**21-355** **Advanced Calculus I** **Fall 2001**

**Assignment 7**
**Due on Tuesday, December 11**

Let $a, b \in \mathbb{R}$ with $a < b$ be given.

1. For each $r \in \mathbb{Q}\backslash\{0\}$ there are unique integers $p(r)$ and $q(r)$ such that $q(r) > 0$, $p(r)$ and $q(r)$ have no common divisors, and $r = \dfrac{p(r)}{q(r)}$. Put $q(0) = 1$. Define $f : [0,1] \to \mathbb{R}$ by

$$f(x) = \begin{cases} 0 & \text{for all } x \in [0,1]\backslash\mathbb{Q} \\ \dfrac{1}{q(x)} & \text{for all } x \in [0,1] \cap \mathbb{Q}. \end{cases}$$

Show that $f \in \mathcal{R}[0,1]$.

2. Give an example of $f \in \mathcal{R}[0,1]$ and an increasing function $\varphi : \mathbb{R} \to \mathbb{R}$ such that $\varphi \circ f \notin \mathcal{R}[0,1]$.

3.* Let $f : [a,b] \to \mathbb{R}$ be given and assume that $f$ is continuous on $[a,b]$. Show that if $\displaystyle\int_a^b f^2 = 0$ then $f(x) = 0$ for all $x \in [a,b]$. (Here $f^2 : [a,b] \to \mathbb{R}$ is the function defined by $f^2(x) = f(x)^2$ for all $x \in [a,b]$.)

4.* Let $f \in \mathcal{B}[a,b]$ be given. Show that if $f$ is Riemman integrable on $[c,b]$ for every $c \in (a,b)$ then $f$ is Riemann integrable on $[a,b]$.

5.* Let $g : [a,b] \to \mathbb{R}$ be given and let $\{f_n\}_{n=1}^{\infty}$ be a sequence of functions such that $f_n \in \mathcal{R}[a,b]$ for every $n \in \mathbb{N}$. Let $\{\alpha_n\}_{n=1}^{\infty}$ be a real sequence such that $\alpha_n > 0$ for every $n \in \mathbb{N}$ and $\alpha_n \to 0$ as $n \to \infty$. Assume that

$$|f_n(x) - g(x)| < \alpha_n \quad \forall n \in \mathbb{N},\ x \in [a,b].$$

Show that $g \in \mathcal{R}[a,b]$ and that

$$\int_a^b g = \lim_{n\to\infty} \int_a^b f_n.$$

6.* Let $f \in \mathcal{R}[a,b]$ be given and define $F : [a,b] \to \mathbb{R}$ by

$$F(x) = \int_a^x f(t)dt \quad \forall x \in [a,b].$$

Show that $F$ is uniformly continuous on $[a,b]$.

7.* Let $f, g \in \mathcal{R}[a,b]$ be given. Show that

$$\int_a^b fg \leq \left(\int_a^b f^2\right)^{1/2} \left(\int_a^b g^2\right)^{1/2}.$$

[Suggestion: Study the function $H : \mathbb{R} \to \mathbb{R}$ defined by

$$H(\lambda) = \int_a^b (f(x) - \lambda g(x))^2 dx \quad \forall \lambda \in \mathbb{R}.$$

Notice that $H(\lambda) \geq 0 \quad \forall \lambda \in \mathbb{R}$ and make a "magic choice" for $\lambda$.

*Problems marked with an asterisk should be written up and handed in.