IV. Continuity and Limits

For real-valued functions defined on general subsets of $\mathbb{R}$, the notion of continuity is central. In the most elementary treatments, continuity is sometimes described by a statement such as "a function is continuous provided that its graph can be drawn without lifting the pencil from the paper" or "a function is continuous on an interval provided that its graph has no breaks". While these intuitive geometric descriptions have certain merit, they can also be misleading. It is important to bear in mind that continuity is a very subtle concept. In order to have a serious discussion of this concept we will need to make a careful definition and interpret it literally.

We shall give a definition of continuity that does not rely on any knowledge of sequences or topology. However, we will develop characterizations of continuity in terms of sequences and in terms of open sets. These characterizations allow us to use previous results as tools to develop properties of continuous functions. The following notation will be useful. Let $S \subset \mathbb{R}$ and $f : S \to \mathbb{R}$ be given. For each $U \subset S$ and $V \subset \mathbb{R}$, put

$$f[U] = \{f(x) : x \in U\} \quad \text{and}$$

$$f^{-1}[V] = \{x \in S : f(x) \in V\};$$

$f[U]$ is called the *image of U under f* and $f^{-1}[V]$ is called the *preimage of V under f*. (Notice that $f[S]$ is simply the range of and that $f^{-1}[\mathbb{R}] = S$.) It is straightforward to verify the following 4 statements.

(1) $f[U_1] \subset f[U_2] \quad \forall U_1, U_2$ with $U_1 \subset U_2 \subset S$

(2) $f^{-1}[V_1] \subset f^{-1}[V_2] \quad \forall V_1, V_2$ with $V_1 \subset V_2 \subset S$

(3) $f[f^{-1}(V)] \subset V \quad \forall V \subset \mathbb{R}$

(4) $f^{-1}[f[U]] \supset U \quad \forall U \subset S$.

A. Definitions

Let $S \subset \mathbb{R}$ and $f : S \to \mathbb{R}$ be given.

Definition 1: Let $y \in S$ be given. We say that f is continuous at y provided that $\forall \epsilon > 0$, $\exists \delta > 0$ such that $|f(x) - f(y)| < \epsilon$ for all $x \in S$ with $|x - y| < \delta$ (i.e. $f[B_\delta(y) \cap S] \subset B_\epsilon(f(y))$). We say that f is continuous (or continuous on S) if f is continuous at every point in S.

Definition 2: We say that f is uniformly continuous (or uniformly continuous on S) provided that $\forall \epsilon > 0$, $\exists \delta > 0$ such that $|f(x) - f(y)| < \epsilon$ for all $x, y \in S$ with $|x - y| < \delta$.

Definition 3: Let $x_0, L \in \mathbb{R}$ be given and assume that x_0 is a limit point of S. We say that L is a limit of f at x_0 provided that $\forall \epsilon > 0,\ \exists \delta > 0$ such that $|f(x) - L| < \epsilon$ for all $x \in S$ with $0 < |x - x_0| < \delta$. It is straightforward to show that f has at most one limit at x_0. If L is a limit of f at x_0 we write $\lim\limits_{x \to x_0} f(x) = L$, and we refer to L as the limit of f at x_0.

B. Some Key Results

IV.1 Proposition: Let $S \subset \mathbb{R},\ y \in S$, and $f : S \to \mathbb{R}$ be given. Then f is continuous at y if and only if $\lim\limits_{n\to\infty} f(x_n) = f(y)$ for every sequence $\{x_n\}_{n=1}^{\infty}$ such that $x_n \in S$ for every $n \in \mathbb{N}$ and $\lim\limits_{n\to\infty} x_n = y$.

IV.2 Proposition: Let $S \subset \mathbb{R}$ and $f : S \to \mathbb{R}$ be given. Then f is uniformly continuous on S if and only if $\lim\limits_{n\to\infty} (f(x_n) - f(y_n)) = 0$ for every pair $\{x_n\}_{n=1}^{\infty},\ \{y_n\}_{n=1}^{\infty}$ of sequences such that $x_n,\ y_n \in S$ for every $n \in \mathbb{N}$ and $\lim\limits_{n\to\infty} (x_n - y_n) = 0$.

IV.3 Theorem: Let $S \subset \mathbb{R},\ y \in S,\ \alpha \in \mathbb{R}$, and $f, g : S \to \mathbb{R}$ be given.

(a) If f and g are continuous at y then $f + g,\ \alpha f$, and fg are continuous at y.

(b) If f and g are continuous at y and $g(y) \neq 0$ then $\dfrac{f}{g}$ is continuous at y. [Here we take the domain of $\dfrac{f}{g}$ to be $\{x \in S : g(x) \neq 0\}$.]

(c) If f and g are uniformly continuous on S then $f + g$ and αf are uniformly continuous on S.

IV.4 Theorem: Let $S, T \subset \mathbb{R},\ y \in S$, and $f : T \to \mathbb{R},\ g : S \to \mathbb{R}$ be given. Assume that $g[S] \subset T$.

(a) If g is continuous at y and f is continuous at $g(y)$ then $f \circ g$ is continuous at y.

(b) If g is uniformly continuous on S and f is uniformly continuous on T then $f \circ g$ is uniformly continuous on S.

IV.5 Theorem: Let $f : \mathbb{R} \to \mathbb{R}$ be given. Then f is continuous on $\mathbb{R}$ if and only if $f^{-1}[V]$ is open for every open set $V \subset \mathbb{R}$.

IV.6 Theorem: Let S be a subset of $\mathbb{R},\ y \in S$ and $f : S \to \mathbb{R}$ be given. Then f is continuous at y if and only if for every open set $V \subset \mathbb{R}$ with $f(y) \in V$ there is an open set $U \subset \mathbb{R}$ such that $f^{-1}[V] = S \cap U$.

IV.7 Theorem: Let $S \subset \mathbb{R}$ and $f : S \to \mathbb{R}$ be given. Assume that S is nonempty and compact and that f is continuous on S. Then f attains a maximum and a minimum on S, i.e. $\exists \alpha, \beta \in S$ such that

$$f(\alpha) \leq f(x) \leq f(\beta) \quad \forall x \in S.$$

IV.8 Theorem: Let $S \subset \mathbb{R}$ and $f : S \to \mathbb{R}$ be given. Assume that S is compact and that f is continuous on S. Then f is uniformly continuous on S.

IV.9 Intermediate Value Theorem: Let $a, b, \gamma \in \mathbb{R}$ with $a < b$ and $f : [a, b] \to \mathbb{R}$ be given. Assume that f is continuous on $[a, b]$ and that$f(a) < \gamma < f(b)$ or that $f(a) > \gamma > f(b)$ Then there is at least one $c \in (a, b)$ such that $f(c) = \gamma$.

IV. 10 Theorem: Let S be a subset of $\mathbb{R}$ and $f : S \to \mathbb{R}$ be given. Assume that S is compact and that f is continuous on S and injective. Let $T = f[S]$ and $g : T \to \mathbb{R}$ be the inverse function of f, i.e.

$$f(g(y)) = y \quad \forall y \in T, \quad \text{and}$$

$$g(f(x)) = x \quad \forall x \in S.$$

Then g is continuous on T.

IV.11 Proposition: Let $S \subset \mathbb{R}$, x_0, $L \in \mathbb{R}$, and $f : S \to \mathbb{R}$ be given. Assume that x_0 is a limit point of S. Then $\lim\limits_{x \to x_0} f(x) = L$ if and only if $\lim\limits_{n \to \infty} f(x_n) = L$ for every sequence $\{x_n\}_{n=1}^{\infty}$ such that $x_n \in S \backslash \{x_0\}$ for every $n \in \mathbb{N}$ and $\lim\limits_{n \to \infty} x_n = x_0$.

IV.12 Theorem: Let $S \subset \mathbb{R}$, $x_0, l, L, \alpha \in \mathbb{R}$ and $f, g : S \to \mathbb{R}$ be given. Assume that x_0 is a limit point of S and that $\lim\limits_{x \to x_0} f(x) = l$, $\lim\limits_{x \to x_0} g(x) = L$. Then

(i) $\lim\limits_{x \to x_0} (f(x) + g(x)) = l + L$

(ii) $\lim\limits_{x \to x_0} (\alpha f(x)) = \alpha l$

(iii) $\lim\limits_{x \to x_0} (f(x)g(x)) = lL$

(iv) $\lim\limits_{x \to x_0} \dfrac{f(x)}{g(x)} = \dfrac{l}{L}$ provided that $L \neq 0$. [Here we take the domain of $\dfrac{f}{g}$ to be $\{x \in S : g(x) \neq 0\}$.]

C. Some Remarks

IV.13 Remark: Although sums and constant multiples of uniformly continuous functions are uniformly continuous, it is important to note that products and quotients of uniformly continuous functions need not be uniformly continuous.

IV.14: Using Theorem IV.5, the fact that a set $V \subset \mathbb{R}$ is closed if and only if its compliment is open, together with the observation that $f^{-1}[\mathbb{R} \backslash V] = \mathbb{R} \backslash f^{-1}[V]$, we conclude that a function $f : \mathbb{R} \to \mathbb{R}$ is continuous if and only if $f^{-1}[W]$ is closed for every closed set $W \subset \mathbb{R}$.

IV.15 Remark: Let $S \subset \mathbb{R}$, $f : S \to \mathbb{R}$ and $x_0 \in S$ be given.

(i) If x_0 is not a limit point of S then f is continuous at x_0.

(ii) If x_0 is a limit point of S then f is continuous at x_0 if and only if $\lim_{x\to x_0} f(x) = f(x_0)$.

IV.16 Remark: Let S be a compact subset of $\mathbb{R}$. If $f : S \to \mathbb{R}$ is continuous then $f[S]$ is compact.

D. Some Proofs

In order to prove Propositions IV.1 and IV.2 it is convenient to introduce some terminology. Let $S \subset \mathbb{R}$, $f : S \to \mathbb{R}$ and $y \in S$ be given. We say that f is *sequentially continuous* at y provided that $\lim_{n\to\infty} f(x_n) = f(y)$ for every sequence $\{x_n\}_{n=1}^{\infty}$ such that $x_n \in S$ for every $n \in \mathbb{N}$ and $\lim_{n\to\infty} x_n = y$. We say that f is *sequentially uniformly continuous* on S provided that $\lim_{n\to\infty}(f(x_n) - f(y_n)) = 0$ for every pair $\{x_n\}_{n=1}^{\infty}$, $\{y_n\}_{n=1}^{\infty}$ of sequences such that $x_n, y_n \in S$ for every $n \in \mathbb{N}$ and $\lim_{n\to\infty}(x_n - y_n) = 0$.

Proof of IV.1: Assume first that f is continuous at y. We want to show that f is sequentially continuous at y. Let $\{x_n\}_{n=1}^{\infty}$ be a sequence such that $x_n \in S$ for every $n \in \mathbb{N}$ and $\lim_{n\to\infty} x_n = y$. We need to show that $\lim_{n\to\infty} f(x_n) = f(y)$. Let $\epsilon > 0$ be given. Since f is continuous at y we may choose $\delta > 0$ such that

$$|f(x) - f(y)| < \epsilon \quad \forall x \in S,\ |x - y| < \delta. \tag{5}$$

Since $\lim_{n\to\infty} x_n = y$ we may choose $N \in \mathbb{N}$ such that

$$|x_n - y| < \delta \quad \forall n \in \mathbb{N},\ n \geq N. \tag{6}$$

Since $x_n \in S$ for every $n \in \mathbb{N}$, it follows that

$$|f(x_n) - f(y)| < \epsilon \quad \forall n \in \mathbb{N},\ n \geq N \tag{7}$$

and consequently $\lim_{n\to\infty} f(x_n) = f(y)$.

It remains to show that if f is sequentially continuous at y then f is continuous at y. To prove this implication, we shall prove the contrapositive, i.e. we shall show that if f is not continuous at y then it is not sequentially continuous at y. For this purpose, we assume that f is not continuous at y. Then we may choose $\epsilon > 0$ such that

(8) $$\forall \delta > 0,\ \exists x \in S \cap B_\delta(y),\ |f(x) - f(y)| \geq \epsilon.$$

For each $n \in \mathbb{N}$ we may choose $x_n \in B_{\frac{1}{n}}(y) \cap S$ such that $|f(x_n) - f(y)| \geq \epsilon$. It follows easily that $x_n \to y$ as $n \to \infty$, but $f(x_n) \nrightarrow f(y)$ as $n \to \infty$, i.e. f is not sequentially continuous at y. ■

Proof of IV.2: Assume first that f is uniformly continuous. We want to show that f is sequentially uniformly continuous. Let $\{x_n\}_{n=1}^\infty$, $\{y_n\}_{n=1}^\infty$ be sequences such that $x_n,\ y_n \in S$ for every $n \in \mathbb{N}$ and $\lim_{n\to\infty}(x_n - y_n) = 0$. We need to show that $\lim_{n\to\infty}(f(x_n) - f(y_n)) = 0$. Let $\epsilon > 0$ be given. Since f is uniformly continuous we may choose $\delta > 0$ such that

(9) $$|f(x) - f(y)| < \epsilon \quad \forall x,\ y \in S,\ |x - y| < \delta.$$

Since $\lim_{n\to\infty}(x_n - y_n) = 0$ we may choose $N \in \mathbb{N}$ such that

(10) $$|x_n - y_n| < \delta \quad \forall n \in \mathbb{N},\ n \geq N.$$

It follows that

(11) $$|f(x_n) - f(y_n)| < \epsilon \quad \forall n \in \mathbb{N},\ n \geq N,$$

and consequently $\lim_{n\to\infty}(f(x_n) - f(y_n)) = 0$.

To show that if f is sequentially uniformly continuous then f is uniformly continuous, we shall establish the contrapositive implication, i.e. we shall show that if f is not uniformly continuous then f is not sequentially uniformly continuous. For this purpose assume that f is not uniformly continuous. Then we may choose $\epsilon > 0$ such that

(12) $$\forall \delta > 0,\ \exists x, y \in S,\ |x - y| < \delta,\ |f(x) - f(y)| \geq \epsilon.$$

For each $n \in \mathbb{N}$ we may choose $x_n,\ y_n \in S$ with $|x_n - y_n| < \frac{1}{n}$ and $|f(x_n) - f(y_n)| \geq \epsilon$. It follows easily that $x_n - y_n \to 0$ as $n \to \infty$, but $f(x_n) - f(y_n) \nrightarrow 0$ as $n \to \infty$, i.e. f is not sequentially uniformly continuous. ■

Proof of IV.3: Theorem IV.3 follows directly from Propositions II.3, IV.1, and IV.2. ■

Proof of IV.4: Part (a) follows directly from Proposition IV.1 and part (b) follows directly from Proposition IV.2. ■

Proof of IV.5: Assume first that f is continuous on $\mathbb{R}$. Let V be an open subset of $\mathbb{R}$. We want to show that $f^{-1}[V]$ is open. For this purpose let $y \in f^{-1}[V]$ be given. We want to show that $y \in \text{ int } (f^{-1}[V])$. Observe that $f(y) \in V$. Since V is open we may choose $\epsilon > 0$ such that

$$B_\epsilon(f(y)) \subset V. \tag{13}$$

Since f is continuous at y, we may choose $\delta > 0$ such that $|f(x) - f(y)| < \epsilon$ for all $x \in B_\delta(y)$, i.e.

$$f[B_\delta(y)] \subset B_\epsilon(f(y)). \tag{14}$$

It follows from (13) and (14) that

$$f[B_\delta(y)] \subset V \tag{15}$$

Applying (2) to (15) yields

$$f^{-1}[f[B_\delta(y)]] \subset f^{-1}[V]. \tag{16}$$

Finally, by applying (4) to (16) we conclude that

$$B_\delta(y) \subset f^{-1}[V], \tag{17}$$

i.e. $y \in \text{int}(f^{-1}[V])$

Conversely, assume now that $f^{-1}[V]$ is open for every open set $V \subset \mathbb{R}$. We want to show that f is continuous on $\mathbb{R}$. Let $y \in \mathbb{R}$ be given. Let $\epsilon > 0$ be given and notice that $B_\epsilon(f(y))$ is open. Consequently $f^{-1}[B_\epsilon(f(y))]$ is open. Moreover, $y \in f^{-1}[B_\delta(f(y))]$. Therefore we may choose $\delta > 0$ such that

$$B_\delta(y) \subset f^{-1}[B_\epsilon(f(y))]. \tag{18}$$

By applying (1) and (3) to (18) we conclude that

$$f[B_\delta(y)] \subset B_\epsilon(f(y)) \tag{19}$$

which implies that f is continuous at y. ■

Proof of IV.7: We shall prove the existence of a maximum for f. [The existence of a minimum for f then follows from the existence of a maximum for the continuous function $-f$ on S.]

Claim: The set $f[S]$ is bounded above.
Proof of Claim: Suppose that $f[S]$ is not bounded above. Then, for every $n \in \mathbb{N}$ we may choose $y_n \in S$ such that

$$f(y_n) > n. \tag{20}$$

Since S is compact, the sequence $\{y_n\}_{n=1}^{\infty}$ is bounded. By the Bolzano-Weierstrass Theorem, we may choose a convergent subsequence $\{y_{n_k}\}_{k=1}^{\infty}$. Let $l = \lim_{k\to\infty} y_{n_k}$. Since S is compact, it is closed, and consequently $l \in S$. Since f is continuous at l, it follows from Proposition IV.1 that $f(y_{n_k}) \to f(l)$ as $k \to \infty$; in particular $\{f(y_{n_k})\}_{k=1}^{\infty}$ is bounded. On the other hand, by virtue of (20) we have

$$f(y_{n_k}) > n_k \geq k \quad \forall k \in \mathbb{N}, \tag{21}$$

which gives a contradiction and completes the proof of the claim. //

The set $f[S]$ is nonempty and bounded above; let

$$M = \sup(f[S]). \tag{22}$$

We shall show that $\exists \beta \in S$ with $f(\beta) = M$. For each $n \in \mathbb{N}$ we may choose $x_n \in S$ such that

$$M - \frac{1}{n} < f(x_n) \leq M. \tag{23}$$

It follows that

$$f(x_n) \to M \quad \text{as} \quad n \to \infty. \tag{24}$$

The sequence $\{x_n\}_{n=1}^{\infty}$ is bounded since S is compact. By the Bolzano-Weierstrass Theorem we may choose a convergent subsequence $\{x_{n_k}\}_{k=1}^{\infty}$. Let $\beta = \lim_{k\to\infty} x_{n_k}$ and notice that $\beta \in S$ because S is closed. It follows from (24) that

$$f(x_{n_k}) \to M \quad \text{as} \quad n \to \infty. \tag{25}$$

Since f is continuous, it follows from Proposition IV.1 that

$$f(x_{n_k}) \to f(\beta) \quad \text{as} \quad k \to \infty. \tag{26}$$

By uniqueness of limits we conclude that $f(\beta) = M$. Since $M = \sup(f[S])$ we see that

$$f(x) \leq f(\beta) \quad \forall x \in S. \quad \blacksquare \tag{27}$$

Proof of IV.8: The result is immediate if $S = \emptyset$, so we assume that S is nonempty. Let $\epsilon > 0$ be given. For each $x \in S$ we may choose $\delta_x > 0$ such that

$$|f(x) - f(y)| < \frac{\epsilon}{2} \quad \forall y \in B_{\delta_x}(x) \cap S. \tag{28}$$

Now, for each $x \in S$ let

$$\mathcal{O}_x = B_{\frac{1}{2}\delta_x}(x) \tag{29}$$

and observe that $\{\mathcal{O}_x : x \in S\}$ is a collection of open sets that covers S. By the Heine-Borel Theorem we may choose a (nonempty) finite set $F \subset S$ such that the collection $\{\mathcal{O}_x : x \in F\}$ covers S. Let

$$\delta = \frac{1}{2} \min \{\delta_x : x \in F\} \tag{30}$$

and notice that $\delta > 0$.

Let $y, z \in S$ with $|y - z| < \delta$ be given. We shall show that $|f(y) - f(z)| < \epsilon$ which establishes uniform continuity of f. Since the collection $\{\mathcal{O}_x : x \in F\}$ covers S we may choose $x_* \in F$ such that $y \in \mathcal{O}_{x_*}$. Notice that $|y - x_*| < \frac{1}{2}\delta_{x_*}$ and consequently

$$|z - x_*| \leq |z - y| + |y - x_*| < \delta + \frac{1}{2}\delta_{x_*} < \delta_{x_*} \tag{31}$$

because $\delta \leq \frac{1}{2}\delta_{x^*}$ by virtue of (30). Using (28) we find that

$$|f(y) - f(x_*)| < \frac{\epsilon}{2}, \tag{32}$$

$$|f(z) - f(x_*)| < \frac{\epsilon}{2} \tag{33}$$

We conclude that

$$|f(y) - f(z)| \leq |f(y) - f(x_*)| + |f(z) - f(x_*)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \quad \blacksquare \tag{34}$$

Proof of IV.9: Assume that

$$f(a) < \gamma < f(b). \tag{35}$$

Define $F : [a, b] \to \mathbb{R}$ by

$$F(x) = f(x) - \gamma \quad \forall x \in [a, b]. \tag{36}$$

Notice that F is continuous on $[a, b]$ and that

$$F(a) < 0, \quad F(b) > 0. \tag{37}$$

Let $S = \{t \in (a, b] : F(x) < 0 \text{ for all } x \in (a, t)\}$ and observe that S is bounded above by b. By Problem 3 from Assignment 4 we may choose $\delta_1 > 0$ such that $F(x) < 0$ for all $x \in B_{\delta_1}(a) \cap [a, b]$. It follows that S is nonempty; let

$$c = \sup(S). \tag{38}$$

We shall show that $c \in (a, b)$ and that $F(c) = 0$. It is immediate that $c > a$ since $S \subset (a, b]$. It is also immediate that $c \leq b$. Since $F(b) > 0$ we may choose $\delta_2 > 0$ such that $F(x) > 0$ for all $x \in B_{\delta_2}(b) \cap [a, b]$. (Here we have made use of Problem 3 from Assignment 4 again.) We conclude that $(b - \delta_2, b] \cap S = \emptyset$ and consequently $b \neq \sup(S)$.

To show that $F(c) = 0$ we shall show that it is impossible to have $F(c) > 0$ or $F(c) < 0$. Suppose $F(c) > 0$. Then we may choose $\delta_3 > 0$ such that $F(x) > 0$ for all $x \in B_{\delta_3}(c) \cap [a, b]$. This implies that there are numbers strictly less than c that are upper bounds for S which is a contradiction. Suppose that $F(c) < 0$. Then we may choose $\delta_4 > 0$ such that $F(x) < 0$ for all $x \in B_{\delta_4}(c) \cap [a, b]$. This implies that there are elements of S that are strictly greater than c, which is also a contradiction. We conclude that $F(c) = 0$. It follows that $f(c) - \gamma = 0$ so that $f(c) = \gamma$. The situation when $f(a) > \gamma > f(b)$ can be handled by applying the above results to the continuous function $-f$. ■

Proof of IV.10: We begin the proof with a lemma about sequences.

Lemma: Let $\{z_n\}_{n=1}^{\infty}$ be a real sequence and let $l \in \mathbb{R}$ be given. Assume that every subsequence $\{z_{n_k}\}_{k=1}^{\infty}$ has a subsequence $\{z_{n_{k_j}}\}_{j=1}^{\infty}$ such that $z_{n_{k_j}} \to l$ as $j \to \infty$. Then $z_n \to l$ as $n \to \infty$.

Proof of Lemma: Suppose that $z_n \nrightarrow l$ as $n \to \infty$. Then we may choose $\epsilon > 0$ with the following property: $\forall\, N \in \mathbb{N},\ \exists n \in \mathbb{N}$ with $n \geq N$ such that $|z_n - l| \geq \epsilon$. In other words: $\{n \in \mathbb{N} : |z_n - l| \geq \epsilon\}$ is infinite. Therefore we may choose a subsequence $\{z_{n_k}\}_{k=1}^{\infty}$ such that

$$|z_{n_k} - l| \geq \epsilon \quad \forall k \in \mathbb{N}. \tag{39}$$

By assumption, there is a subsequence $\{z_{n_{k_j}}\}_{j=1}^{\infty}$ such that

$$z_{n_{k_j}} \to l \quad \text{as} \quad j \to \infty. \tag{40}$$

On the other hand, it follows from (39) that

$$|z_{n_{k_j}} - l| \geq \epsilon \quad \forall j \in \mathbb{N}, \tag{41}$$

which contradicts (40) and completes the proof of the lemma. //

To prove the theorem, let $y \in T$ be given and let $\{y_n\}_{n=1}^{\infty}$ be a sequence such that $y_n \in T$ for every $n \in \mathbb{N}$ and $y_n \to y$ as $n \to \infty$. We want to show that $g(y_n) \to g(y)$ as $n \to \infty$. We shall do so by employing the lemma. For each $n \in \mathbb{N}$, let

$$x_n = g(y_n) \tag{42}$$

and notice that $x_n \in S$ and

$$f(x_n) = y_n. \tag{43}$$

We want to show that $x_n \to g(y)$ as $n \to \infty$. By the lemma, it suffices to show that every subsequence of $\{x_n\}_{n=1}^{\infty}$ has a subsequence that converges to $g(y)$. Let $\{x_{n_k}\}_{k=1}^{\infty}$ be a subsequence of $\{x_n\}_{n=1}^{\infty}$. Since S is compact, the sequence $\{x_{n_k}\}_{k=1}^{\infty}$ is bounded. By the Bolzano-Weierstrass Theorem we may choose a convergent subsequence $\{x_{n_{k_j}}\}_{j=1}^{\infty}$. Let $l = \lim\limits_{j \to \infty} x_{n_{k_j}}$. It remains to show that $l = g(y)$. Since S is closed we know that $l \in S$. Furthermore, since f is continuous we know that

$$f(x_{n_{k_j}}) \to f(l) \quad \text{as} \quad j \to \infty. \tag{44}$$

Since $y_n \to y$ as $n \to \infty$ we know that $y_{n_{k_j}} \to y$ as $j \to \infty$. Since $f(x_{n_{k_j}}) = y_{n_{k_j}}$ for every $j \in \mathbb{N}$ and $y_{n_{k_j}} \to y$ as $j \to \infty$ we deduce from (44) that

$$y = f(l) \tag{45}$$

by uniqueness of limits. Finally, since f is injective and $f(g(y)) = y$, we deduce from (45) that $l = g(y)$. ■