Assignment 6

Due on Thursday, November 29

1.* Let $n \in \mathbb{N}$ and $a_0, a_1, \ldots,\ a_n \in \mathbb{R}$ be given. Show that if

$$a_0 + \frac{a_1}{2} + \frac{a_2}{3} + \ldots + \frac{a_{n-1}}{n} + \frac{a_n}{n+1} = 0$$

then the equation

$$a_0 + a_1 x + \ldots + a_{n-1}x^{n-1} + a_n x^n = 0$$

has at least one solution $x \in (0, 1)$.

2. Give an example of two functions $f, g : \mathbb{R} \to \mathbb{R}$ such that f and g are differentiable on $\mathbb{R}$, $f(0) = g(0) = 0$, $g'(x) \neq 0$ for all $x \in \mathbb{R}$, $\lim_{x\to 0} \dfrac{f(x)}{g(x)}$ exists, but $\lim_{x\to 0} \dfrac{f'(x)}{g'(x)}$ does not exist.

3.* Let $f : \mathbb{R} \to \mathbb{R}$ be given. Show that if f is differentiable on $\mathbb{R}$ and f' is bounded on $\mathbb{R}$ then f is uniformly continuous on $\mathbb{R}$.

4. Let $f : \mathbb{R} \to \mathbb{R}$ be given and assume that f is twice differentiable on $\mathbb{R}$. Show that if f and f'' are bounded on $\mathbb{R}$ then f' is bounded on $\mathbb{R}$.

5.*a. Let $f : \mathbb{R} \to \mathbb{R}$ and assume that f is twice differentiable on $\mathbb{R}$ and that f'' is continuous on $\mathbb{R}$. Show that for every $x \in \mathbb{R}$,

$$\lim_{h\to 0} \frac{f(x+h) + f(x-h) - 2f(x)}{h^2} = f''(x).$$

b. Give an example of a function $g : \mathbb{R} \to \mathbb{R}$ such that

$$\lim_{h\to 0} \frac{g(h) + g(-h) - 2g(0)}{h^2}$$

exists, but g does not have a second derivative at 0.

6. Let $g : \mathbb{R} \to \mathbb{R}$, $\alpha, \beta, r \in \mathbb{R}$ with $\beta > \alpha > r$ be given. Assume that g is strictly increasing and that $g(r) = 0$. Consider the sequence defined recursively by $x_1 = \beta, x_2 = \alpha$,

$$x_{n+2} = x_{n+1} - g(x_{n+1}) \left[\frac{x_{n+1} - x_n}{g(x_{n+1}) - g(x_n)} \right]$$

for all $n \in \mathbb{N}$. Formulate and prove a theorem ensuring that $x_n \to r$ as $n \to \infty$.

7.* Let $f : [-1, 1] \to \mathbb{R}$ be given. Assume that f is continuous on $[-1, 1]$ and that f is three times differentiable on $(-1, 1)$. Assume further that $f(-1) = 0$, $f(0) = 0$, $f(1) = 1$, and $f'(0) = 0$. Show that there exists $c \in (-1, 1)$ such that $f'''(c) \geq 3$. [Suggestion: Use Taylor's Theorem with $n = 2$, $x_0 = 0$, and $x = \pm 1$.]