21-355 **Advanced Calculus I** **Fall 2001**

Assignment 4

Due on Monday, October 29

1. Determine whether or not f is uniformly continuous on S.

   (a) $S = [0, \infty)$, $\quad f(x) = \sqrt{x}$ for all $x \in S$.

   (b) $S = \mathbb{R}$, $\quad f(x) = \sin(x^2)$ for all $x \in S$.

   (c) $S = \mathbb{R}$, $\quad f(x) = \dfrac{1}{1+x^2}$ for all $x \in S$.

   (d) $S = (0, 1)$, $\quad f(x) = \dfrac{\sin x}{x}$ for all $x \in S$.

   (e) $S = (0, 1)$, $\quad f(x) = \dfrac{1}{\sqrt{x}}$ for all $x \in S$.

2.* Assume that $f : (0, 1) \to \mathbb{R}$ is uniformly continuous. Show that f is bounded, i.e. $\exists M \in \mathbb{R}$ such that $|f(x)| \leq M$ for all $x \in (0, 1)$.

3*. Let $S \subset \mathbb{R}$, $y \in S$, and $f : S \to \mathbb{R}$ be given. Assume that f is continuous at y and that $f(y) > 0$. Show that there exists $\delta > 0$ such that $f(x) > 0$ for all $x \in B_\delta(y) \cap S$.

4.* Assume that $f : \mathbb{R} \to \mathbb{R}$ is continuous. Assume further that $\forall \epsilon > 0$, $\exists M \in \mathbb{R}$ such that

$$|f(x)| < \epsilon \quad \text{for all } x \in \mathbb{R},\ |x| > M.$$

Show that f is uniformly continuous.

5. Let $f, g : \mathbb{R} \to \mathbb{R}$ and $y \in \mathbb{R}$ be given.

   (a) Show that if g is continuous at y and f is continuous at $g(y)$ then $f \circ g$ is continuous at y.

   (b) Show that if f and g are uniformly continuous on $\mathbb{R}$ then $f \circ g$ is uniformly continuous on $\mathbb{R}$.

6.* Let S be a subset of $\mathbb{R}$ and let $g : S \to \mathbb{R}$, $f : \mathbb{R} \to \mathbb{R}$ be given. Assume that g is uniformly continuous and that f is continuous. Show that if g is bounded, then $f \circ g$ is uniformly continuous on S.

7. Assume that $f : [0, 1] \to \mathbb{R}$ is continuous. Show that if $0 \leq f(x) \leq 1$ for all $x \in [0, 1]$ then there exists $y \in [0, 1]$ such that $y = f(y)$.

8.* Let $S \subset \mathbb{R}$ and assume that $f, g : S \to \mathbb{R}$ are continuous. Let $T \subset S$ such that $S \subset cl(T)$. Show that if $f(x) = g(x)$ for all $x \in T$, then $f(x) = g(x)$ for all $x \in S$.

*Problems marked with an asterisk should be written up and handed in.