\begin{aligned}
[\text{div}\,\mathbf{T}]^r &= \frac{1}{r^2}(r^2\mathbf{T}^{rr})_{;r} + \frac{1}{\sin\circ\theta}((\sin\circ\theta)\mathbf{T}^{r\theta})_{;\theta} + \mathbf{T}^{r\varphi}{}_{;\varphi}\\
&\quad - r\mathbf{T}^{\theta\theta} - r(\sin\circ\theta)^2\mathbf{T}^{\varphi\varphi},\\
[\text{div}\,\mathbf{T}]^\theta &= \frac{1}{r^2}(r^2\mathbf{T}^{\theta r})_{;r} + \frac{1}{\sin\circ\theta}((\sin\circ\theta)\mathbf{T}^{\theta\theta})_{;\theta} + \mathbf{T}^{\theta\varphi}{}_{;\varphi}\\
&\quad + \frac{1}{r}(\mathbf{T}^{r\theta} + \mathbf{T}^{\theta r}) - ((\sin\cos)\circ\theta)\mathbf{T}^{\varphi\varphi},\\
[\text{div}\,\mathbf{T}]^\varphi &= \frac{1}{r^2}(r^2\mathbf{T}^{\varphi r})_{;r} + \frac{1}{\sin\circ\theta}((\sin\circ\theta)\mathbf{T}^{\varphi\theta})_{;\theta} + \mathbf{T}^{\varphi\varphi}{}_{;\varphi}\\
&\quad + \frac{1}{r}(\mathbf{T}^{r\varphi} + \mathbf{T}^{\varphi r}) + \frac{1}{\tan\circ\theta}(\mathbf{T}^{\varphi\theta} + \mathbf{T}^{\theta\varphi}).
\end{aligned} \tag{74.41}$$

(Brackets are omitted on the right sides to avoid clutter.)

Notes 74

(1) Unfortunately, there is no complete agreement in the literature on what letters to use for which coordinates. Often, the letter φ is used for our θ and vice versa. In cylindrical coordinates, the letter ρ is often used for our r. In spherical coordinates, one sometimes finds ω for our φ.

(2) Most of the literature is very vague about how one should choose the domain $\mathcal{D}$ for each of the curvilinear coordinate systems discussed in this section.

75 Problems for Chapter 7

(1) Let Γ be a coordinate system as in Def.1 of Sect.71 and let $\boldsymbol{\beta}$ be the dual basis field of Γ. Let $p : I \to \mathcal{D}$ be a twice differentiable process on some interval $I \in \text{Sub}\,\mathbb{R}$. Define the component-functions of $p^\bullet$ and $p^{\bullet\bullet}$ by

$$[p^\bullet]^c := (\boldsymbol{\beta}^c \circ p)p^\bullet, \quad [p^{\bullet\bullet}]^c := (\boldsymbol{\beta}^c \circ p)p^{\bullet\bullet} \tag{P7.1}$$

for all $c \in \Gamma$.

(a) Show that

$$[p^\bullet]^c = (c \circ p)^\bullet \tag{P7.2}$$

and

$$[p^{\bullet\bullet}]^c = (c \circ p)^{\bullet\bullet} + \textstyle\sum_{(d,e)\in\Gamma^2} ({C_d}^c{}_e \circ p)(d \circ p)^\bullet (e \circ p)^\bullet \tag{P7.3}$$

for all $c \in \Gamma$, where C denotes the family of connection components.

(b) Write out the formula $(P7.3)$ for cylindrical and spherical coordinates.

(2) Let Γ be a coordinate system as in Def.1 of Sect.71 and let $\mathbf{b}$ be the basis field and $\boldsymbol{\beta}$ the dual basis field of Γ. If $\mathbf{F}$ is a field whose codomain is $\mathrm{Lin}(\mathcal{V}, \mathrm{Lin}\mathcal{V}) \cong \mathrm{Lin}_2(\mathcal{V}^2, \mathcal{V})$, then the component family $[\mathbf{F}] \in (\mathrm{Map}\,(\mathrm{Dom}\,\mathbf{F}, \mathbb{R}))^{\Gamma^3}$ of $\mathbf{F}$ is given by

$$[\mathbf{F}]^c{}_{de} := \boldsymbol{\beta}^c \mathbf{F}(\mathbf{b}_e, \mathbf{b}_d) \quad \text{for all} \quad c, d, e \in \Gamma. \tag{P7.4}$$

(a) Show: The components of the gradient $\nabla\mathbf{T}$ of a differentiable lineon field $\mathbf{T}$ are given by

$$[\nabla\mathbf{T}]^c{}_{de} = [\mathbf{T}]^c{}_{d;e} + \textstyle\sum_{f\in\Gamma} \left([\mathbf{T}]^f{}_d \; {C_f}^c{}_e - [\mathbf{T}]^c{}_f \; {C_d}^f{}_e\right) \tag{P7.5}$$

for all $c, d, e \in \Gamma$.

(b) Show that if the connection components are of class C^1, they satisfy

$${C_c}^d{}_{e;f} - {C_c}^d{}_{f;e} + \textstyle\sum_{g\in\Gamma} \left({C_c}^g{}_e \; {C_g}^d{}_f - {C_c}^g{}_f \; {C_g}^d{}_e\right) = 0 \tag{P7.6}$$

for all $c, d, e, f \in \Gamma$. (Hint: Apply the Theorem on Symmetry of Second Gradients to $\mathbf{b}_c$ and use Part (a).)

(3) Let Γ be a coordinate system on an open subset $\mathcal{D}$ of a genuine Euclidean space $\mathcal{E}$ with translation space $\mathcal{V}$ and let $\mathbf{b}$ be the basis field of

Γ. Assume that the system is orthogonal in the sense that $\mathbf{b}_c \bullet \mathbf{b}_d = 0$ for all $c, d \in \Gamma$ with $c \neq d$. Define

$$b_c := |\mathbf{b}_c| \quad \text{for all} \quad c \in \Gamma \tag{P7.7}$$

and

$$\mathbf{e}_c := \tfrac{\mathbf{b}_c}{b_c} \quad \text{for all} \quad c \in \Gamma, \tag{P7.8}$$

so that, for each $x \in \mathcal{D}$, $\mathbf{e}(x) := (\mathbf{e}_c(x) \,|\, c \in \Gamma)$ is an orthonormal basis of $\mathcal{V}$. If $\mathbf{h}$ is a vector field with $\operatorname{Dom} \mathbf{h} \subset \mathcal{D}$, we define the family $\langle \mathbf{h} \rangle := (\langle \mathbf{h} \rangle_c \,|\, c \in \Gamma)$ of **physical components** of $\mathbf{h}$ by $\langle \mathbf{h} \rangle(x) := \operatorname{lnc}^{-1}_{\mathbf{e}(x)} \mathbf{h}(x)$ for all $x \in \mathcal{D}$, so that

$$\langle \mathbf{h} \rangle_c = \mathbf{h} \bullet \mathbf{e}_c \quad \text{for all} \quad c \in \Gamma. \tag{P7.9}$$

Physical components of fields of other types are defined analogously.

(a) Derive a set of formulas that express the connection components in terms of the functions b_c, and $b_{c;d}$, $c, d \in \Gamma$. (Hint: Use the Theorem on Connection Components of Sect.73.)

(b) Show that the physical components of a vector field $\mathbf{h}$ are related to the components $[\mathbf{h}]^c$ and $[\mathbf{h}]_c$, $c \in \Gamma$, by

$$[\mathbf{h}]_c = b_c \langle \mathbf{h} \rangle_c, \quad [\mathbf{h}]^c = \tfrac{1}{b_c} \langle \mathbf{h} \rangle_c \quad \text{for all} \quad c \in \Gamma \tag{P7.10}$$

(c) Show that the physical components of the gradient $\nabla \mathbf{h}$ of a differentiable vector field $\mathbf{h}$ are given by

$$\begin{aligned} \langle \nabla \mathbf{h} \rangle_{c,d} &= \tfrac{1}{b_d} \langle \mathbf{h} \rangle_{c;d} - \tfrac{b_{d;c}}{b_d b_c} \langle \mathbf{h} \rangle_d \qquad \text{if } c \neq d, \\ \langle \nabla \mathbf{h} \rangle_{c,c} &= \tfrac{1}{b_c} \langle \mathbf{h} \rangle_{c;c} + \sum_{e \in \Gamma \setminus \{c\}} \tfrac{b_{c;e}}{b_c b_e} \langle \mathbf{h} \rangle_e \end{aligned} \tag{P7.11}$$

for all c, $d \in \Gamma$.

(4) Let a 2-dimensional genuine inner-product space $\mathcal{E}$ be given. Assume that a point $q \in \mathcal{E}$, an orthonormal basis $(\mathbf{e}, \mathbf{f})$ and a number $\varepsilon \in \mathbb{P}^\times$ have been prescribed. Consider the mapping $\Psi : (\varepsilon + \mathbb{P}^\times) \times]0, \varepsilon[\to \mathcal{E}$ defined by

$$\Psi(s,t) := q + \tfrac{1}{\varepsilon}(st\,\mathbf{e} + \sqrt{(s^2-\varepsilon^2)(\varepsilon^2-t^2)}\,\mathbf{f}). \tag{P7.12}$$

(a) Compute the partial derivatives $\Psi_{,1}$ and $\Psi_{,2}$ and show that Ψ is locally invertible.

(b) Show that Ψ is injective and hence that $\mathcal{D} := \operatorname{Rng}\Psi$ is open and that $\psi := \Psi|^{\operatorname{Rng}}$ is invertible.

(c) Put $\gamma := \psi^{\leftarrow}$ and define the functions λ, μ from $\mathcal{D}$ to $\mathbb{R}$ by $(\lambda, \mu) := \gamma|^{\mathbb{R}^2}$. Show that $\Gamma := \{\lambda, \mu\}$ is a coordinate-system on $\mathcal{D}$; it is called an **elliptical coordinate system**.

(d) Show that the coordinate curves corresponding to the coordinates λ and μ are parts of ellipses and hyperbolas, respectively, whose foci are $q - \varepsilon\mathbf{e}$ and $q + \varepsilon\mathbf{e}$. Show that $\mathcal{D} = q + \mathbb{P}^{\times}\mathbf{e} + \mathbb{P}^{\times}\mathbf{f}$.

(e) Using Part (a), write down the basis field $(\mathbf{b}_i \mid i \in \{\lambda, \mu\})$ of the system $\{\lambda, \mu\}$, and show that the inner-product components are given by

$$G_{\lambda,\mu} = 0, \quad G_{\lambda,\lambda} = \tfrac{\lambda^2-\mu^2}{\lambda^2-\varepsilon^2}, \quad G_{\mu,\mu} = \tfrac{\lambda^2-\mu^2}{\varepsilon^2-\mu^2}. \tag{P7.13}$$

(f) Show that the Laplacian of a twice differentiable scalar field f with $\operatorname{Dom} f \subset \mathcal{D}$ is given by

$$\Delta f = \frac{\sqrt{(\lambda^2-\varepsilon^2)(\varepsilon^2-\mu^2)}}{\lambda^2-\mu^2}\left(\left(\sqrt{\tfrac{\lambda^2-\varepsilon^2}{\varepsilon^2-\mu^2}}\,f_{;\lambda}\right)_{;\lambda} + \left(\sqrt{\tfrac{\varepsilon^2-\mu^2}{\lambda^2-\varepsilon^2}}\,f_{;\mu}\right)_{;\mu}\right). \tag{P7.14}$$

(g) Compute the connection components of the system $\{\lambda, \mu\}$.

(5) Let a 3-dimensional genuine inner-product space $\mathcal{E}$ with translation space $\mathcal{V}$ be given. Assume that $q, \mathbf{f}$ and $\mathbf{J}$ are prescribed as in Example (C) of Sect.74 and that $\mathbf{h} : \mathbb{R} \to \mathcal{V}$ is defined by (74.4). Define $\Psi : \mathbb{P}^{\times} \times \mathbb{P}^{\times} \times \mathbb{R} \to \mathcal{E}$ by

$$\Psi(s,t,u) := q + \tfrac{1}{2}(s^2 - t^2)\mathbf{f} + st\,\mathbf{h}(u). \tag{P7.15}$$

(a) Compute the partial derivatives of Ψ and show that Ψ is locally invertible.

(b) Specify an open subset $\overline{\mathcal{D}}$ of $\mathbb{P}^{\times} \times \mathbb{P}^{\times} \times \mathbb{R}$ and an open subset $\mathcal{D}$ of $\mathcal{E}$ such that $\psi := \Psi|_{\overline{\mathcal{D}}}^{\mathcal{D}}$ is invertible.

(c) Show that $\gamma := \psi^{\leftarrow}$ and $(\alpha, \beta, \theta) := \gamma|^{\mathbb{R}^3}$ define a coordinate system $\Gamma := \{\alpha, \beta, \theta\}$ on $\mathcal{D}$; it is called a **paraboloidal coordinate system**.

(d) Show that the coordinate curves corresponding to the coordinates α and β are parabolas with focus q.

(e) Using Part (a), write down the basis field $(\mathbf{b}_i \mid i \in \{\alpha, \beta, \theta\})$ of the system and compute the inner-product components.

(f) Find the formula for the Laplacian of a twice differentiable scalar field in paraboloidal coordinates.

(g) Compute the connection components of the system $\{\alpha, \beta, \theta\}$.

(6) Let Γ be a coordinate system on an open subset $\mathcal{D}$ of a Euclidean space $\mathcal{E}$ and let $\boldsymbol{\beta}$ be the dual basis field of Γ. Let $\overline{G}^{cd}$, $c, d \in \Gamma$, be defined by (73.2) and g by (73.15).

(a) Show that the Laplacian of the coordinate $c \in \Gamma$ is given by

$$\Delta c = \frac{1}{g} \sum_{d \in \Gamma} (g \overline{G}^{dc})_{;d}. \tag{P7.16}$$

(b) Given $c \in \Gamma$, show that the components of the Laplacian of $\boldsymbol{\beta}^c$ are given by

$$[\Delta \boldsymbol{\beta}^c]_d = \left(\frac{1}{g} \sum_{e \in \Gamma} \left(g \overline{G}^{ec} \right)_{;e} \right)_{;d}. \tag{P7.17}$$

(Hint: Use $\boldsymbol{\beta}^c = \nabla c$ and Part (a)).

(c) Let $\mathbf{h}$ be a twice differentiable vector field with $\operatorname{Dom} \mathbf{h} \subset \mathcal{D}$. Show that the components of the Laplacian of $\mathbf{h}$ are given by

$$[\triangle \mathbf{h}]_c = \triangle [\mathbf{h}]_c +$$

$$\sum_{(d,e) \in \Gamma^2} \left(\left(\frac{1}{g} (g \overline{G}^{ed})_{;e} \right)_{;c} [\mathbf{h}]_d - 2 \sum_{f \in \Gamma} C_c{}^d{}_f \, \overline{G}^{bfe} [\mathbf{h}]_{d;e} \right) \tag{P7.18}$$

for all $c \in \Gamma$, where $C_c{}^d{}_f$, $c, d, f \in \Gamma$, are the connection components of Γ. (Hint: Use Prop.5 of Sect.67.)

(d) Write out the formula $(P7.18)$ for cylindrical coordinates.