V. Differentiation

This section contains basic results from differential calculus. The key observation about differentiation is that a function f is differentiable at a point x_0 if and only if there is a line through the point $(x_0, f(x_0))$ that approximates the graph of f near $(x_0, f(x_0))$ in a sense that is made precise in Proposition V.1.

A. Definitions

Let S be a subset of $\mathbb{R}$ and $f : S \to \mathbb{R}$ be given.

Definition 1: Let $x_0 \in \text{int}(S)$ be given. We say that f is differentiable at x_0 if

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

exists in $\mathbb{R}$. In this case we write

$$f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} \tag{1}$$

and we call $f'(x_0)$ the derivative of f at x_0. If V is an open subset of S, we say that f is differentiable on V provided that f is differentiable at each $x \in V$.

Definition 2: Let $x_0 \in S$ be given. We say that f attains

(i) a local minimum at x_0 if $\exists \delta > 0$ such that $f(x) \geq f(x_0)$ for all $x \in B_\delta(x_0) \cap S$.

(ii) a local maximum at x_0 if $\exists \delta > 0$ such that $f(x) \leq f(x_0)$ for all $x \in B_\delta(x_0) \cap S$.

Higher Derivatives: If f is differentiable on an open set V and f' is itself differentiable, we denote the derivative of f' by f'' and call it the second derivative of f. Continuing in this manner we obtain functions

$$f, f', f'', f''', f'''', \cdots$$

each of which is the derivative of the preceding one. For derivatives of order n, we often write $f^{(n)}$ rather than f followed by n super-scripted primes. We make the convention that $f^{(0)} = f$.

B. Some Key Results

Let $a, b \in \mathbb{R}$ with $a < b$, sets $S, T \subset \mathbb{R}$, and $x_0 \in \text{int}(S)$ be given.

V.1 Proposition: Let $f : S \to \mathbb{R}$ and $\alpha \in \mathbb{R}$ be given. Then f is differentiable at x_0 if and only if there exists a function $e(\cdot; x_0) : S \to \mathbb{R}$ such that $e(x_0; x_0) = 0$, $\lim_{x \to x_0} e(x; x_0) = 0$, and

$$f(x) = f(x_0) + \alpha(x - x_0) + e(x; x_0)(x - x_0) \quad \forall x \in S. \tag{2}$$

V.2 Proposition: If $f : S \to \mathbb{R}$ is differentiable at x_0 then f is continuous at x_0.

V.3 Theorem: Let $f, g : S \to \mathbb{R}$ and $\alpha \in \mathbb{R}$ be given. Assume that f and g are differentiable at x_0. Then

(i) $f + g$ is differentiable at x_0 and $(f + g)'(x_0) = f'(x_0) + g'(x_0)$

(ii) αf is differentiable at x_0 and $(\alpha f)'(x_0) = \alpha f'(x_0)$

(iii) fg is differentiable at x_0 and $(fg)'(x_0) = f(x_0)g'(x_0) + f'(x_0)g(x_0)$

(iv) $\left(\frac{f}{g}\right)$ is differentiable at x_0 and $\left(\frac{f}{g}\right)'(x_0) = \dfrac{g(x_0)f'(x_0) - f(x_0)g'(x_0)}{g(x_0)^2}$ provided $g(x_0) \neq 0$.

V.4 Theorem (Chain Rule): Let $g : S \to \mathbb{R}$ be given. Assume that $g[S] \subset T$, $g(x_0) \in \operatorname{int}(T)$, g is differentiable at x_0, and that f is differentiable at $g(x_0)$. Then $f \circ g$ is differentiable at x_0 and

$$(f \circ g)'(x_0) = f'(g(x_0))g'(x_0).$$

V.5 Proposition: Assume tht f attains a local maximum or a local minimum at x_0. Then $f'(x_0) = 0$.

V.6 Lemma (Rolle's Theorem): Assume that f is continuous on $[a, b]$, differentiable on (a, b), and that $f(a) = f(b)$. Then, there exists $c \in (a, b)$ such that $f'(c) = 0$.

V.7 Cauchy's Mean Value Theorem: Assume that f, g are continuous on $[a, b]$ and differentiable on (a, b). Then there exists $c \in (a, b)$ such that

$$f'(c)[g(b) - g(a)] = g'(c)[f(b) - f(a)].$$

V.8 Corollary (Mean Value Theorem): Assume that f is continuous on $[a, b]$ and differentiable on (a, b). Then there exists $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

V.9 Corollary: Assume that f is continuous on $[a, b]$ and differentiable on (a, b).

(i) If $f'(x) \geq 0$ for all $x \in (a,b)$ then $f(x_2) \geq f(x_1)$ for all $x_1, x_2 \in [a,b]$ with $x_1 \leq x_2$.

(ii) If $f'(x) > 0$ for all $x \in (a,b)$ then $f(x_2) > f(x_1)$ for all $x_1, x_2 \in [a,b]$ with $x_1 < x_2$.

(iii) If $f'(x) \leq 0$ for all $x \in (a,b)$ then $f(x_2) \leq f(x_1)$ for all $x_1, x_2 \in [a,b]$ with $x_1 \leq x_2$.

(iv) If $f'(x) < 0$ for all $x \in (a,b)$ then $f(x_2) < f(x_1)$ for all $x \in [a,b]$ with $x_1 < x_2$.

(v) If $f'(x) = 0$ for all $x \in (a,b)$ then f is constant on $[a,b]$.

V.10 Theorem (L'Hôpital's Rule): Let $\eta > 0, \ell \in \mathbb{R}$, and $f, g : B^*_\eta(x_0) \to \mathbb{R}$ be given. Assume that $\lim_{x\to x_0} f(x) = \lim_{x\to x_0} g(x) = 0$, that f and g are differentiable on $B^*_\eta(x_0)$ and $g'(x) \neq 0$ for all $x \in B^*_\eta(x_0)$. If $\lim_{x\to x_0} \frac{f'(x)}{g'(x)} = \ell$, then $\lim_{x\to x_0} \frac{f(x)}{g(x)} = \ell$.

V.11 Taylor's Theorem: Let $f : [a,b] \to \mathbb{R}$, $n \in \mathbb{N} \cup \{0\}$, and $x_* \in (a,b)$ be given. Assume that f is continuous on $[a,b]$ and $(n+1)$-times differentiable on (a,b). Define $P_n(\cdot; x_*),\ R_n(\cdot; x_*) : [a,b] \to \mathbb{R}$ by

$$\begin{aligned} P_n(x; x_*) &= \sum_{k=0}^{n} \frac{f^{(k)}(x_*)}{k!}(x - x_*)^k \quad \forall x \in [a,b] \\ R_n(x; x_*) &= f(x) - P_n(x) \quad \forall x \in [a,b]. \end{aligned}$$

Then for each $x \in [a,b] \backslash \{x_*\}$ there exist c between x_* and x such that

$$R_n(x; x_*) = \frac{f^{(n+1)}(c)}{(n+1)!}(x - x_*)^{n+1}.$$

C. Some Remarks

V.12 Remark: It is useful to note that the definition of derivative can be rewritten so that (1) becomes

$$f'(x_0) = \lim_{h\to 0} \frac{f(x_0 + h) - f(x_0)}{h}. \tag{1*}$$

V.13 Remark: L'Hôpital's Rule can be adapted to handle indeterminant forms of the type $\frac{\infty}{\infty}$ and also limits as $x \to \pm\infty$ as well as one-sided limits.

V.14 Remark: The function $P_n(\cdot, x_*)$ in Theorem V.11 is called the Taylor polynomial of order n for f about x_0. The function $R_n(\cdot, x_*)$ is called the remainder. There are other useful expressions for the remainder. The one given here is referred to as the Lagrange form.

D. Some Proofs

Proof of V.1: Assume first that such a function $e(\cdot; x_0)$ exists. Then for all $x \in S\backslash\{x_0\}$ we have

$$\frac{f(x)-f(x_0)}{x-x_0} = \alpha + e(x; x_0). \tag{3}$$

Taking the limit as $x \to x_0$ in (3) we obtain

$$\begin{aligned} f'(x_0) &= \lim_{x\to x_0} \frac{f(x)-f(x_0)}{x-x_0} = \lim_{x\to x_0} (\alpha + e(x; x_0)) \\ &= \alpha + \lim_{x\to x_0} e(x; x_0) = \alpha \end{aligned} \tag{4}$$

Assume now that f is differentiable at x_0 and $f'(x_0) = \alpha$.
Define $e(\cdot; x_0) : S \to \mathbb{R}$ by

$$e(x; x_0) = \begin{cases} \dfrac{f(x)-f(x_0)}{x-x_0} - \alpha & \forall x \in S\backslash\{x_0\} \\ 0, & x = x_0. \end{cases} \tag{5}$$

Observe that

$$\begin{aligned} \lim_{x\to x_0} e(x; x_0) &= \lim_{x\to x_0)} \frac{f(x)-f(x_0)}{x-x_0} - \alpha \\ &= f'(x_0) - \alpha = 0. \end{aligned} \tag{6}$$

It follows readily from (5) that (2) holds. ■

Proof of V.2: Assume that f is differentiable at x_0. By Proposition V.1 we may choose a function $e(\cdot; x_0) : S \to \mathbb{R}$ such that $\lim_{x\to x_0} e(x; x_0) = 0$ and

$$f(x) = f(x_0) + f'(x_0)(x-x_0) + e(x; x_0)(x-x_0) \ \forall x \in S. \tag{7}$$

Taking the limit as $x \to x_0$ in (7) we find that

$$\lim_{x\to x_0} f(x) = f(x_0) + 0 + 0 = f(x_0) \tag{8}$$

which implies that f is continuous at x_0. ■

Proof of V.4: By Proposition V.1 we may choose a function $e(\cdot; g(x_0)) : T \to \mathbb{R}$ such that

$$
\begin{aligned}
f(u) &= f(g(x_0)) + f'(g(x_0))(u - g(x_0)) \\
&\quad + e(u; g(x_0))(u - g(x_0)) \ \forall u \in T
\end{aligned} \tag{9}
$$

and

$$
\lim_{u \to g(x_0)} e(u; g(x_0)) = e(g(x_0); g(x_0)) = 0. \tag{10}
$$

It follows from (9) that

$$
\begin{aligned}
f(g(x)) &= f(g(x_0)) + f'(g(x_0))(g(x) - g(x_0)) \\
&\quad + e(g(x); g(x_0))(g(x) - g(x_0)) \ \ \forall x \in S
\end{aligned} \tag{11}
$$

Consequently, we have

$$
\begin{aligned}
\frac{f(g(x)) - f(g(x_0))}{x - x_0} &= f'(g(x_0))\frac{(g(x) - g(x_0))}{x - x_0} \\
&\quad + e(g(x); g(x_0))\frac{(g(x) - g(x_0))}{x - x_0} \ \ \forall x \in S \backslash \{x_0\}.
\end{aligned} \tag{12}
$$

Since g is differentiable at x_0, it is continuous at x_0 so that $\lim_{x \to x_0} g(x) = g(x_0)$. Using (10) we see that

$$
\lim_{x \to x_0} e(g(x); g(x_0)) = 0. \tag{13}
$$

Taking the limit as $x \to x_0$ in (12) we find that

$$
\begin{aligned}
\lim_{x \to x_0} \frac{f(g(x)) - f(g(x_0))}{x - x_0} &= f'(g(x_0))g'(x_0) + 0 \cdot g'(x_0) \\
&= f'(g(x_0))g'(x_0).
\end{aligned} \tag{14}
$$

It follows that $f \circ g$ is differentiable at x_0 and

$$
(f \circ g)'(x_0) = f'(g(x_0))g'(x_0). \ \blacksquare \tag{15}
$$

Proof of V.5: Assume that f attains a local minimum at x_0. We may choose $\delta > 0$ such that $B_\delta(x_0) \subset S$ and

$$f(x) \geq f(x_0) \quad \forall x \in B_\delta(x_0). \tag{16}$$

It follows from (16) that

$$\frac{f(x) - f(x_0)}{x - x_0} \geq 0 \quad \forall x \in (x_0, x_0 + \delta) \tag{17}$$

and

$$\frac{f(x) - f(x_0)}{x - x_0} \leq 0 \quad \forall x \in (x_0 - \delta, x_0). \tag{18}$$

Choose sequences $\{y_n\}_{n=1}^\infty$, and $\{z_n\}_{n=1}^\infty$ such that $y_n \in (x_0, x_0 + \delta)$, $z_n \in (x_0 - \delta, x_n)$ for every $n \in \mathbb{N}$ and $\lim\limits_{n\to\infty} y_n = \lim\limits_{n\to\infty} z_n = x_0$. Then, by (17) and (18) we have

$$\frac{f(y_n) - f(x_0)}{y_n - x_0} \geq 0 \quad \forall n \in \mathbb{N} \tag{19}$$

$$\frac{f(z_n) - f(x_0)}{z_n - x_0} \leq 0 \quad \forall n \in \mathbb{N}. \tag{20}$$

Since f is differentiable at x_0 and $\lim\limits_{n\to\infty} y_n = \lim\limits_{n\to\infty} z_n = x_0$ we know that

$$f'(x_0) = \lim_{n\to\infty} \frac{f(y_n) - f(x_0)}{y_n - x_0} \tag{21}$$

and

$$f'(x_0) = \lim_{n\to\infty} \frac{f(z_n) - f(x_0)}{z_n - x_0}. \tag{22}$$

It follows from (19) and (21) that $f'(x_0) \geq 0$. It follows from (20) and (22) that $f'(x_0) \leq 0$. We conclude that $f'(x_0) = 0$. If f attains a local maximum at x_0, then $-f$ attains a local minimum at x_0 and $(-f)'(x_0) = -f'(x_0) = 0$. ■

Proof of V.6: Since f is continuous on $[a, b]$ and $[a, b]$ is nonempty and compact we may choose $\alpha, \beta \in [a, b]$ such that

$$f(\alpha) \leq f(x) \leq f(\beta) \quad \forall x \in [a, b]. \tag{23}$$

If $\{\alpha, \beta\} \subset \{a, b\}$ then $f(\alpha) = f(\beta)$ (since $f(a) = f(b)$) and f is constant on $[a, b]$. It follows that $f'(x) = 0$ for all $x \in (a, b)$. If $\{\alpha, \beta\} \not\subset \{a, b\}$ then f attains a local maximum or a local minimum at a point $c \in (a, b)$. By Proposition V.5, $f'(c) = 0$. ■

Proof of V.7: Define $F : [a, b] \to \mathbb{R}$ by

(24) $$F(x) = f(x)[g(b) - b(a)] - g(x)[f(b) - f(a)] \quad \forall x \in [a, b].$$

It follows easily that F is continuous on $[a, b]$, differentiable on (a, b) and that

(25) $$F'(x) = f'(x)[g(b) - g(a)] - g'(x)[f(b) - f(a)] \quad \forall x \in (a, b).$$

Using (24) we find that

(26) $$F(a) = f(a)g(b) - g(a)f(b) = F(b).$$

By Rolle's Theorem, we may choose $c \in (a, b)$ such that $F'(c) = 0$. It follows from (25) that

(27) $$f'(c)[g(b) - g(a)] - g'(c)[f(b) - f(a)] = 0. \blacksquare$$

Proof of V.9: Apply Cauchy's Mean Value Theorem in the special case when $g(x) = x$ for all $x \in [a, b]$ and notice that $g(b) - g(a) = b - a$ and that $g'(x) = 1$ for all $x \in (a, b)$. ■

Proof of V.10: Define $F, G : B_\eta(x_0) \to \mathbb{R}$ by

(28) $$F(t) = \begin{cases} f(t) & \forall t \in B_\eta^*(x_0) \\ 0 & t = x_0 \end{cases}$$

(29) $$G(t) = \begin{cases} g(t) & \forall t \in B_\eta^*(x_0) \\ 0 & t = x_0 \end{cases}$$

Notice that F and G are continuous on $B_\eta(x_0)$, differentiable on $B_\eta^*(x_0)$, and that

(30) $$F'(t) = f'(t), \quad G'(t) = g'(t) \quad \forall t \in B_\eta^*(x_0).$$

[Indeed, F and G are differentiable on $B_\eta^*(x_0)$ and (30) holds by virtue of the fact that $F(t) = f(t)$ and $G(t) = g(t)$ for all $t \in B_\eta^*(x_0)$. The continuity of F and G at x_0 follows from the fact that $0 = F(x_0) = \lim_{t \to x_0} F(t) = \lim_{t \to x_0} G(t) = G(x_0)$.]

Let $\varepsilon > 0$ be given. Then we may choose $\delta > 0$ with $\delta \leq \eta$ such that

$$\left| \frac{f'(z)}{g'(z)} - \ell \right| < \varepsilon \quad \forall z \in B^*_\delta(x_0). \tag{31}$$

We shall show that

$$g(x) \neq 0 \quad \forall x \in B^*_\delta(x_0) \tag{32}$$

and

$$\left| \frac{f(x)}{g(x)} - \ell \right| < \varepsilon \quad \forall x \in B^*_\delta(x_0) \tag{33}$$

For this purpose, let $x \in B^*_\delta(x_0)$ be given. If $g(x) = 0$ then $G(x) = 0$ and we may apply Rolle's Theorem to G to deduce the existence of a point c between x_0 and x such that $G'(c) = g'(c) = 0$. This is a contradiction and consequently $g(x) \neq 0$. By Cauchy's Mean Value Theorem we may choose c_x between x_0 and x such that

$$[F(x) - F(x_0)]\, G'(c_x) = [G(x) - G(x_0)]\, F'(c_x). \tag{34}$$

Using (28), (29), and (30) we may rewrite (34) as

$$f(x)g'(c_x) = g(x)f'(c_x). \tag{35}$$

Since $g(x) \neq 0$, and $g'(c_x) \neq 0$, we deduce from (35) that

$$\frac{f(x)}{g(x)} = \frac{f'(c_x)}{g'(c_x)}. \tag{36}$$

Since c_x is between x_0 and x and $x \in B^*_\delta(x_0)$ we conclude that $c_x \in B^*_\delta(x_0)$. Combining (31) and (36) we arrive at

$$\left| \frac{f(x)}{g(x)} - \ell \right| < \varepsilon. \blacksquare \tag{37}$$

Proof of V.11: Let $x \in [a, b] \backslash \{x_*\}$ be given and put

$$M = \frac{f(x) - P_n(x; x_*)}{(x - x_*)^{n+1}}. \tag{38}$$

Define $F : [a, b] \to \mathbb{R}$ by

(39) $$F(t) = f(t) - P_n(t; x_*) - M(t - x_*)^{n+1} \quad \forall t \in [a, b].$$

It is not difficult to verify that F is continuous on $[a, b]$, $(n + 1)$-times differentiable on (a, b) and that

(40) $$F^{(k)}(x_*) = 0, \quad k = 0, 1, 2, \ldots, n$$

(41) $$F^{(n+1)}(t) = f^{(n+1)}(t) - M(n + 1)! \quad \forall t \in (a, b)$$

Moreover, it follows easily from (38) and (39) that

(42) $$F(x) = 0.$$

By Rolle's Theorem, we may choose c_1 between x_* and x such that $F'(c_1) = 0$. Applying Rolle's Theorem to F' we may choose c_z between x_* and c_1 such that $F''(c_2) = 0$. Continuing in this fashion we may eventually apply Rolle's Theorem to $F^{(n)}$ to choose a point c_{n+1} between x_* and x such that

(43) $$F^{(n+1)}(c_{n+1}) = 0.$$

It follows from (40) and (43) that

(44) $$f^{(n+1)}(c_{n+1}) = M(n + 1)!, \quad \text{i.e.,}$$

(45) $$M = \frac{f^{(n+1)}(c_{n+1})}{(n + 1)!}$$

Combining (38) and (45), we arrive at

(46) $$R_n(x; x_*) = \frac{f^{(n+1)}(c_{n+1})(x - x_*)^{n+1}}{(n + 1)!}. \blacksquare$$