Chapter 3

Flat Spaces

In this chapter, the term "linear space" will be used as a shorthand for "linear space over the real field $\mathbb{R}$". (Actually, many definitions remain meaningful and many results remain valid when $\mathbb{R}$ is replaced by an arbitrary field. The interested reader will be able to decide for himself when this is the case.)

31 Actions of Groups

Let $\mathcal{E}$ be a set and consider the permutation group $\operatorname{Perm}\mathcal{E}$, which consists of all invertible mappings from $\mathcal{E}$ onto itself. Perm $\mathcal{E}$ is a group under composition, and the identity mapping $1_{\mathcal{E}}$ is the neutral of this group (see Sect. 06).

Definition 1: *Let $\mathcal{E}$ be a non-empty set and $\mathcal{G}$ a group. By an* **action of $\mathcal{G}$ on $\mathcal{E}$** *we mean a group homomorphism $\tau : \mathcal{G} \to \operatorname{Perm}\mathcal{E}$.*

We write τ_g for the value of τ at $g \in \mathcal{G}$. To say that τ is an action of $\mathcal{G}$ on $\mathcal{E}$ means that $\tau_{\operatorname{cmb}(g,h)} = \tau_g \circ \tau_h$ for all $g, h \in \mathcal{G}$, where cmb denotes the combination of $\mathcal{G}$ (see Sect. 06). If n is the neutral of $\mathcal{G}$, then $\tau_n = 1_{\mathcal{E}}$, and if $\operatorname{rev}(g)$ is the group-reverse of $g \in \mathcal{G}$, then $\tau_{\operatorname{rev}(g)} = \tau_g^{\leftarrow}$.

Examples:

1. If $\mathcal{G}$ is any subgroup of $\operatorname{Perm}\mathcal{E}$, then the inclusion mapping $1_{\mathcal{G}\subset\operatorname{Perm}\mathcal{E}}$ is an action of $\mathcal{G}$ on $\mathcal{E}$. For subgroups of $\operatorname{Perm}\mathcal{E}$ it is always understood that the action is this inclusion.

2. If $\mathcal{E}$ is a set with a prescribed structure, then the automorphisms of $\mathcal{E}$ form a subgroup of $\operatorname{Perm}\mathcal{E}$ which acts on $\mathcal{E}$ by inclusion in $\operatorname{Perm}\mathcal{E}$ as

explained in Example 1. For instance, if $\mathcal{V}$ is a linear space, then the lineon group Lis$\mathcal{V}$ acts on $\mathcal{V}$.

3. If $\mathcal{V}$ is a linear space, define $\tau : \mathcal{V} \to \operatorname{Perm}\mathcal{V}$ by letting $\tau_{\mathbf{v}}$ be the operation of adding $\mathbf{v} \in \mathcal{V}$, so that $\tau_{\mathbf{v}}(\mathbf{u}) := \mathbf{u} + \mathbf{v}$ for all $\mathbf{u} \in \mathcal{V}$. It is easily seen that τ is a homomorphism of the additive group of $\mathcal{V}$ into the permutation group Perm$\mathcal{V}$, i.e., τ is an action of $\mathcal{V}$ on itself.

4. If $\mathcal{V}$ is a linear space, define $\operatorname{mult} : \mathbb{P}^\times \to \operatorname{Perm}\mathcal{V}$ by letting $\operatorname{mult}_\lambda := \lambda\mathbf{1}_\mathcal{V}$ be the operation of taking the λ-multiple, so that $\operatorname{mult}_\lambda(\mathbf{v}) = \lambda\mathbf{v}$ for all $\mathbf{v} \in \mathcal{V}$. It is evident that mult is a homomorphism of the multiplicative group $\mathbb{P}^\times$ into Perm$\mathcal{V}$ and hence an action of $\mathbb{P}^\times$ on $\mathcal{V}$. $\operatorname{mult}_>(\mathbb{P}^\times)$ is a subgroup of Lis$\mathcal{V}$ that is isomorphic to $\mathbb{P}^\times$; namely, $\operatorname{mult}_>(\mathbb{P}^\times) = \mathbb{P}^\times\mathbf{1}_\mathcal{V}$. ■

Let τ be an action of $\mathcal{G}$ on $\mathcal{E}$. We define a relation $\sim_\tau$ on $\mathcal{E}$ by

$$x \sim_\tau y \quad :\Longleftrightarrow \quad y = \tau_g(x) \quad \text{for some} \quad g \in \mathcal{G}. \tag{31.1}$$

It is easily seen that $\sim_\tau$ is an equivalence relation. The equivalence classes are called the **orbits** in $\mathcal{E}$ under the action τ of $\mathcal{G}$. If $x \in \mathcal{E}$, then $\{\tau_g(x) \mid g \in \mathcal{G}\}$ is the orbit to which x belongs, also called *the orbit of* x.

Definition 2: *An action τ of a group $\mathcal{G}$ on a non-empty set $\mathcal{E}$ is said to be* **transitive** *if for all x, $y \in \mathcal{E}$ there is $g \in \mathcal{G}$ such that $\tau_g(x) = y$. The action is said to be* **free** *if τ_g, with $g \in \mathcal{G}$, can have a fixed point* (see Sect.03) *only if $g = n$, the neutral of $\mathcal{G}$.*

To say that an action is transitive means that all of $\mathcal{E}$ is the only orbit under the action. Of course, since $\tau_n = 1_\mathcal{E}$, all points in $\mathcal{E}$ are fixed points of τ_n.

If the action τ of $\mathcal{G}$ on $\mathcal{E}$ is free, one easily sees that $\tau : \mathcal{G} \to \operatorname{Perm}\mathcal{E}$ must be injective, and hence that $\tau_>(\mathcal{G})$ is a subgroup of Perm$\mathcal{E}$ that is an isomorphic image of $\mathcal{G}$.

Proposition 1: *An action τ of a group $\mathcal{G}$ on a non-empty set $\mathcal{E}$ is both transitive and free if and only if for all x, $y \in \mathcal{E}$ there is exactly one $g \in \mathcal{G}$ such that $\tau_g(x) = y$.*

Proof: Assume the action is both transitive and free and that x, $y \in \mathcal{E}$ are given. If $\tau_g(x) = y$ and $\tau_h(x) = y$, then $x = (\tau_g)^{\leftarrow}(y) = \tau_{\operatorname{rev}(g)}(y) = \tau_{\operatorname{rev}(g)}(\tau_h(x)) = (\tau_{\operatorname{rev}(g)} \circ \tau_h)(x) = \tau_{\operatorname{cmb}(\operatorname{rev}(g),h)}(x)$. Since the action is free, it follows that $\operatorname{cmb}(\operatorname{rev}(g), h) = n$ and hence $g = h$. Thus there can be at most one $g \in \mathcal{G}$ such that $\tau_g(x) = y$. The transitivity of the action assures the existence of such g.

Assume now that the condition is satisfied. The action is then transitive. Let $\tau_g(x) = x$ for some $x \in \mathcal{E}$. Since $\tau_n(x) = 1_{\mathcal{E}}(x) = x$ it follows from the uniqueness that $g = n$. Hence the action is free. ■

32 Flat Spaces and Flats

Definition 1: *A* **flat space** *is a non-empty set $\mathcal{E}$ endowed with structure by the prescription of*

(i) *a commutative subgroup $\mathcal{V}$ of* $\mathrm{Perm}\,\mathcal{E}$ *whose action is transitive.*

(ii) *a mapping* $\mathrm{sm} : \mathbb{R} \times \mathcal{V} \to \mathcal{V}$ *which makes $\mathcal{V}$ a linear space when composition is taken as the addition and* sm *as the scalar multiplication in $\mathcal{V}$.*

The linear space $\mathcal{V}$ is then called the **translation space** *of $\mathcal{E}$.*

The elements of $\mathcal{E}$ are called **points** and the elements of $\mathcal{V}$ **translations** or **vectors**. If $\xi \in \mathbb{R}$ and $\mathbf{v} \in \mathcal{V}$ we write, as usual, $\xi\mathbf{v}$ for $\mathrm{sm}(\xi, \mathbf{v})$. In $\mathcal{V}$, we use additive notation for composition, i.e., we write $\mathbf{u} + \mathbf{v}$ for $\mathbf{v} \circ \mathbf{u}$, $-\mathbf{v}$ for $\mathbf{v}^{\leftarrow}$, and $\mathbf{0}$ for $1_{\mathcal{E}}$.

Proposition 1: *The action of the translation space $\mathcal{V}$ on the flat space $\mathcal{E}$ is free.*

Proof: Suppose that $\mathbf{v} \in \mathcal{V}$ satisfies $\mathbf{v}(x) = x$ for some $x \in \mathcal{E}$. Let $y \in \mathcal{E}$ be given. Since the action of $\mathcal{V}$ is transitive, we may choose $\mathbf{u} \in \mathcal{V}$ such that $y = \mathbf{u}(x)$. Using the commutativity of the group $\mathcal{V}$, we find that

$$\mathbf{v}(y) = \mathbf{v}(\mathbf{u}(x)) = (\mathbf{v} \circ \mathbf{u})(x) = (\mathbf{u} \circ \mathbf{v})(x) = \mathbf{u}(\mathbf{v}(x)) = \mathbf{u}(x) = y.$$

Since $y \in \mathcal{E}$ was arbitrary, it follows that $\mathbf{v} = 1_{\mathcal{E}}$. ■

This Prop.1 and Prop.1 of Sect.31 have the following immediate consequence:

Proposition 2: *There is exactly one mapping*

$$\mathrm{diff} : \mathcal{E} \times \mathcal{E} \longrightarrow \mathcal{V}$$

such that

$$(\mathrm{diff}(x, y))(y) = x \quad \text{for all} \quad x, y \in \mathcal{E}. \tag{32.1}$$

We call $\mathrm{diff}(x, y) \in \mathcal{V}$ *the* **point-difference** *between x and y and use the following simplified notations:*

$$\begin{aligned} x - y &:= \mathrm{diff}(x, y) \quad \text{when } x, y \in \mathcal{E}, \\ x + \mathbf{v} &:= \mathbf{v}(x) \quad \text{when } x \in \mathcal{E}, \mathbf{v} \in \mathcal{V}. \end{aligned}$$

The following rules are valid for all $x, y, z \in \mathcal{E}$ and for all $\mathbf{u}, \mathbf{v} \in \mathcal{V}$.

$$\begin{array}{rcl} x - x & = & \mathbf{0}, \\ x - y & = & -(y - x), \\ (x - y) + (y - z) & = & x - z, \\ (x + \mathbf{u}) + \mathbf{v} & = & x + (\mathbf{u} + \mathbf{v}), \\ x + \mathbf{v} = y & \Longleftrightarrow & \mathbf{v} = y - x. \end{array}$$

In short, an equation involving points and vectors is valid if it is valid according to the rules of ordinary algebra, provided that all expressions occurring in the equation make sense. Of course, an expression such as $x + y$ does *not* make sense when $x, y \in \mathcal{E}$.

The following notations, which are similar to notations introduced in Sect.06., are very suggestive and useful. They apply when $x \in \mathcal{E}, \mathbf{v} \in \mathcal{V}$, $\mathcal{H}, \mathcal{Y} \in \operatorname{Sub}\mathcal{E}$, $\mathcal{S} \in \operatorname{Sub}\mathcal{V}$.

$$\begin{array}{rcl} x + \mathcal{S} & := & \{x + \mathbf{s} \mid \mathbf{s} \in \mathcal{S}\}, \\ \mathcal{H} + \mathbf{v} & := & \mathbf{v}_{>}(\mathcal{H}) = \{x + \mathbf{v} \mid x \in \mathcal{H}\}, \\ \mathcal{H} + \mathcal{S} & := & \bigcup_{\mathbf{s} \in \mathcal{S}} \mathbf{s}_{>}(\mathcal{H}) \;=\; \{x + \mathbf{s} \mid x \in \mathcal{H}, \mathbf{s} \in \mathcal{S}\}, \\ \mathcal{H} - \mathcal{Y} & := & \{x - y \mid x \in \mathcal{H}, y \in \mathcal{Y}\}. \end{array}$$

The first three of the above are subsets of $\mathcal{E}$, the last is a subset of $\mathcal{V}$. We have $\mathcal{V} = \mathcal{E} - \mathcal{E}$ and we sometimes use $\mathcal{E}$ - $\mathcal{E}$ as a *notation* for the translation space of $\mathcal{E}$.

Flat spaces serve as mathematical models for physical planes and spaces. The vectors, i.e. the mappings in $\mathcal{V}$ (except the identity mapping $\mathbf{0} = 1_{\mathcal{E}}$), are to be interpreted as parallel shifts (hence the term “translation”). If a vector $\mathbf{v} \in \mathcal{V}$ is given, we can connect points $x, y, \ldots$, with their images $x + \mathbf{v}, y + \mathbf{v}, \ldots$, by drawing arrows as shown in Figure 1. In this sense, vectors can be represented pictorially by arrows. The commutativity of $\mathcal{V}$ is illustrated in Figure 2. The assumption that the action of $\mathcal{V}$ is transitive corresponds to the fact that given any two points x and y, there exists a vector that carries x to y.

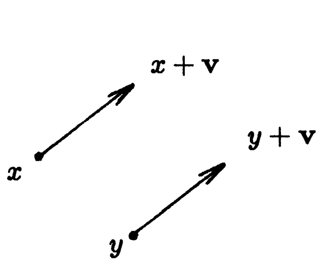


Figure 1.

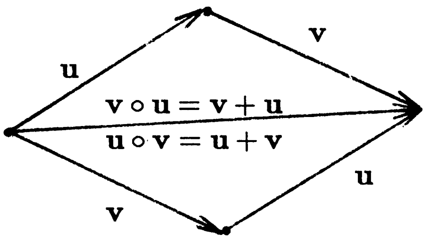


Figure 2.

It often happens that a set $\mathcal{E}$, a linear space $\mathcal{V}$, and an action of the additive group of $\mathcal{V}$ on $\mathcal{E}$ are given. If this action is transitive and injective, then $\mathcal{E}$ acquires the structure of a flat space whose translation space is the isomorphic image of $\mathcal{V}$ in $\operatorname{Perm}\mathcal{E}$ under the given action. Under such circumstances, we identify $\mathcal{V}$ with its isomorphic image and, using poetic license, call $\mathcal{V}$ itself the translation space of $\mathcal{E}$. If we wish to emphasize the fact that $\mathcal{V}$ becomes the translation space only after such an identification, we say that $\mathcal{V}$ acts as an **external translation space** of $\mathcal{E}$.

Let $\mathcal{V}$ be a linear space. The action $\tau : \mathcal{V} \to \operatorname{Perm}\mathcal{V}$ described in Example 3 of the previous section is easily seen to be transitive and injective. Hence we have the following trivial but important result:

Proposition 3: *Every linear space $\mathcal{V}$ has the natural structure of a flat space. The space $\mathcal{V}$ becomes its own external translation space by associating with each $\mathbf{v} \in \mathcal{V}$ the mapping $\mathbf{u} \mapsto \mathbf{u} + \mathbf{v}$ from $\mathcal{V}$ to itself.*

The linear-space structure of $\mathcal{V}$ embodies more information than its flat-space structure. As a linear space, $\mathcal{V}$ has a distinguished element, $\mathbf{0}$, but as a flat space it is homogeneous in the sense that all of its elements are of equal standing. Roughly, the flat structure of $\mathcal{V}$ is obtained from its linear structure by forgetting where $\mathbf{0}$ is.

Note that the operations involving points and vectors introduced earlier, when applied to the flat structure of a linear space, reduce to ordinary linear-space operations. For example, point-differences reduce to ordinary differences.

Of course, the set $\mathbb{R}$ of reals has the natural structure of a flat space whose (external) translation space is $\mathbb{R}$ itself, regarded as a linear space.

We now assume that a flat space $\mathcal{E}$ with translation space $\mathcal{V}$ is given. A subset $\mathcal{F}$ of $\mathcal{E}$ will inherit from $\mathcal{E}$ the structure of a flat space if the translations of $\mathcal{E}$ that leave $\mathcal{F}$ invariant can serve, after adjustment (see Sect.03), as translations of $\mathcal{F}$. The precise definition is this:

Definition 2: *A non-empty subset $\mathcal{F}$ of $\mathcal{E}$ is called flat subspace of $\mathcal{E}$, or simply a* **flat** *in $\mathcal{E}$, if the set*

$$\mathcal{U} := \{\mathbf{u} \in \mathcal{V} \mid \mathcal{F} + \mathbf{u} \subset \mathcal{F}\} \tag{32.2}$$

is a linear subspace of $\mathcal{V}$ whose additive group acts transitively on $\mathcal{F}$ under the action which associates with every $\mathbf{u} \in \mathcal{U}$ the mapping $\mathbf{u}_{|\mathcal{F}}$ of $\mathcal{F}$ onto itself.

The action of $\mathcal{U}$ on $\mathcal{F}$ described in this definition endows $\mathcal{F}$ with the natural structure of a flat space whose (external) translation space is $\mathcal{U}$. We say that $\mathcal{U}$ is the **direction space** of $\mathcal{F}$. The following result is immediate from Def.2.

Proposition 4: *Let $\mathcal{U}$ be a subspace of $\mathcal{V}$. A non-empty subset $\mathcal{F}$ of $\mathcal{E}$ is a flat with direction space $\mathcal{U}$ if and only if*

$$\mathcal{F} - \mathcal{F} \subset \mathcal{U} \text{ and } \mathcal{F} + \mathcal{U} \subset \mathcal{F}. \tag{32.3}$$

Let $\mathcal{U}$ be a subspace of $\mathcal{V}$. The inclusion of the additive group of $\mathcal{U}$ in the group $\text{Perm}\,\mathcal{E}$ is an action of $\mathcal{U}$ on $\mathcal{E}$. This action is transitive only when $\mathcal{U} = \mathcal{V}$. The orbits in $\mathcal{E}$ under the action of $\mathcal{U}$ are exactly the flats with direction space $\mathcal{U}$. Since $x + \mathcal{U}$ is the orbit of $x \in \mathcal{E}$ we obtain:

Proposition 5: *Let $\mathcal{U}$ be a subspace of $\mathcal{V}$. A subset $\mathcal{F}$ of $\mathcal{E}$ is a flat with direction space $\mathcal{U}$ if and only if it is of the form*

$$\mathcal{F} = x + \mathcal{U} \tag{32.4}$$

for some $x \in \mathcal{E}$.

We say that two flats are **parallel** if the direction space of one of them is included in the direction space of the other. For example, two flats with the same direction space are parallel; if they are distinct, then their intersection is empty.

The following result is an immediate consequence of Prop.4.

Proposition 6: *The intersection of any collection of flats in $\mathcal{E}$ is either empty or a flat in $\mathcal{E}$. If it is not empty, then the direction space of the intersection is the intersection of the direction spaces of the members of the collection.*

In view of the remarks on span-mappings made in Sect.03, we have the following result.

Proposition 7: *Given any non-empty subset $\mathcal{S}$ of $\mathcal{E}$, there is a unique smallest flat that includes $\mathcal{S}$. More precisely: There is a unique flat that includes $\mathcal{S}$ and is included in every flat that includes $\mathcal{S}$. It is called the* **flat**

span *of* $\mathcal{S}$ *and is denoted by* $\mathrm{Fsp}\,\mathcal{S}$*. We have* $\mathrm{Fsp}\,\mathcal{S} = \mathcal{S}$ *if and only if* $\mathcal{S}$ *is a flat.*

If x and y are two distinct points in $\mathcal{E}$, then

$$\overleftrightarrow{xy} := \mathrm{Fsp}\{x, y\} \tag{32.5}$$

is called the **line** *passing through* x *and* y.

It is easily seen that, if $\mathcal{S} \subset \mathcal{E}$ and if $q \in \mathcal{S}$, then $\mathrm{Lsp}(\mathcal{S} - q)$ is the direction space of $\mathrm{Fsp}\,\mathcal{S}$ and hence, by Prop.5,

$$\mathrm{Fsp}\,\mathcal{S} = q + \mathrm{Lsp}(\mathcal{S} - q). \tag{32.6}$$

Let $\mathcal{E}_1, \mathcal{E}_2$ be flat spaces with translation spaces $\mathcal{V}_1, \mathcal{V}_2$. The set-product $\mathcal{E}_1 \times \mathcal{E}_2$ then has the natural structure of a flat space as follows: The (external) translation space of $\mathcal{E}_1 \times \mathcal{E}_2$ is the product space $\mathcal{V}_1 \times \mathcal{V}_2$. The action of $\mathcal{V}_1 \times \mathcal{V}_2$ on $\mathcal{E}_1 \times \mathcal{E}_2$ is defined by associating with each $(\mathbf{v}_1, \mathbf{v}_2) \in \mathcal{V}_1 \times \mathcal{V}_2$ the mapping $(x_1, x_2) \mapsto (x_1 + \mathbf{v}_1, x_2 + \mathbf{v}_2)$ of $\mathcal{E}_1 \times \mathcal{E}_2$ onto itself. More generally, if $(\mathcal{E}_i \mid i \in I)$ is any family of flat spaces, then $\times(\mathcal{E}_i \mid i \in I)$ has the natural structure of a flat space whose (external) translation space is the product-space $\times(\mathcal{V}_i \mid i \in I)$, where $\mathcal{V}_i$ is the translation space of $\mathcal{E}_i$ for each $i \in I$. The action of $\times(\mathcal{V}_i \mid i \in I)$ on $\times(\mathcal{E}_i \mid i \in I)$ is defined by

$$(x_i \mid i \in I) + (\mathbf{v}_i \mid i \in I) := (x_i + \mathbf{v}_i \mid i \in I). \tag{32.7}$$

We say that a flat space $\mathcal{E}$ is **finite-dimensional** if its translation space is finite-dimensional. If this is the case, we define the **dimension** of $\mathcal{E}$ by

$$\dim\,\mathcal{E} := \dim\,(\mathcal{E} - \mathcal{E}). \tag{32.8}$$

Let $\mathcal{E}$ be any flat space. The only flat in $\mathcal{E}$ whose direction space is $\mathcal{V} := \mathcal{E} - \mathcal{E}$ is $\mathcal{E}$ itself. The singleton subsets of $\mathcal{E}$ are the zero-dimensional flats, their direction space is the zero-subspace $\{\mathbf{0}\}$ of $\mathcal{V}$. The one-dimensional flats are called **lines**, the two-dimensional flats are called **planes**, and, if $n := \dim \mathcal{E} \in \mathbb{N}^\times$, then the $(n-1)$-dimensional flats are called **hyperplanes**. The only n-dimensional flat is $\mathcal{E}$ itself.

Let $\mathcal{V}$ be a linear space, which is its own translation space when regarded as a flat space (see Prop.3). A subset $\mathcal{U}$ of $\mathcal{V}$ is then a subspace of $\mathcal{V}$ if and only if it is a flat in $\mathcal{V}$ and contains $\mathbf{0}$.

Notes 32

(1) The traditional term for our "flat space" is "affine space". I believe that the term "flat space" is closer to the intuitive content of the concept.

(2) A fairly rigorous definition of the concept of a flat space, close to the one given here, was given 65 years ago by H. Weyl (Raum, Zeit, Materie; Springer, 1918). After all these years, the concept is still not used as much as it should be. One finds explanations (often bad ones) of the concept in some recent geometry and abstract algebra textbooks, but the concept is almost never introduced in analysis. One of my reasons for writing this book is to remedy this situation, for I believe that flat spaces furnish the most appropriate conceptual background for analysis.

(3) Strictly speaking, the term "point" for an element of a flat space and the term "vector" for an element of its translation space are appropriate only if the flat space is used as a primitive structure to describe some geometrical-physical reality. Sometimes a flat space may come up as a derived structure, and then the terms "point" and "vector" may not be appropriate to physical reality. Nevertheless, we will use these terms when dealing with the general theory.

(4) What we call a "flat" or a "flat subspace" is sometimes called a "linear manifold", "translated subspace", or "affine subset", especially when it is a subset of a linear space.

(5) What we call "flat span" is sometimes called "affine hull" and the notation $\mathrm{aff}\mathcal{S}$ is then used instead of $\mathrm{Fsp}\,\mathcal{S}$.

33 Flat Mappings

Let $\mathcal{E}$, $\mathcal{E}'$ be flat spaces with translation spaces $\mathcal{V}$, $\mathcal{V}'$. We say that a mapping $\alpha : \mathcal{E} \to \mathcal{E}'$ is *flat* if, roughly, it preserves translations and scalar multiples of translations. This means that if any two points in $\mathcal{E}$ are related by a given translation $\mathbf{v}$, their α-values must be related by a corresponding translation $\mathbf{v}'$, and if $\mathbf{v}$ is replaced by $\xi\mathbf{v}$ then $\mathbf{v}'$ can be replaced by $\xi\mathbf{v}'$. The precise statement is this:

Definition 1: *A mapping $\alpha : \mathcal{E} \to \mathcal{E}'$ is called a* **flat mapping** *if for every $\mathbf{v} \in \mathcal{V}$, there is a $\mathbf{v}' \in \mathcal{V}'$ such that*

$$\alpha \circ (\xi\mathbf{v}) = (\xi\mathbf{v}') \circ \alpha \quad \text{for all } \xi \in \mathbb{R}. \tag{33.1}$$

If we evaluate (33.1) at $x \in \mathcal{E}$ and take $\xi := 1$, we see that $\mathbf{v}' = \alpha(x+\mathbf{v}) - \alpha(x)$ is uniquely determined by $\mathbf{v}$. Thus, the following result is immediate from the definition.

Proposition 1: *A mapping $\alpha : \mathcal{E} \to \mathcal{E}'$ is flat if and only if there is a mapping $\nabla\alpha : \mathcal{V} \to \mathcal{V}'$ such that*

$$(\nabla\alpha)(\mathbf{v}) = \alpha(x+\mathbf{v}) - \alpha(x) \quad \text{for all} \quad x \in \mathcal{E} \quad \text{and } \mathbf{v} \in \mathcal{V}, \tag{33.2}$$

and

$$(\nabla\alpha)(\xi\mathbf{v}) = \xi(\nabla\alpha(\mathbf{v})) \quad \text{for all} \quad \mathbf{v} \in \mathcal{V} \text{ and } \xi \in \mathbb{R}. \tag{33.3}$$

The mapping $\nabla\alpha$ is uniquely determined by α and is called the **gradient** *of α.*

Note that condition (33.2) is equivalent to

$$\alpha(x) - \alpha(y) = (\nabla\alpha)(x - y) \quad \text{for all} \quad x, y \in \mathcal{E}. \tag{33.4}$$

Proposition 2: *The gradient $\nabla\alpha$ of a flat mapping α is linear, i.e., $\nabla\alpha \in \mathrm{Lin}(\mathcal{V}, \mathcal{V}')$.*

Proof: Choose $x \in \mathcal{E}$. Using condition (33.2) three times, we see that

$$\begin{aligned}(\nabla\alpha)(\mathbf{v}+\mathbf{u}) &= \alpha(x + (\mathbf{v}+\mathbf{u})) - \alpha(x) \\ &= (\alpha((x+\mathbf{v})+\mathbf{u}) - \alpha(x+\mathbf{v})) + (\alpha(x+\mathbf{v}) - \alpha(x)) \\ &= (\nabla\alpha)(\mathbf{u}) + (\nabla\alpha)(\mathbf{v})\end{aligned}$$

is valid for all $\mathbf{u}, \mathbf{v} \in \mathcal{V}$. Hence $\nabla\alpha$ preserves addition. The condition (33.3) states that $\nabla\alpha$ preserves scalar multiplication. ∎

Theorem on Specification of Flat Mappings: *Let $q \in \mathcal{E}$, $q' \in \mathcal{E}'$ and $\mathbf{L} \in \mathrm{Lin}(\mathcal{V}, \mathcal{V}')$ be given. Then there is exactly one flat mapping $\alpha : \mathcal{E} \to \mathcal{E}'$ such that $\alpha(q) = q'$ and $\nabla\alpha = \mathbf{L}$. It is given by*

$$\alpha(x) := q' + \mathbf{L}(x - q) \quad \text{for all} \quad x \in \mathcal{E}. \tag{33.5}$$

Proof: Using (33.4) with y replaced by q we see that (33.5) must be valid when α satisfies the required conditions. Hence α is uniquely determined. Conversely, if $\alpha : \mathcal{E} \to \mathcal{E}'$ is defined by (33.5), one easily verifies that it is flat and satisfies the conditions. ∎

Examples:

1. All constant mappings from one flat space into another are flat. A flat mapping is constant if and only if its gradient is zero.

2. If $\mathcal{F}$ is a flat in $\mathcal{E}$, then the inclusion $1_{\mathcal{F}\subset\mathcal{E}}$ is flat. Its gradient is the inclusion $\mathbf{1}_{\mathcal{U}\subset\mathcal{V}}$ of the direction space $\mathcal{U}$ of $\mathcal{F}$ in the translation space $\mathcal{V}$ of $\mathcal{E}$.

3. A function $a : \mathbb{R} \to \mathbb{R}$ is flat if and only if it is of the form $a = \xi\iota + \eta$ with $\xi, \eta \in \mathbb{R}$. The gradient of a is $\xi\iota \in \mathrm{Lin}\mathbb{R}$, which one identifies with the number $\xi \in \mathbb{R}$ (see Sect.25). The graph of a is a straight line with slope ξ. This explains our use of the term "gradient" (gradient means slope or inclination).

4. The translations $\mathbf{v} \in \mathcal{V}$ of a flat space $\mathcal{E}$ are flat mappings of $\mathcal{E}$ into itself. They all have the same gradient, namely $\mathbf{1}_{\mathcal{V}}$. In fact, a flat mapping from $\mathcal{E}$ into itself is a translation if and only if its gradient is $\mathbf{1}_{\mathcal{V}}$.

5. Let $\mathcal{V}$ and $\mathcal{V}'$ be linear spaces, regarded as flat spaces (see Prop.3 of Sect.32). A mapping $\mathbf{L} : \mathcal{V} \to \mathcal{V}'$ is linear if and only if it is flat and preserves zero, i.e. $\mathbf{L0} = \mathbf{0}'$. If this is the case, then $\mathbf{L}$ is its own gradient.

6. Let $(\mathcal{E}_i \mid i \in I)$ be a family of flat spaces and let $(\mathcal{V}_i \mid i \in I)$ be the family of their translation spaces. Given $j \in I$, the evaluation mapping $\mathrm{ev}_j^{\mathcal{E}} : \times(\mathcal{E}_i \mid i \in I) \to \mathcal{E}_j$ (see Sect.04) is a flat mapping when $\times(\mathcal{E}_i \mid i \in I)$ is endowed with the natural flat-space structure described in the previous section. The gradient of $\mathrm{ev}_j^{\mathcal{E}}$ is the evaluation mapping $\mathrm{ev}_j^{\mathcal{V}} : \times(\mathcal{V}_i \mid i \in I) \to \mathcal{V}_j$ (see Sect.14).

7. If $\mathcal{E}$ is a flat space and $\mathcal{V} := \mathcal{E} - \mathcal{E}$, then $((x, \mathbf{v}) \mapsto x + \mathbf{v}) : \mathcal{E} \times \mathcal{V} \to \mathcal{E}$ is a flat mapping. Its gradient is the vector-addition $\mathcal{V} \times \mathcal{V} \to \mathcal{V}$.

8. The point-difference mapping defined by (32.1) is flat. Its gradient is the vector-difference mapping

$$((\mathbf{u}, \mathbf{v}) \mapsto (\mathbf{u} - \mathbf{v})) : \mathcal{V} \times \mathcal{V} \longrightarrow \mathcal{V}. \blacksquare$$

Proposition 3: *Let $\alpha : \mathcal{E} \to \mathcal{E}'$ be a flat mapping, $\mathcal{F}$ a flat in $\mathcal{E}$ with direction space $\mathcal{U}$, and $\mathcal{F}'$ a flat in $\mathcal{E}'$ with direction space $\mathcal{U}'$. Then:*

(i) *$\alpha_>(\mathcal{F})$ is a flat in $\mathcal{E}'$ with direction space $(\nabla\alpha)_>(\mathcal{U})$.*

(ii) *$\alpha^<(\mathcal{F}')$ is either empty or a flat in $\mathcal{E}$ with direction space $(\nabla\alpha)^<(\mathcal{U}')$.*

Proof: Using (33.4) and (33.2) one easily verifies that the conditions of Prop.4 of Sect.32 are verified in the situations described in (i) and (ii). $\blacksquare$

If $\mathcal{F}' := \{x'\}$ is a singleton in Prop.3, then $\mathcal{U}' = \{\mathbf{0}\}$, and (ii) states that $\alpha^<(\{x'\})$ is either empty or a flat in $\mathcal{E}$ whose direction space is $\mathrm{Null}\,\nabla\alpha$. If $\mathcal{F} := \mathcal{E}$, then $\mathcal{U} = \mathcal{V}$ and (i) states that $\mathrm{Rng}\,\alpha$ is a flat in $\mathcal{E}'$ whose direction space is $\mathrm{Rng}\,\nabla\alpha$. Therefore, the following result is an immediate consequence of Prop.3.

Proposition 4: *A flat mapping is injective [surjective] if and only if its gradient is injective [surjective].*

The following two rules describe the behavior of flat mappings with respect to composition and inversion. They follow immediately from Prop.1.

Chain Rule for Flat Mappings: *The composite of two flat mappings is again flat, and the gradient of their composite is the composite of their gradients. More precisely: If* $\mathcal{E}, \mathcal{E}', \mathcal{E}''$ *are flat spaces and* $\alpha : \mathcal{E} \to \mathcal{E}'$ *and* $\beta : \mathcal{E}' \to \mathcal{E}''$ *are flat mappings, then* $\beta \circ \alpha : \mathcal{E} \to \mathcal{E}''$ *is again a flat mapping and*

$$\nabla(\beta \circ \alpha) = \nabla\beta\nabla\alpha. \tag{33.6}$$

Inversion Rule for Flat Mappings: *A flat mapping* $\alpha : \mathcal{E} \to \mathcal{E}'$ *is invertible if and only if its gradient* $\nabla\alpha \in \mathrm{Lin}(\mathcal{V}, \mathcal{V}')$ *is invertible. If this is the case, then the inverse* $\alpha^{\leftarrow} : \mathcal{E}' \to \mathcal{E}$ *is also flat and*

$$\nabla(\alpha^{\leftarrow}) = (\nabla\alpha)^{-1}. \tag{33.7}$$

Hence every invertible flat mapping is a **flat isomorphism***.*

The set of all flat automorphisms of flat space $\mathcal{E}$, i.e. all flat isomorphism from $\mathcal{E}$ to itself, is denoted by $\mathrm{Fis}\,\mathcal{E}$. It is a subgroup of $\mathrm{Perm}\,\mathcal{E}$. Let $\mathcal{V}$ be the translation space of $\mathcal{E}$ and let $h : \mathrm{Fis}\,\mathcal{E} \to \mathrm{Lis}\mathcal{V}$ be defined by $h(\alpha) := \nabla\alpha$ for all $\alpha \in \mathrm{Fis}\,\mathcal{E}$. It follows from the Chain Rule and Inversion Rule for Flat Mappings that h is a group-homomorphism. It is easily seen that h is surjective and that the kernel of h is the translation group $\mathcal{V}$ (see Sect.06).

Notes 33

(1) The traditional terms for our "flat mapping" are "affine mapping" or "affine transformation".

(2) In one recent textbook, the notation $\alpha^{\sharp}$ and the unfortunate term "trace" are used for what we call the "gradient" $\nabla\alpha$ of a flat mapping α.

34 Charge Distributions, Barycenters, Mass-Points

Let $\mathcal{E}$ be a flat space with translation space $\mathcal{V}$. By a **charge distribution** γ on $\mathcal{E}$ we mean a family in $\mathbb{R}$, indexed on $\mathcal{E}$ with finite support, i.e., $\gamma \in \mathbb{R}^{(\mathcal{E})}$. We may think of the term γ_x of γ at x as an electric charge placed at the point x. The *total charge* of the distribution γ is its sum (see (15.2))

$$\mathrm{sum}_{\mathcal{E}}\gamma = \sum_{x\in\mathcal{E}} \gamma_x = \sum_{x\in \mathrm{Supt}\,\gamma} \gamma_x. \tag{34.1}$$

Definition 1: *We say that the charge distributions* $\gamma, \beta \in \mathbb{R}^{(\mathcal{E})}$ *are* **equivalent**, *and we write* $\gamma \sim \beta$, *if* γ *and* β *have the same total charge, i.e.* $\mathrm{sum}_{\mathcal{E}}\gamma = \mathrm{sum}_{\mathcal{E}}\beta$, *and if*

$$\sum_{x\in\mathcal{E}} \gamma_x(x-q) = \sum_{y\in\mathcal{E}} \beta_y(y-q) \quad \text{for all} \quad q \in \mathcal{E}. \tag{34.2}$$

It is easily verified that the relation $\sim$ thus defined is indeed an equivalence relation on $\mathbb{R}^{(\mathcal{E})}$. Moreover, we have $\gamma \sim \beta$ if and only if $\gamma - \beta \sim 0$. Also, one easily proves the following result:

Proposition 1: *If* $\gamma, \beta \in \mathbb{R}^{(\mathcal{E})}$ *have the same total charge, i.e.* $\mathrm{sum}_{\mathcal{E}}\gamma = \mathrm{sum}_{\mathcal{E}}\beta$, *and if (34.2) holds for some* $q \in \mathcal{E}$, *then* γ, β *are equivalent, i.e., (34.2) holds for all* $q \in \mathcal{E}$.

Given any $x \in \mathcal{E}$ we define $\delta_x \in \mathbb{R}^{(\mathcal{E})}$ by

$$(\delta_x)_y = \left\{ \begin{array}{lll} 0 & \text{if} & x \neq y \\ 1 & \text{if} & x = y \end{array} \right\}; \tag{34.3}$$

in words, the distribution δ_x places a unit charge at x and no charge anywhere else. Actually, δ_x is the same as the x-term $\delta_x^{\mathcal{E}}$ of the standard basis $\delta^{\mathcal{E}}$ of $\mathbb{R}^{(\mathcal{E})}$ as defined in Sect.16.

The following result is an immediate consequence of Prop.1.

Proposition 2: *Given* $\gamma \in \mathbb{R}^{(\mathcal{E})}$, $b \in \mathcal{E}$ *and* $\sigma \in \mathbb{R}$, *we have* $\gamma \sim \sigma\delta_b$ *if and only if*

$$\sigma = \mathrm{sum}_{\mathcal{E}}\gamma \quad \text{and} \quad \sum_{x\in\mathcal{E}} \gamma_x(x-b) = 0.$$

The next result states that every charge distribution with a non-zero total charge is equivalent to a single charge.

Theorem on Unique Existence of Barycenters: *Let* $\gamma \in \mathbb{R}^{(\mathcal{E})}$ *be given so that* $\sigma := \mathrm{sum}_{\mathcal{E}}\gamma \neq 0$. *Then there is exactly one* $b \in \mathcal{E}$ *such that* $\gamma \sim \sigma\delta_b$. *We have*

$$b = q + \frac{1}{\sigma}\sum_{x\in\mathcal{E}} \gamma_x(x-q) \quad \text{for all} \quad q \in \mathcal{E}. \tag{34.4}$$

The point b *is called the* **barycenter** *of the distribution* γ.

Proof: Assume that $b \in \mathcal{E}$ is such that $\gamma \sim \sigma\delta_b$. Using (34.2) with $\beta := \sigma\delta_b$ we get $\sum(\gamma_x(x-q) \mid x \in \mathcal{E}) = \sigma(b-q)$ for all $q \in \mathcal{E}$. Since $\sigma \neq 0$, it follows that (34.4) must hold for each $q \in \mathcal{E}$. This proves the uniqueness of $b \in \mathcal{E}$. To prove existence, we choose $q \in \mathcal{E}$ arbitrarily and *define* $b \in \mathcal{E}$ by (34.4). Using Prop.1, it follows immediately that $\gamma \sim \sigma\delta_b$. ∎

Proposition 3: *The barycenter of a charge distribution with non-zero total charge belongs to the flat span of the support of the distribution. More precisely, if* $\gamma \in \mathbb{R}^{(\mathcal{E})}, b \in \mathcal{E}$ *and* $\sigma \in \mathbb{R}^{\times}$ *are such that* $\gamma \sim \sigma\delta_b$*, then* $b \in \operatorname{Fsp}(\operatorname{Supt}\gamma)$.

Proof: Let $\mathcal{F}$ be any flat such that $\operatorname{Supt}\gamma \subset \mathcal{F}$. Then $\gamma|_{\mathcal{F}} \in \mathbb{R}^{(\mathcal{F})}$ is a charge distrubution on $\mathcal{F}$ and has a barycenter in $\mathcal{F}$. But this is also the barycenter b in $\mathcal{E}$ and hence we have $b \in \mathcal{F}$. Since $\mathcal{F}$, with $\operatorname{Supt}\gamma \subset \mathcal{F}$, was arbitrary, it follows that $b \in \operatorname{Fsp}(\operatorname{Supt}\gamma)$. ∎

We call a pair $(\mu, p) \in \mathbb{P}^{\times} \times \mathcal{E}$ a **mass-point**. We may think of it as describing a particle of mass μ placed at p. By the *barycenter* b of a non-empty finite family $((\mu_i, p_i) \mid i \in I)$ of mass-points we mean the barycenter of the distribution $\sum(\mu_i\delta_{p_i} \mid i \in I)$. It is characterized by

$$\sum_{i\in I} \mu_i(p_i - b) = 0 \tag{34.5}$$

and satisfies

$$b = q + \frac{1}{\operatorname{sum}_I \mu} \sum_{i\in I} \mu_i(p_i - q) \tag{34.6}$$

for all $q \in \mathcal{E}$.

Proposition 4: *Let* Π *be a partition of a given non-empty finite set* I *and let* $((\mu_i, p_i) \mid i \in I)$ *be a family of mass-points. For every* $J \in \Pi$*, let* $\lambda_J := \operatorname{sum}_J(\mu|_J)$ *and let* q_J *be the barycenter of the family* $((\mu_j, p_j) \mid j \in J)$*. Then* $((\mu_i, p_i) \mid i \in I)$ *and* $((\lambda_J, q_J) \mid J \in \Pi)$ *have the same barycenter.*

Proof: Using (34.6) with I replaced by J, b by q_J and q by the barycenter b of $((\mu_i, p_i) \mid i \in I)$, we find that

$$\lambda_J(q_J - b) = \sum_{j\in J} \mu_j(p_j - b)$$

holds for all $J \in \Pi$. Using the summation rule (07.5), we obtain

$$\sum_{J\in\Pi} \lambda_J(q_J - b) = \sum_{i\in I} \mu_i(p_i - b) = \mathbf{0},$$

which proves that b is the barycenter of $((\lambda_J, q_J) \mid J \in \Pi)$. ∎

By the **centroid** of a non-empty finite family $(p_i \mid i \in I)$ of points we mean the barycenter of the family $((1, p_i) \mid i \in I)$.

Examples and Remarks:

1. If z is the barycenter of the pair $((\mu, x), (\lambda, y)) \in (\mathbb{P}^{\times} \times \mathcal{E})^2$ i.e. of the distribution $\mu\delta_x + \lambda\delta_y$, we say that z **divides** (x, y) **in the ratio** $\lambda : \mu$.

By (34.5) z is characterized by $\mu(x-z)+\lambda(y-z)=\mathbf{0}$ (see Figure 1). If $\lambda=\mu$, we say that z is the **midpoint** of (x,y).

Thus, the midpoint of (x,y) is just the centroid of (x,y).

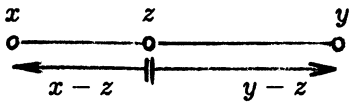


Figure 1.

2. Consider a triple (x_1,x_2,x_3) of points and let c be centroid of this triple. If we apply Prop.4 to the partition $\{\{1\},\{2,3\}\}$ of $3^{]}$ and to the case when $\mu_i=1$ for all $i\in 3^{]}$, we find that c is also the barycenter of $((1,x_1),(2,z))$, where z is the midpoint of (x_2,x_3). In other words, c divides (x_1,z) in the ratio 2:1. (see Figure 2). We thus obtain the following well known theorem of elementary geometry: The three medians of a triangle all meet at the centroid, which divides each in the ratio 2:1.

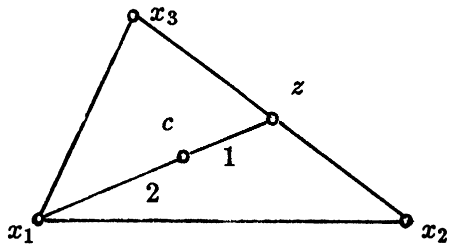


Figure 2.

3. Let c be the centroid of a quadruple (x_1,x_2,x_3,x_4) of points. If we apply Prop.4 to the partition $\{\{1,2\},\{3,4\}\}$ of $4^{]}$ and to the case when $\mu_i=1$ for all $i\in 4^{]}$, we find that c is also the midpoint of (y,z), where y is the midpoint of (x_1,x_2) and z the midpoint of (x_3,x_4) (see Figure 3). We thus obtain the following geometrical theorem: The four line-segments that join the midpoints of opposite edges of a tetrahedron all meet at the centroid of the tetrahedron, which is the midpoint of all four.

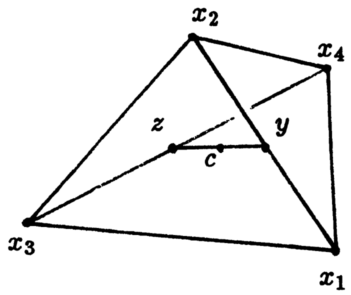


Figure 3.

If Prop.4 is applied to the partition $\{\{1,2,3\},\{4\}\}$ of $4^{]}$ and to the case when $\mu_i = 1$ for all $i \in 4^{]}$ one obtains the following geometric theorem: The four line segments that join the vertices of a tetrahedron to the centroids of the opposite faces all meet at the centroid of the tetrahedron, which divides each in the ratio 3:1.

4. If a non-empty finite family $((\mu_i, p_i) \mid i \in I)$ in $\mathbb{P}^\times \times \mathcal{E}$ is interpreted physically as a *system of point-particles* then the barycenter of the family is the *center of mass* of the system. If the system moves, then the places p_i will change in time, and so will the center of mass. Newtonian mechanics teaches that the center of mass moves on a straight line with constant speed if no external forces act on the system.

5. Let $\mathcal{E}$, with $\dim \mathcal{E} = 2$, be a mathematical model for a rigid, plane, horizontal plate. A distribution $\gamma \in \mathbb{R}^{(\mathcal{E})}$ can then be interpreted as a *system of vertical forces*. The term γ_x with $x \in \operatorname{Supt}\gamma$ gives the magnitude and direction of a force acting at x. We take γ_x as positive if the force acts downwards, negative if it acts upwards. If $\sigma := \operatorname{sum}_{\mathcal{E}}\gamma$ is not zero and if b is the barycenter of γ, then the force system is statically equivalent to a single *resultant force* σ applied at b; i.e. the plate can be held in equilibrium by applying on opposite force $-\sigma$ at b.

6. We can define an "addition" add: $(\mathbb{P}^\times \times \mathcal{E})^2 \to \mathbb{P}^\times \times \mathcal{E}$ on $\mathbb{P}^\times \times \mathcal{E}$ by

$$\operatorname{add}((\mu, x), (\lambda, y)) := (\mu + \lambda, z),$$

where z is the barycenter of the pair $((\mu, x), (\lambda, y))$, i.e., the point that divides (x, y) in the ratio $\lambda : \mu$. It is obvious that this addition is commutative. Application of Prop.4 to triples easily shows that this addition is also associative and hence endows $\mathbb{P}^\times \times \mathcal{E}$ with the structure of a commutative pre-monoid (see Sect.06). We use additive notation, i.e. we write

$$(\mu, x) + (\lambda, y) := \mathrm{add}((\mu, x), (\lambda, y)).$$

For any non-empty finite family $((\mu_i, p_i) \mid i \in I)$ in $\mathbb{P}^\times \times \mathcal{E}$ we then have

$$\sum_{i \in I} (\mu_i, p_i) = (\mathrm{sum}_I \mu, b),$$

where b is the barycenter of the family. It is easily shown that the pre-monoid $\mathbb{P}^\times \times \mathcal{E}$ is cancellative. ■

Pitfall: The statements "z divides (x, y) in the ratio $\lambda : \mu$" and "z is the midpoint of (x, y)" should *not* be interpreted as statements about the distances from z to x and y. There is no concept of distance in a flat space unless it is endowed with additional structure as in Chap.4. ■

Notes 34

(1) The terms "weight" or "mass" are often used instead of our "charge". The trouble with "weight" and "mass" is that they lead one to assume that their values are positive.

(2) There is no agreement in the literature about the terms "barycenter" and "centroid". Sometimes "centroid" is used for what we call "barycenter" and sometimes "barycenter" for what we call "centroid".

35 Flat Combinations

We assume that a flat space $\mathcal{E}$ with translation space $\mathcal{V}$ and a non-empty index set I are given. Recall that the summation mapping $\mathrm{sum}_I : \mathbb{R}^{(I)} \to \mathbb{R}$ defined by (15.2) is linear and hence flat. For each $\nu \in \mathbb{R}$, we write

$$(\mathbb{R}^{(I)})_\nu := \mathrm{sum}_I^{<}(\{\nu\}) = \{\lambda \in \mathbb{R}^{(I)} \mid \mathrm{sum}_I \lambda = \nu\}. \qquad (35.1)$$

It follows from Prop. 3 of Sect. 33 that $(\mathbb{R}^{(I)})_\nu$ is a flat in $\mathbb{R}^{(I)}$ whose direction space is $(\mathbb{R}^{(I)})_0$.

Definition 1: *The mapping*

$$\mathrm{flc}_p \;:\; (\mathbb{R}^{(I)})_1 \longrightarrow \mathcal{E},$$

defined so that $\text{flc}_p(\lambda)$ *is the barycenter of the distribution* $\sum(\lambda_i\delta_{p_i} \mid i \in I)$*, is called the* **flat-combination** *mapping for* p*. The value* $\text{flc}_p(\lambda)$ *is called the flat combination of* p *with coefficient family* λ*.*

By the Theorem on the Unique Existence of Barycenters of Sect.34, the mapping flc_p is characterized by

$$\sum_{i\in I} \lambda_i(p_i - \text{flc}_p(\lambda)) = 0 \quad \text{for all} \quad \lambda \in (\mathbb{R}^{(I)})_1 \tag{35.2}$$

and we have, for every $q \in \mathcal{E}$,

$$\text{flc}_p(\lambda) = q + \sum_{i\in I} \lambda_i(p_i - q) \quad \text{for all} \quad \lambda \in (\mathbb{R}^{(I)})_1. \tag{35.3}$$

If $\mathcal{F}$ is a flat in $\mathcal{E}$ and if $\text{Rng}\, p \subset \mathcal{F}$, one can apply Def.1 to $\mathcal{F}$ instead of $\mathcal{E}$ and obtain the flat-combination mapping $\text{flc}_p^{\mathcal{F}}$ for p relative to $\mathcal{F}$. It is easily seen that $\text{flc}_p^{\mathcal{F}}$ coincides with the adjustment $\text{flc}_p|^{\mathcal{F}}$.

Proposition 1: *The flat-combination mapping* flc_p *is a flat mapping. Its gradient is the restriction to* $(\mathbb{R}^{(I)})_0$ *of the linear-combination mapping* $\text{lnc}_{(p-q)}$ *for the family* $p - q := (p_i - q \mid i \in I)$ *in* $\mathcal{V}$*, no matter how* $q \in \mathcal{E}$ *is chosen.*

Proof: It follows from (35.3) that

$$\text{flc}_p(\lambda) - \text{flc}_p(\mu) = \sum_{i\in I}(\lambda_i - \mu_i)(p_i - q) = \text{lnc}_{p-q}(\lambda - \mu)$$

holds for all $\lambda, \mu \in (\mathbb{R}^{(I)})_1$. Comparison of this result with (33.4) gives the desired conclusion. ∎

If $\mathcal{S}$ is a non-empty subset of $\mathcal{E}$, we identify $\mathcal{S}$ with the family $(x \mid x \in \mathcal{S})$ indexed on $\mathcal{S}$ itself (see Sect.02). Thus, the flat combination mapping $\text{flc}_{\mathcal{S}} : (\mathbb{R}^{(\mathcal{S})})_1 \to \mathcal{E}$ has the following interpretation: For every $\gamma \in (\mathbb{R}^{(\mathcal{S})})_1$, which can be viewed as a charge distribution with total charge 1 and support included in $\mathcal{S}$, $\text{flc}_{\mathcal{S}}(\gamma)$ is the barycenter of γ.

Flat Span Theorem: *The set of all flat combinations of a non-empty family* p *of points in* $\mathcal{E}$ *is the flat span of the range of* p*,* i.e. $\text{Rng}\,\text{flc}_p = \text{Fsp}(\text{Rng}\, p)$*. In particular, if* $\mathcal{S}$ *is a non-empty subset of* $\mathcal{E}$*, then* $\text{Rng}\,\text{flc}_{\mathcal{S}} = \text{Fsp}\,\mathcal{S}$*.*

Proof: Noting that $\text{Rng}\,\text{flc}_p = \text{Rng}\,\text{flc}_{\mathcal{S}}$ if $\mathcal{S} = \text{Rng}\, p$, we see that it is sufficient to prove that $\text{Rng}\,\text{flc}_{\mathcal{S}} = \text{Fsp}\,\mathcal{S}$ for every non-empty $\mathcal{S} \in \text{Sub}\,\mathcal{E}$.

Let $x \in \mathcal{S}$. Then $\delta_x \in (\mathbb{R}^{(\mathcal{S})})_1$ and $\text{flc}_{\mathcal{S}}(\delta_x) = x$. Hence $x \in \text{Rng}\,\text{flc}_{\mathcal{S}}$. Since $x \in \mathcal{S}$ was arbitrary, it follows that $\mathcal{S} \subset \text{Rng}\,\text{flc}_{\mathcal{S}}$. Since $\text{flc}_{\mathcal{S}}$ is a flat mapping by Prop.1, it follows, by Prop.3 of Sect.33, that $\mathrm{Rng}\,\mathrm{flc}_{\mathcal{S}}$ is a flat in $\mathcal{E}$ and hence that $\mathcal{S} \subset \mathrm{Fsp}\,\mathcal{S} \subset \mathrm{Rng}\,\mathrm{flc}_{\mathcal{S}}$. On the other hand, by Prop.3 of Sect.34, we have $\mathrm{flc}_{\mathcal{S}}(\gamma) \in \mathrm{Fsp}(\mathrm{Supt}\,\gamma) \subset \mathrm{Fsp}\,\mathcal{S}$ for every $\gamma \in (\mathbb{R}^{(\mathcal{S})})_1$ and hence $\mathrm{Rng}\,\mathrm{flc}_{\mathcal{S}} \subset \mathrm{Fsp}\,\mathcal{S}$. ■

Applying the Flat Span Theorem to the case when $p := (x, y)$ is a pair of distinct points in $\mathcal{E}$, we find that the line $\overleftrightarrow{xy}$ passing through x and y (see (32.5)) is just the set of all flat combinations of (x, y):

$$\overleftrightarrow{xy} = \{\mathrm{flc}_{(x,y)}(\lambda, \mu) \mid \lambda, \mu \in \mathbb{R},\ \lambda + \mu = 1\}. \tag{35.4}$$

If $\mathcal{V}$ is a linear space, regarded as a flat space that is its own translation space, and if $\mathbf{f} := (\mathbf{f}_i \mid i \in I)$ is a family of elements of $\mathcal{V}$, then the flat combination mapping $\mathrm{flc}_{\mathbf{f}}$ for $\mathbf{f}$ is nothing but the restriction to $(\mathbb{R}^{(I)})_1$ of the linear combination mapping for $\mathbf{f}$, i.e. $\mathrm{flc}_{\mathbf{f}} = \mathrm{lnc}_{\mathbf{f}}|_{(\mathbb{R}^{(I)})_1}$. In other words we have

$$\sum_{i \in I} \lambda_i \mathbf{f}_i = \mathrm{flc}_{\mathbf{f}}(\lambda) \quad \text{for all} \quad \lambda \in (\mathbb{R}^{(I)})_1.$$

If $\mathcal{E}$ is a flat space that does not carry the structure of a linear space, and if $p := (p_i \mid i \in I)$ is a family in $\mathcal{E}$, we often use the **symbolic notation**

$$\sum_{i \in I} \lambda_i p_i := \mathrm{flc}_p(\lambda) \quad \text{for all} \quad \lambda \in (\mathbb{R}^{(I)})_1, \tag{35.5}$$

even though the terms $\lambda_i p_i$ do not make sense by themselves. If $p := (p_1, p_2)$ is a pair of points, we write

$$\lambda_1 p_1 + \lambda_2 p_2 := \mathrm{flc}_{(p_1,p_2)}(\lambda_1, \lambda_2) \tag{35.6}$$

for all $(\lambda_1, \lambda_2) \in (\mathbb{R}^2)_1$, i.e. for all $\lambda_1, \lambda_2 \in \mathbb{R}$ with $\lambda_1 + \lambda_2 = 1$. Similar notations are used for other lists with a small number of terms.

If p is a family of points in $\mathcal{E}$ and $\mathbf{f}$ a family of vectors in $\mathcal{V}$, both with the same index set I, then

$$p + \mathbf{f} := (p_i + \mathbf{f}_i \mid i \in I) \tag{35.7}$$

is also a family of points in $\mathcal{E}$.

Definition 2: *A non-empty family p of points in $\mathcal{E}$ is said to be* **flatly independent, flatly spanning**, *or a* **flat basis** *of $\mathcal{E}$ if the flat-combination mapping flc_p is injective, surjective, or invertible, respectively.*

By the Flat Span Theorem, the family p is flatly spanning if and only if the flat span of its range is all of $\mathcal{E}$.

Proposition 2: *Let $p = (p_i \mid i \in I)$ be a non-empty family of points in $\mathcal{E}$ and let $k \in I$. Then p is flatly independent, flatly spanning, or a flat basis of $\mathcal{E}$ according as the family $\mathbf{f} := (p_i - p_k \mid i \in I \setminus \{k\})$ of vectors is linearly independent, spanning, or a basis of $\mathcal{V}$, respectively.*

Proof: Using (35.3) with the choice $q := p_k$ we see that

$$\mathrm{flc}_p(\lambda) = p_k + \mathrm{lnc}_{\mathbf{f}}(\lambda|_{I\setminus\{k\}}) \quad \text{for all} \quad \lambda \in (\mathbb{R}^{(I)})_1. \tag{35.8}$$

Since the mapping $(\lambda \mapsto \lambda\,|_{I\setminus\{k\}}) \;:\; (\mathbb{R}^{(I)})_1 \to \mathbb{R}^{(I\setminus\{k\})}$ is easily seen to be invertible, it follows from (35.8) that flc_p is injective, surjective, or invertible according as $\mathrm{lnc}_{\mathbf{f}}$ is injective, surjective, or invertible, respectively. In view of Def.2 of Sect.15, this is the desired result. ■

The following result follows from Prop.2 and the Theorem on Characterization of Dimension (Sect.17).

Proposition 3: *Let $\mathcal{E}$ be a finite-dimensional flat space and let $p := (p_i \mid i \in I)$ be a family of points in $\mathcal{E}$.*

(a) *If p is flatly independent then I is finite and $\sharp I \le \dim \mathcal{E} + 1$, with equality if and only if p is a flat basis.*

(b) *If p is flatly spanning, then $\sharp I \ge \dim \mathcal{E} + 1$, with equality if and only if p is a flat basis.*

Let $p := (p_i \mid i \in I)$ be a flat basis of $\mathcal{E}$. Then, by Def.2, for every $x \in \mathcal{E}$ there is a unique $\lambda \in (\mathbb{R}^{(I)})_1$ such that $x = \mathrm{flc}_p(\lambda)$. The family λ is called the family of **barycentric coordinates** of x relative to the flat basis p.

The following result states that the flat mappings are exactly those that "preserve" flat combinations. We leave its proof to the reader (see Problem 7 below).

Proposition 4: *Let $\mathcal{E}, \mathcal{E}'$ be flat spaces. A mapping $\alpha : \mathcal{E} \to \mathcal{E}'$ is flat if and only if for every non-empty family p of points in $\mathcal{E}$ we have*

$$\alpha \circ \mathrm{flc}_p = \mathrm{flc}_{\alpha\circ p}. \tag{35.9}$$

The following result is an analogue, and a consequence, of Prop.2 of Sect.16.

Proposition 5: *Let $\mathcal{E}, \mathcal{E}'$ be flat spaces. Let $p := (p_i \mid i \in I)$ be a flat basis of $\mathcal{E}$ and let $p' := (p'_i \mid i \in I)$ be a family of points in $\mathcal{E}'$. Then there is a unique flat mapping $\alpha : \mathcal{E} \to \mathcal{E}'$ such that $\alpha \circ p = p'$. This α is injective, surjective, or invertible depending on whether p' is flatly independent, flatly spanning, or a flat basis, respectively.*

Notes 35

(1) In the case when p is a list, the flat-combination $\text{flc}_p(\lambda)$ is sometimes called the "average" of p with "weights" λ.

(2) The term "frame" is sometimes used for what we call a "flat basis".

36 Flat Functions

In this section, we assume that $\mathcal{E}$ is a *finite-dimensional* flat space with translation space $\mathcal{V}$. We denote the set of all flat mappings from $\mathcal{E}$ into $\mathbb{R}$ by $\text{Flf}\,\mathcal{E}$ and call its members **flat functions**. Given any $q \in \mathcal{E}$, we use the notation

$$\text{Flf}_q\mathcal{E} := \{a \in \text{Flf}\,\mathcal{E} \mid a(q) = 0\} \tag{36.1}$$

for the set of all flat functions that have the value 0 at the point q. The set of all real-valued constants with domain $\mathcal{E}$ will be denoted by $\mathbb{R}_{\mathcal{E}}$. Note that the gradient ∇a of a flat function a belongs to the dual $\mathcal{V}^* := \text{Lin}(\mathcal{V}, \mathbb{R})$ of the translation space $\mathcal{V}$.

The following two basic facts are easily verified using Prop.1 of Sect.33.

Proposition 1: $\text{Flf}\,\mathcal{E}$ *is a subspace of* $\text{Map}\,(\mathcal{E}, \mathbb{R})$*. The set* $\mathbb{R}_{\mathcal{E}}$ *is a one-dimensional subspace of* $\text{Flf}\,\mathcal{E}$ *and, for each* $q \in \mathcal{E}$*,* $\text{Flf}_q\,\mathcal{E}$ *is a supplement of* $\mathbb{R}_{\mathcal{E}}$ *in* $\text{Flf}\,\mathcal{E}$*, so that* $\mathbb{R}_{\mathcal{E}} \cap \text{Flf}_q\,\mathcal{E} = \{0\}$ *and*

$$\text{Flf}\,\mathcal{E} = \mathbb{R}_{\mathcal{E}} + \text{Flf}_q\mathcal{E}. \tag{36.2}$$

Proposition 2: *The mapping* $\mathbf{G}_{\mathcal{E}} : \text{Flf}\,\mathcal{E} \to \mathcal{V}^*$ *defined by* $\mathbf{G}_{\mathcal{E}}a := \nabla a$ *is linear and surjective and has the nullspace* $\text{Null}\ \mathbf{G}_{\mathcal{E}} = \mathbb{R}_{\mathcal{E}}$*.*

The next result follows from Prop.2 by applying the Theorem on Dimension of Range and Nullspace (Sect.14) and (21.1).

Proposition 3: *We have*

$$\dim \mathcal{E} = \dim(\text{Flf}_q\mathcal{E}) = \dim(\text{Flf}\,\mathcal{E}) - 1 \tag{36.3}$$

for all $q \in \mathcal{E}$*.*

The following result states, among other things, that every flat function defined on a flat in $\mathcal{E}$ can be extended to a flat function on all of $\mathcal{E}$.

Proposition 4: *Let* $\mathcal{F}$ *be a flat with direction space* $\mathcal{U}$*. The restriction mapping* $(a \mapsto a|_{\mathcal{F}}) : \text{Flf}\,\mathcal{E} \to \text{Flf}\,\mathcal{F}$ *is linear and surjective. Its nullspace* $\mathcal{N} := \{a \in \text{Flf}\,\mathcal{E} \mid a|_{\mathcal{F}} = 0\}$ *has the following properties:*

(i) *For every* $q \in \mathcal{F}$ *we have* $\mathcal{N} = \{a \in \text{Flf}_q\,\mathcal{E} \mid \nabla a \in \mathcal{U}^{\perp}\}$*.*

(ii) *For every $q \in \mathcal{E} \setminus \mathcal{F}$ there is an $a \in \mathcal{N}$ such that $a(q) = 1$.*

Proof: The linearity of $a \mapsto a|_{\mathcal{F}}$ is evident. Now let $q \in \mathcal{F}$ be given. If $b \in \operatorname{Flf}\mathcal{F}$ is given we can choose, by Prop.6 of Sect.21, $\boldsymbol{\lambda} \in \mathcal{V}^*$ such that $\boldsymbol{\lambda}|_{\mathcal{U}} = \nabla b$. By the Theorem on Specification of Flat Mappings (Sect.33), we can define $a \in \operatorname{Flf}\mathcal{E}$ by $a(x) := b(q) + \boldsymbol{\lambda}(x - q)$ for all $x \in \mathcal{E}$. We then have $a|_{\mathcal{F}} = b$. Since $b \in \operatorname{Flf}\mathcal{F}$ was arbitrary, this shows the surjectivity of $a \mapsto a|_{\mathcal{F}}$.

The property (i) of $\mathcal{N}$ is an immediate consequence of Prop.1 of Sect.33.

Assume now that $q \in \mathcal{E} \setminus \mathcal{F}$ is given and choose $z \in \mathcal{F}$. Then $\mathbf{v} := q - z \notin \mathcal{U}$ and hence $\mathcal{U} \subsetneqq \mathcal{U} + \mathbb{R}\mathbf{v}$. Using Prop.4 of Sect.22, it follows that $(\mathcal{U} + \mathbb{R}\mathbf{v})^{\perp} \subsetneqq \mathcal{U}^{\perp}$. Hence we can choose $\boldsymbol{\lambda} \in \mathcal{U}^{\perp}$ such that $\boldsymbol{\lambda}\mathbf{v} \neq 0$. By the Theorem on Specification of Flat Mappings, we can define $a \in \operatorname{Flf}\mathcal{E}$ by

$$a(x) := \frac{1}{\boldsymbol{\lambda}\mathbf{v}}\boldsymbol{\lambda}(x - z) \quad \text{for all} \quad x \in \mathcal{E}.$$

It is evident that $a \in \mathcal{N}$ and $a(q) = 1$. ∎

Consider now a non-constant flat function $a : \mathcal{E} \to \mathbb{R}$. As stated in Example 1 of Sect.33, we then have $\nabla a \neq 0$. It follows from Prop.3 of Sect.33 that the direction space of $a^{<}(\{0\})$ is $\{\nabla a\}^{\perp} = (\mathbb{R}\nabla a)^{\perp}$. Since $\dim(\mathbb{R}\nabla a) = 1$, it follows from the Formula for Dimension of Annihilators (21.15) that $\dim a^{<}(\{0\}) = \dim\{\nabla a\}^{\perp} = n - 1$, i.e. that $a^{<}(\{0\})$ is a hyperplane. We say that $a^{<}(\{0\})$ is the **hyperplane determined by** a. Given any hyperplane $\mathcal{F}$ in $\mathcal{E}$, it follows from Prop.4, (ii) that there is a non-constant $a \in \operatorname{Flf}\mathcal{E}$ such that $\mathcal{F}$ is the hyperplane determined by a. Also, Prop.4, (ii), and Prop.6 of Sect.32 have the following immediate consequence.

Proposition 5: *Every flat in $\mathcal{E}$ other than $\mathcal{E}$ itself is the intersection of all hyperplanes that include it.*

Proposition 6: *The evaluation mapping* ev $: \mathcal{E} \to (\operatorname{Flf}\mathcal{E})^*$ *defined by*

$$\operatorname{ev}(x)a := a(x) \quad \text{for all} \quad a \in \operatorname{Flf}\mathcal{E},\ x \in \mathcal{E} \tag{36.4}$$

is flat and injective. Its gradient $\nabla\operatorname{ev} \in \operatorname{Lin}(\mathcal{V}, (\operatorname{Flf}\mathcal{E})^*)$ coincides with the transpose of the linear mapping $\mathbf{G}_{\mathcal{E}} \in \operatorname{Lin}(\operatorname{Flf}\mathcal{E}, \mathcal{V}^*)$ defined in Prop.2.

Proof: Using (33.2) and (36.4) we see that

$$(\nabla a)\mathbf{v} = a(x + \mathbf{v}) - a(x) = (\operatorname{ev}(x + \mathbf{v}) - \operatorname{ev}(x))a$$

holds for all $a \in \operatorname{Flf}\mathcal{E}$, $x \in \mathcal{E}$ and $\mathbf{v} \in \mathcal{V}$. Hence, if we define $\mathbf{L} : \mathcal{V} \to (\operatorname{Flf}\mathcal{E})^*$ by

$$\mathbf{L}(\mathbf{v})a := (\nabla a)\mathbf{v} \quad \text{for all} \quad a \in \operatorname{Flf}\mathcal{E}, \mathbf{v} \in \mathcal{V}, \tag{36.5}$$

we obtain

$$\mathbf{L}(\mathbf{v}) = \mathrm{ev}(x+\mathbf{v}) - \mathrm{ev}(x) \quad \text{for all} \quad x \in \mathcal{E},\ \mathbf{v} \in \mathcal{V}.$$

It is clear from (36.5) that $\mathbf{L}(\xi\mathbf{v}) = \xi\mathbf{L}(\mathbf{v})$ for all $\mathbf{v} \in \mathcal{V}, \xi \in \mathbb{R}$. Thus, ev satisfies the two conditions of Prop.1 of Sect.33 and $\mathbf{L}$ is the gradient of ev, i.e. $\nabla\mathrm{ev} = \mathbf{L}$.

The equation (36.5) states that

$$((\nabla\mathrm{ev})\mathbf{v})a = (\mathbf{G}_{\mathcal{E}}a)\mathbf{v} \quad \text{for all} \quad a \in \mathrm{Flf}\,\mathcal{E},\ \mathbf{v} \in \mathcal{V} \tag{36.6}$$

In view of the characterization (22.3) of the transpose, this means that $\nabla\mathrm{ev} = \mathbf{G}_{\mathcal{E}}^{\top}$. By the Theorem on Annihilators and Transposes, (21.13), and by Prop.2, it follows that Null $\nabla\mathrm{ev} = (\mathrm{Rng}\,\mathbf{G}_{\mathcal{E}})^{\perp} = \mathcal{V}^{*\perp} = \{\mathbf{0}\}$. Therefore $\nabla\mathrm{ev}$ is injective and so is ev (see Prop.4 of Sect.33). ∎

If we write

$$\tilde{x} := \mathrm{ev}(x), \quad \tilde{\mathbf{v}} := (\nabla\mathrm{ev})\mathbf{v} \tag{36.7}$$

when $x \in \mathcal{E}, \mathbf{v} \in \mathcal{V}$, we can, by Prop.6, regard $x \mapsto \tilde{x}$ as a flat isomorphism from $\mathcal{E}$ onto the flat $\tilde{\mathcal{E}} := \mathrm{ev}_{>}(\mathcal{E})$ in $(\mathrm{Flf}\,\mathcal{E})^*$ and $\mathbf{v} \mapsto \tilde{\mathbf{v}}$ as a linear isomorphism from $\mathcal{V}$ onto the subspace $\tilde{\mathcal{V}} := (\nabla\mathrm{ev})_{>}(\mathcal{V})$ of $(\mathrm{Flf}\,\mathcal{E})^*$. This subspace $\tilde{\mathcal{V}}$ is the direction space of the flat $\tilde{\mathcal{E}}$. For all $x, y \in \mathcal{E}$, $\mathbf{v} \in \mathcal{V}$, we have

$$x - y = \mathbf{v} \iff \tilde{x} - \tilde{y} = \tilde{\mathbf{v}}$$

and

$$x + \mathbf{v} = y \iff \tilde{x} + \tilde{\mathbf{v}} = \tilde{y}.$$

Hence, if points in $\mathcal{E}$ and translations in $\mathcal{V}$ are replaced by their images in $\tilde{\mathcal{E}}$ and $\tilde{\mathcal{V}}$, a point-difference becomes an ordinary difference in $(\mathrm{Flf}\,\mathcal{E})^*$ and the symbolic sum of a point and a translation becomes an ordinary sum in $(\mathrm{Flf}\,\mathcal{E})^*$. If points in $\mathcal{E}$ are replaced by their images in $\tilde{\mathcal{E}}$, then flat combinations become linear combinations. In other words, the image in $\tilde{\mathcal{E}}$ of a symbolic sum of the form (35.5) is the ordinary sum $\sum_{i\in I} \lambda_i \tilde{p}_i$.

It is not hard to show that every element of $(\mathrm{Flf}\,\mathcal{E})^*$ is either of the form $\xi\tilde{x}$ with $x \in \mathcal{E}$, $\xi \in \mathbb{R}^{\times}$, or of the form $\tilde{\mathbf{v}}$ with $\mathbf{v} \in \mathcal{V}$.

Notes 36

(1) In accord with Note 1 to Sect.32, the usual term for our "flat function" is "affine function". However, in very elementary texts (especially those for high-school use), the term "linear fnction" is often found. Of course, such usage clashes with our (and the standard) use of "linear".

37 Convex Sets

Let $\mathcal{E}$ be a flat space with translation space $\mathcal{V}$. Given any points $x, y \in \mathcal{E}$ we define the **segment** *joining* x *and* y to be the set of all flat combinations of (x, y) with positive coefficients and denote it by

$$[x, y] := \{\lambda x + \mu y \mid \lambda, \mu \in \mathbb{P},\ \lambda + \mu = 1\}, \tag{37.1}$$

where the sumbolic-sum notation (35.6) is used. Comparing this definition with (35.4), we see that, if $x \neq y$, the segment joining x and y is a subset of the line passing through x and y, as one would expect.

Definition 1: *A subset $\mathcal{C}$ of $\mathcal{E}$ is said to be* **convex** *if the segment joining any two points in $\mathcal{C}$ is included in $\mathcal{C}$, i.e. if $x, y \in \mathcal{C}$ implies $[x, y] \subset \mathcal{C}$.*

It is evident that the empty set $\emptyset$ and all flats in $\mathcal{E}$, and in particular $\mathcal{E}$ itself, are convex. Also, all segments are convex. If $\mathbb{R}$ is regarded as a one-dimensional flat space, then its convex sets are exactly the intervals (see Sect.08). If $s, t \in \mathbb{R}$ and $s \leq t$, the notation (37.1) is consistent with the notation $[s, t] := \{r \in \mathbb{R} \mid s \leq r \leq t\}$ (see (08.4)).

The following result is an immediate consequence of Def.1.

Proposition 1: *The intersection of any collection of convex sets is again a convex set.*

In view of the remarks on span-mappings made in Sect.03 we have the following result.

Proposition 2: *Given any subset $\mathcal{S}$ of $\mathcal{E}$, there is a unique smallest convex set that includes $\mathcal{S}$. More precisely, there is a unique convex set that includes $\mathcal{S}$ and is included in every convex set that includes $\mathcal{S}$. It is called the* **convex hull** *of $\mathcal{S}$ and is denoted by* $\mathrm{Cxh}\mathcal{S}$*. We have $\mathcal{S} = \mathrm{Cxh}\mathcal{S}$ if and only if $\mathcal{S}$ is convex.*

It is clear that the convex hull of two points is just the segment joining the two points; more precisely $\mathrm{Cxh}\{x, y\} = [x, y]$ for all $x, y \in \mathcal{E}$.

The following result follows from the fact (Prop.4 of Sect.35) that flat mappings preserve flat combinations.

Proposition 3: *Images and pre-images of convex sets under flat mappings are again convex. In particular, if $x, y \in \mathcal{E}$ and if α is a flat mapping with domain $\mathcal{E}$, then*

$$\alpha_{>}([x, y]) = [\alpha(x), \alpha(y)]. \tag{37.2}$$

Using (35.7) for pairs $p := (x, y)$ and $\mathbf{f} := (\mathbf{u}, \mathbf{v})$ we immediately obtain the following result.

Proposition 4: *Let $\mathcal{C}$ be a convex subset of $\mathcal{E}$, and $\mathcal{B}$ a convex subset of $\mathcal{V}$. Then $\mathcal{C} + \mathcal{B}$ is a convex subset of $\mathcal{E}$. In particular, $\mathcal{C} + \mathbf{v}$ is convex for all $\mathbf{v} \in \mathcal{V}$ and $x + \mathcal{B}$ is convex for all $x \in \mathcal{E}$.*

Given any non-empty set I, we use the notation

$$(\mathbb{P}^{(I)})_1 := (\mathbb{R}^{(I)})_1 \cap \mathbb{P}^{(I)}$$

for the set of all families of positive numbers that have finite support and add up to 1. It is easily seen that $(\mathbb{P}^{(I)})_1$ is a convex subset of the flat space $(\mathbb{R}^{(I)})_1$.

Definition 3: *Let $p := (p_i \mid i \in I)$ be a non-empty family of points in $\mathcal{E}$. The restriction of the flat-combination mapping for p to $(\mathbb{P}^{(I)})_1$ is called the* **convex-combination** *mapping for p and is denoted by*

$$\mathrm{cxc}_p := \mathrm{flc}_p \,|_{(\mathbb{P}^{(I)})_1}. \tag{37.3}$$

Convex Hull Theorem: *For every non-empty family p of points in $\mathcal{E}$, we have*

$$\mathrm{Rng}\, \mathrm{cxc}_p = \mathrm{Cxh}(\mathrm{Rng}\, p). \tag{37.4}$$

In particular, a subset $\mathcal{C}$ of $\mathcal{E}$ is convex if and only if $\mathrm{Rng}\, \mathrm{cxc}_{\mathcal{C}} = \mathcal{C}$.

Proof: We have, for all $j \in I$, $\delta_j \in (\mathbb{P}^{(I)})_1$ and hence $p_j = \mathrm{flc}_p(\delta_j) = \mathrm{cxc}_p(\delta_j)$. Since $j \in I$ was arbitrary, it follows that $\mathrm{Rng}\, p \subset \mathrm{Rng}\, \mathrm{cxc}_p$. Since $(\mathbb{P}^{(I)})_1$ is convex and flc_p flat, it follows from Prop.3 that $\mathrm{Rng}\, \mathrm{cxc}_p = (\mathrm{flc}_p)_>((\mathbb{P}^{(I)})_1)$ is a convex subset of $\mathcal{E}$ and so $\mathrm{Cxh}(\mathrm{Rng}\, p) \subset \mathrm{Rng}\, \mathrm{cxc}_p$.

To prove the reverse inclusion $\mathrm{Rng}\, \mathrm{cxc}_p \subset \mathrm{Cxh}(\mathrm{Rng}\, p)$, we must show that $\mathrm{cxc}_p(\lambda) \in \mathrm{Cxh}(\mathrm{Rng}\, p)$ for all $\lambda \in (\mathbb{P}^{(I)})_1$. We do so by induction over $\sharp\mathrm{Supt}\, \lambda$. If $\sharp\mathrm{Supt}\, \lambda = 1$, then

$$\lambda = \delta_j \quad \text{for some} \quad j \in I \quad \text{and } \mathrm{cxc}_p(\lambda) = p_j \in \mathrm{Rng}\, p \subset \mathrm{Cxh}(\mathrm{Rng}\, p).$$

Assume, then, that $\lambda \in (\mathbb{P}^{(I)})_1$ with $\sharp\mathrm{Supt}\, \lambda > 1$ is given, and that $\mathrm{cxc}_p(\mu) \in \mathrm{Cxh}(\mathrm{Rng}\, p)$ holds for all $\mu \in (\mathbb{P}^{(I)})_1$ with $\sharp\mathrm{Supt}\, \mu < \sharp\mathrm{Supt}\, \lambda$. We may and do choose $j \in \mathrm{Supt}\, \lambda$ and we put $\sigma := \sum(\lambda_i \mid i \in I \setminus \{j\}) \in]0, 1[$. We define $\mu \in (\mathbb{P}^{(I)})_1$ by

$$\mu_i := \left\{ \begin{array}{ccc} \frac{1}{\sigma}\lambda_i & \text{if} & i \in I \setminus \{j\} \\ 0 & \text{if} & i = j \end{array} \right\}$$

Clearly, $\sharp\mathrm{Supt}\, \mu = \sharp\mathrm{Supt}\, \lambda - 1$ and $\lambda = \sigma\mu + \lambda_j\delta_j$. Since $\sigma + \lambda_j = \mathrm{sum}_I \lambda = 1$, this states that $\lambda \in [\mu, \delta_j]$ in $(\mathbb{P}^{(I)})_1$. Since flc_p is flat, it follows by (37.2) that

$$\mathrm{cxc}_p(\lambda) = \mathrm{flc}_p(\lambda) \;\in\; [\mathrm{flc}_p(\mu), \mathrm{flc}_p(\delta_j)] = [\mathrm{cxc}_p(\mu), p_j]$$

By the induction hypothesis, $\mathrm{cxc}_p(\mu) \in \mathrm{Cxh}(\mathrm{Rng}\, p)$. Since $p_j \in \mathrm{Rng}\, p \subset \mathrm{Cxh}(\mathrm{Rng}\, p)$ and since $\mathrm{Cxh}(\mathrm{Rng}\, p)$ is convex, it follows by Def.1 that $\mathrm{cxc}_p(\lambda) \in \mathrm{Cxh}(\mathrm{Rng}\, p)$. ∎

Since flat mappings preserve flat combinations (Prop.4 of Sect.35) and hence convex combinations, we immediately obtain the following consequence of the Convex Hull Theorem.

Proposition 5: *If $\mathcal{S}$ is a subset of $\mathcal{E}$ and α a flat mapping with domain $\mathcal{E}$, then*

$$\alpha_>(\mathrm{Cxh}\mathcal{S}) = \mathrm{Cxh}(\alpha_>(\mathcal{S})). \tag{37.5}$$

The following is a refinement of the Convex Hull Theorem.

Strong Convex Hull Theorem: *Let p be a non-empty family of points in $\mathcal{E}$ and let $x \in \mathrm{Cxh}(\mathrm{Rng}\, p)$ be given. Then there is a $\lambda \in (\mathbb{P}^{(I)})_1$ such that $x = \mathrm{cxc}_p(\lambda)$ and such that $p|_{\mathrm{Supt}\,\lambda}$ is flatly independent.*

Proof: By the Convex Hull Theorem, $\mathrm{cxc}_p^<(\{x\})$ is not empty. We choose a $\lambda \in \mathrm{cxc}_p^<(\{x\})$ whose support has minimum cardinal. Put $J := \mathrm{Supt}\,\lambda$.

Assume that $p' := p|_J$ is flatly dependent, i.e. that $\mathrm{flc}_{p'}$ is not injective. Then, by Prop.4 of Sect.33, the gradient of $\mathrm{flc}_{p'}$ is not injective. Hence we can choose $\nu \in ((\mathbb{R}^{(I)})_0)^\times$ such that $\mathrm{Supt}\,\nu \subset J$ and

$$\mathbf{0} = (\nabla\mathrm{flc}_{p'})\nu|_J = (\nabla\mathrm{flc}_p)\nu \tag{37.6}$$

We now choose $k \in J$ such that

$$\frac{\nu_k}{\lambda_k} = \max\{\frac{\nu_i}{\lambda_i} \mid i \in J\}.$$

Since $\mathrm{sum}_J\nu|_J = 0$ and $\nu|_J \neq 0$ and $\lambda_i > 0$ for all $i \in J$, we must have $\nu_k > 0$ and we conclude that

$$\lambda_i \geq \frac{\nu_i}{\nu_k}\lambda_k \quad \text{for all} \quad i \in J.$$

Hence we have $\lambda' := \lambda - \frac{\lambda_k}{\nu_k}\nu \in (\mathbb{P}^{(I)})_1$, so that $\mathrm{cxc}_p\lambda'$ is meaningful. Using (37.6) We obtain

$$\begin{aligned} \mathrm{cxc}_p\lambda' &= \mathrm{flc}_p(\lambda - \tfrac{\lambda_k}{\nu_k}\nu) = \mathrm{flc}_p\lambda - \tfrac{\lambda_k}{\nu_k}(\nabla\mathrm{flc}_p)\nu \\ &= \mathrm{flc}_p\lambda = \mathrm{cxc}_p\lambda = x, \end{aligned}$$

which means that $\lambda' \in \mathrm{cxc}_p^<(\{x\})$. On the other hand, we have $\mathrm{Supt}\,\lambda' \subsetneqq \mathrm{Supt}\,\lambda$ because $\lambda'_k = 0$. It follows that $\sharp\mathrm{Supt}\,\lambda' < \sharp\mathrm{Supt}\,\lambda$ which contradicts the assumption that $\mathrm{Supt}\,\lambda$ has minimum cardinal. Hence $p|_J$ cannot be flatly dependent. ■

The following consequence of the Strong Convex Hull Theorem is obtained by using Prop.3, (a) of Sect.35.

Corollary: *Let $\mathcal{S}$ be a subset of a finite-dimensional flat space $\mathcal{E}$. Then, for every $x \in \mathrm{Cxh}\mathcal{S}$, there is a finite subset $\mathcal{I}$ of $\mathcal{S}$ such that $x \in \mathrm{Cxh}\mathcal{I}$ and* $\sharp\mathcal{I} \leq (\dim \mathcal{E}) + 1$.

Notes 37

(1) Many other notations, for example $\mathrm{Conv}\mathcal{S}$ and $\hat{\mathcal{S}}$, can be found for our $\mathrm{Cxh}\mathcal{S}$.

(2) The Corollary to the Strong Convex Hull Theorem is often called "Carathéodory's Theorem".

38 Half-Spaces

Let $\mathcal{E}$ be a flat space with translation space $\mathcal{V}$. Consider a non-constant flat function $a : \mathcal{E} \to \mathbb{R}$, i.e. a flat function such that $\mathrm{Rng}\, a$ is not a singleton. Since the only flats in $\mathbb{R}$ are the singletons and $\mathbb{R}$ itself, it follows by Prop.3 of Sect.33 that $\mathrm{Rng}\, a = \mathbb{R}$ and hence that $a^{<}(\{0\}), a^{<}(\mathbb{P})$, and $a^{<}(\mathbb{P}^{\times})$ are all non-empty. As we have seen in Sect.36, $a^{<}(\{0\})$ is the hyperplane determined by a if $\mathcal{E}$ is finite-dimensional. By Prop.3 of Sect.37, $a^{<}(\mathbb{P})$ and $a^{<}(\mathbb{P}^{\times})$ are (non-empty) convex subsets of $\mathcal{E}$, called the **half-space** and the **open-half-space** determined by a. The hyperplane $a^{<}(\{0\})$ is called the **boundary** of these half spaces. Of course, the half-space $a^{<}(\mathbb{P})$ is the union of its boundary $a^{<}(\{0\})$ and the open-half-space $a^{<}(\mathbb{P}^{\times})$, and these two are disjoint. If $\xi \in \mathbb{P}^{\times}$, then a and ξa determine the same hyperplane and half-spaces.

If $z \in \mathcal{E}$ and $a \in (\mathrm{Flf}_z\, \mathcal{E})^{\times}$, then a is not constant and z belongs to the boundary of the half-space $a^{<}(\mathbb{P})$.

Half-Space Inclusion Theorem: *Let $\mathcal{E}$ be a finite-dimensional flat space, $\mathcal{C}$ a non-empty convex subset of $\mathcal{E}$ and $q \in \mathcal{E} \setminus \mathcal{C}$. Then there is a half-space that includes $\mathcal{C}$ and has q on its boundary. In other words, there is an $a \in (\mathrm{Flf}_q\, \mathcal{E})^{\times}$ such that $a|_{\mathcal{C}} \geq 0$.*

The proof wll be based on the following preliminary result:

Lemma: *Let $\mathcal{C}$ be a convex subset of $\mathcal{E}$ and let $b \in \mathrm{Flf}\,\mathcal{E}$ be given. If both $\mathcal{C} \cap b^{<}(\mathbb{P}^{\times})$ and $\mathcal{C} \cap b^{<}(-\mathbb{P}^{\times})$ are non-empty, so is $\mathcal{C} \cap b^{<}(\{0\})$. If $c \in \mathrm{Flf}\,\mathcal{E}$ satisfies $c|_{\mathcal{C} \cap b^{<}(\{0\})} \geq 0$, then*

$$\frac{c(x)}{b(x)} \geq \frac{c(y)}{b(y)} \quad \text{for all} \quad x \in \mathcal{C} \cap b^{<}(\mathbb{P}^{\times}),\; y \in \mathcal{C} \cap b^{<}(-\mathbb{P}^{\times}). \tag{38.1}$$

Proof: Let $x \in \mathcal{C} \cap b^{<}(\mathbb{P}^{\times})$, $y \in \mathcal{C} \cap b^{<}(-\mathbb{P}^{\times})$ be given, so that $b(x) > 0$, $b(y) < 0$, and hence $b(x) - b(y) > 0$. We define

$$\lambda := \frac{-b(y)}{b(x) - b(y)}, \quad \mu := \frac{b(x)}{b(x) - b(y)}. \tag{38.2}$$

It is clear that $\lambda, \mu \in \mathbb{P}^{\times}, \lambda + \mu = 1$. Hence $z := \lambda x + \mu y$ belongs to $[x, y]$. Since b preserves convex combinations, we have, by (38.2),

$$b(z) = \lambda b(x) + \mu b(y) = 0$$

and hence $z \in b^{<}(\{0\})$. Since $\mathcal{C}$ is convex, we also have $z \in \mathcal{C}$ and hence $z \in \mathcal{C} \cap b^{<}(\{0\})$. The situation is illustrated in Fig.1.

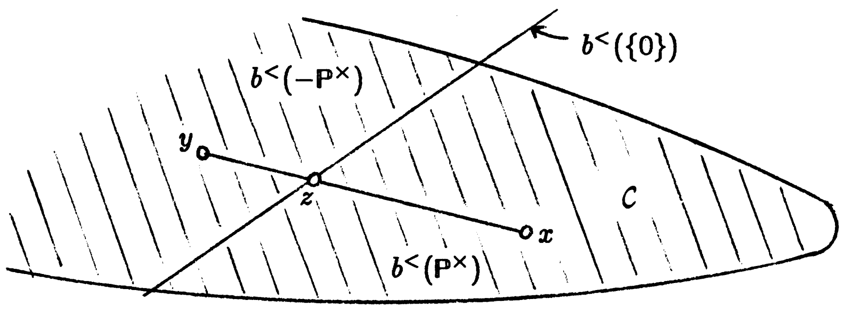


Figure 1.

If $\mathcal{C} \cap b^{<}(\mathbb{P}^{\times})$ and $\mathcal{C} \cap b^{<}(-\mathbb{P}^{\times})$ are both non-empty, we may choose a point in each and obtain a point in $\mathcal{C} \cap b^{<}\{0\}$ by the above method, showing that $\mathcal{C} \cap b^{<}\{0\}$ is not empty.

Now let $c \in \operatorname{Flf} \mathcal{E}$ be such that $c|_{\mathcal{C} \cap b^{<}\{0\}} \geq 0$. Then, if x and y are given as above and z constructed accordingly, we have

$$0 \leq c(z) = \lambda c(x) + \mu c(y).$$

Substituting (38.2) into this inequality and multiplying the result by $b(x) - b(y) \in \mathbb{P}^{\times}$ yields $-b(y)c(x) + b(x)c(y) \geq 0$ which is equivalent to (38.1). ∎

Proof of Theorem: We proceed by induction. The assertion is vacuously valid if $\dim \mathcal{E} = 0$. Assume then, that $\dim \mathcal{E} > 0$, that a non-empty convex $\mathcal{C} \in \operatorname{Sub} \mathcal{E}$ and a $q \in \mathcal{E} \setminus \mathcal{C}$ are given and that the assertion is valid when $\mathcal{E}$ is replaced by a hyperplane $\mathcal{F}$ in $\mathcal{E}$.

Since, by Prop.3 of Sect.36, $\dim(\mathrm{Flf}_q\,\mathcal{E}) = \dim\mathcal{E} > 0$, we may and do choose $b \in (\mathrm{Flf}_q\,\mathcal{E})^\times$ and put $\mathcal{F} := b^<\{0\}$. If $\mathcal{C} \cap b^<(\mathbb{P}^\times)$ or $\mathcal{C} \cap b^<(-\mathbb{P}^\times)$ is empty then $a := -b$ or $a := b$, respectively, fulfills the requirement of the theorem. Therefore, we may assume that both $\mathcal{C} \cap b^<(\mathbb{P}^\times)$ and $\mathcal{C} \cap b^<(-\mathbb{P}^\times)$ are not empty. By the Lemma, $\mathcal{C}\cap\mathcal{F}$ is then also non-empty. Since $\mathcal{C}\cap\mathcal{F}$ is a convex subset of $\mathcal{F}$ and $q \in \mathcal{F}\setminus(\mathcal{F}\cap\mathcal{C})$ we may use the induction hypothesis and choose $d \in (\mathrm{Flf}_q\,\mathcal{F})^\times$ such that $d|_{\mathcal{C}\cap\mathcal{F}} \geq 0$. In view of Prop.4 of Sect.36, we may and do choose a flat extension c of d to $\mathcal{E}$, so that $c \in (\mathrm{Flf}_q\,\mathcal{E})^\times$ and $c|_{\mathcal{C}\cap\mathcal{F}} \geq 0$. We define subsets S and T of $\mathbb{R}$ by

$$S := \{\frac{c(x)}{b(x)} \mid x \in \mathcal{C} \cap b^<(\mathbb{P}^\times)\},$$

$$T := \{\frac{c(y)}{b(y)} \mid y \in \mathcal{C} \cap b^<(-\mathbb{P}^\times)\}.$$

Both S and T are non-empty. Applying the second statement of the Lemma, we see that every number in T is less than every number in S. It follows that $-\infty < \sup T \leq \inf S < \infty$. We choose $\xi \in [\sup T, \inf S]$ and put $a := c - \xi b$. If $x \in \mathcal{C} \cap \mathcal{F} = \mathcal{C} \cap b^<\{0\}$ we have $a(x) = c(x) \geq 0$ since $c|_{\mathcal{C}\cap\mathcal{F}} \geq 0$. If $x \in \mathcal{C} \cap b^<(\mathbb{P}^\times)$ we have $\frac{c(x)}{b(x)} \geq \inf S \geq \xi,\ b(x) > 0$, and hence $a(x) = c(x) - \xi b(x) \geq 0$. Finally, if $x \in \mathcal{C} \cap b^<(-\mathbb{P})$, we have $\frac{c(x)}{b(x)} \leq \sup T \leq \xi,\ b(x) < 0$, and hence $a(x) = c(x) - \xi b(x) \geq 0$. Therefore, since $\mathcal{E} = b^<\{0\} \cup b^<(\mathbb{P}^\times) \cup b^<(-\mathbb{P}^\times)$, we have $a(x) \geq 0$ for all $x \in \mathcal{C}$. Since $\mathcal{F} = b^<\{0\}$ we have $a|_\mathcal{F} = c|_\mathcal{F} = d \neq 0$ and hence $a \neq 0$. Hence a fulfills the requirement of the theorem. ■

Remark 1: The requirement, in the Theorem, that $\mathcal{C}$ be non-empty can obviously be omitted when $\dim\mathcal{E} > 0$. This requirement is needed only to make the assertion vacuously valid when $\dim\mathcal{E} = 0$ and thus to simplify the induction proof. ■

Remark 2: In general, there are many half-spaces that meet the requirements of the Theorem. However, if $\mathcal{C} := a^<(\mathbb{P}^\times) \cup \{z\}$, where $z \in \mathcal{E}$ and $a \in (\mathrm{Flf}_z\,\mathcal{E})^\times$, and if $q \in a^<(\{0\})\setminus\{z\}$, then $a^<(\mathbb{P})$ is the only half-space that meets the requirements (see Fig.2). ■

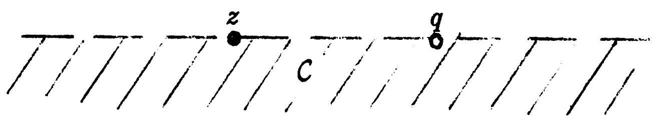


Figure 2.

Remark 3: The Half-Space Inclusion Theorem depends strongly on the assumption that the translation space of $\mathcal{E}$ is a linear space over the *real* field $\mathbb{R}$. Its conclusion is no longer valid if $\mathbb{R}$ is replaced by $\mathbb{Q}$ (see Problem 10, below). ∎

Notes 38

(1) What we call the "Half-Space Inclusion Theorem" is at the root of a number of results, often called "Separation Theorems", in convex analysis. Some of these results are stated in Sect.54.

39 Problems for Chapter 3

(1) Let $\mathcal{S}$ be a subset of a linear space $\mathcal{V}$. Show that

$$\mathrm{Lsp}\mathcal{S} = \mathrm{Fsp}(\mathcal{S} \cup \{\mathbf{0}\}). \tag{P3.1}$$

(2) Let $\mathcal{E}, \mathcal{E}'$ be flat spaces. Show that a mapping $\alpha : \mathcal{E} \to \mathcal{E}'$ is flat if and only if its graph $\mathrm{Gr}(\alpha)$ is a flat in $\mathcal{E} \times \mathcal{E}'$, and, if this is the case, show that the direction space of $\mathrm{Gr}(\alpha)$ is $\mathrm{Gr}(\nabla\alpha)$. (Hint: Use Problem 1 of Chap.1)

(3) Let $\mathcal{E}$ be a flat space and let ε be a flat mapping from $\mathcal{E}$ to itself that is idempotent in the sense that

$$\varepsilon \circ \varepsilon = \varepsilon. \tag{P3.2}$$

Show:

(a) We have

$$\varepsilon|_{\mathrm{Rng}\,\varepsilon} = 1_{\mathrm{Rng}\,\varepsilon \subset \mathcal{E}}. \tag{P3.3}$$

(b) $\nabla\varepsilon$ is an idempotent lineon on $\mathcal{V} := \mathcal{E} - \mathcal{E}$.

(c) We have

$$\mathrm{Rng}\,\varepsilon + \mathrm{Null}\ \nabla\varepsilon = \mathcal{E}. \tag{P3.4}$$

(d) Let $x \in \mathrm{Rng}\,\varepsilon$ be given. Then $\varepsilon^<(\{x\})$ is a flat with direction space $\mathrm{Null}\ \nabla\varepsilon$ and we have

$$\varepsilon^{<}(\{x\}) \cap \operatorname{Rng}\varepsilon = \{x\}, \tag{P3.5}$$

$$\varepsilon^{<}(\{x\}) + \operatorname{Rng}(\nabla\varepsilon) = \mathcal{E}. \tag{P3.6}$$

Moreover, if $\mathcal{E}$ is finite-dimensional, we have

$$\dim \varepsilon^{<}(\{x\}) + \dim\operatorname{Rng}\varepsilon = \dim\mathcal{E}. \tag{P3.7}$$

(4) Let $\mathcal{E}$ be a flat space and let γ be a charge distribution on $\mathcal{E}$ with total charge zero. Show: Given any $p \in \mathcal{E}$ and $\sigma \in \mathbb{P}^{\times}$ there is exactly one $q \in \mathcal{E}$ such that $\gamma \sim \sigma\delta_p - \sigma\delta_q$.

Note: If $p \neq q$, the distribution $\sigma\delta_p - \sigma\delta_q$ is called a **dipole**. ■

(5) (a) Let (x_1, x_2, x_3, x_4) be a quadruple of points in a flat space. Show that the following are equivalent:

(i) The midpoint of (x_1, x_3) coincides with the midpoint of (x_2, x_4),

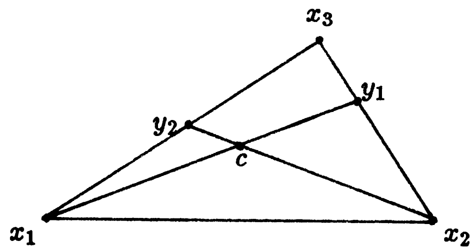
(ii) $x_2 - x_1 = x_3 - x_4$,

(iii) $x_3 - x_2 = x_4 - x_1$.

Remark: If the quadruple is injective and if any and hence all of these conditions are satisfied, then the points are the vertices of a parallelogram. In fact, one may take this to be the definition of "parallelogram". ■

(b) Show that the midpoints of the sides of a quadrilateral (not necessarily plane) in a flat space form a parallelogram.

(6) Consider a flatly independent triple (x_1, x_2, x_3) in a flat space $\mathcal{E}$. Assume that y_1 divides (x_2, x_3) in the ratio $\lambda_1 : \mu_1$ and that y_2 divides (x_1, x_3) in the ratio $\lambda_2 : \mu_2$. Determine the ratios in which the point c of intersection of $\overleftrightarrow{x_1y_1}$ and $\overleftrightarrow{x_2y_2}$ divides (x_1, y_1) and (x_2, y_2) (see Figure).



Figure

(7) Let $\mathcal{E}, \mathcal{E}'$ be flat spaces and let $\alpha : \mathcal{E} \to \mathcal{E}'$ be a mapping which preserves flat combinations of pairs of points, i.e. which satisfies $\alpha \circ \text{flc}_p = \text{flc}_{\alpha \circ p}$ for all pairs $p := (p_1, p_2) \in \mathcal{E}^2$. Prove that α must be a flat mapping.

(8) Let $\mathcal{E}$ be a flat space and consider the pre-monoid $\mathbb{P}^\times \times \mathcal{E}$ described in Example 6 of Sect.34.

(a) Prove that the pre-monoid $\mathbb{P}^\times \times \mathcal{E}$ is cancellative (see Sect.06).

(b) Show that $\mathbb{P}^\times \times \mathcal{E}$ does not contain an element that satisfies the neutrality law (06.2) and hence is not monoidable.

(9) Let $\mathcal{E}$ be a flat space and put $\mathcal{V} := \mathcal{E} - \mathcal{E}$. Let $\boldsymbol{\lambda} \in \mathcal{V}^*$ and $\mathbf{a}, \mathbf{b} \in \mathcal{V}^\times$ be given such that $\boldsymbol{\lambda}\mathbf{a} = 1, \boldsymbol{\lambda}\mathbf{b} = 0$. Put

$$\mathbf{E} := \mathbf{1}_{\mathcal{V}} - (\mathbf{a} \otimes \boldsymbol{\lambda}), \quad \mathbf{N} := \mathbf{b} \otimes \boldsymbol{\lambda}. \tag{P3.8}$$

(a) Show that $\mathbf{E} + \xi\mathbf{N}$ is idempotent for each $\xi \in \mathbb{R}$ and determine $\text{Null}\,(\mathbf{E} + \xi\mathbf{N})$ and $\text{Rng}\,(\mathbf{E} + \xi\mathbf{N})$.

Now let $q \in \mathcal{E}$ be given and define, for each $\mathbf{v} \in \mathcal{V}$, $\varphi_{\mathbf{v}} : \mathcal{E} \to \mathcal{E}$ by

$$\varphi_{\mathbf{v}}(x) := x + (\mathbf{E} + (\boldsymbol{\lambda}(x - q))\mathbf{N})\mathbf{v} \quad \text{for all} \quad x \in \mathcal{E}. \tag{P3.9}$$

(b) Show that, for each $\mathbf{v} \in \mathcal{V}$, $\varphi_{\mathbf{v}}$ is flat and determine $\nabla\varphi_{\mathbf{v}}$; also, show that $\varphi_{\mathbf{v}}$ is invertible, so that $\varphi_{\mathbf{v}} \in \text{Fis}\,\mathcal{E}$.

(c) Show that the mapping

$$\varphi := (\mathbf{v} \mapsto \varphi_{\mathbf{v}}) : \mathcal{V} \to \text{Perm}\,\mathcal{E} \tag{P3.10}$$

is an injective homomorphism from the additive group of $\mathcal{V}$ to $\operatorname{Perm}\mathcal{E}$. (Hence, in view of (b), $\varphi_{>}(\mathcal{V})$ is a subgroup of $\operatorname{Fis}\mathcal{E}$ that is isomorphic to—but different from—the additive group $\mathcal{V}$.)

(d) Is the action of $\mathcal{V}$ on $\mathcal{E}$ defined by $(P3.10)$ free? Is it transitive?

(10) Note that all the definitions of Chap.3 remain meaningful if the field $\mathbb{R}$ of real numbers is replaced by the field $\mathbb{Q}$ of rational numbers. Give an example of a finite-dimensional flat space $\mathcal{E}$ over $\mathbb{Q}$, a non-empty convex subset $\mathcal{C}$ of $\mathcal{E}$ and a point $q \in \mathcal{E} \setminus \mathcal{C}$ such that $a_{>}(\mathcal{C}) \not\subset \mathbb{P}$ for all $a \in (\operatorname{Flf}_q \mathcal{E})^{\times}$, where $\operatorname{Flf}\mathcal{E}$ denotes the set of all flat functions on $\mathcal{E}$ with values in $\mathbb{Q}$. (Thus, the Half-Space Inclusion Theorem does *not* extend to the case when $\mathbb{R}$ is replaced by $\mathbb{Q}$.)