VI. Riemann Integration

A. Definitions

Let $a, b \in \mathbb{R}$ with $a < b$ be given. By a partition of $[a, b]$ we mean a finite set $P \subset [a, b]$ with $a, b \in P$. The set of all partitions of $[a, b]$ will be denoted by $\mathcal{P}[a, b]$. The set of all bounded functions $f : [a, b] \to \mathbb{R}$ will be denoted by $\mathcal{B}[a, b]$.

Given $P \in \mathcal{P}[a, b]$ and $f \in \mathcal{B}[a, b]$ we write $P = \{x_0, x_1, x_2, \ldots x_n\}$ where $a = x_0 < x_1 < x_2 < \ldots < x_n = b$, and put $\Delta x_i = x_i - x_{i-1}$, $m_i(f) = \inf\{f(x) : x_{i-1} \le x \le x_i\}$ and $M_i(f) = \sup\{f(x) : x_{i-1} \le x \le x_i\}$. We define the lower and upper sums of f for the partition P by

$$L(f, P) = \sum_{i=1}^{n} m_i(f)\Delta x_i \qquad \text{and}$$

$$U(f, P) = \sum_{i=1}^{n} M_i(f)\Delta x_i.$$

Notice that for every $P \in \mathcal{P}[a, b]$ we have

$$m(f)(b-a) \le L(f, P) \le U(f, P) \le M(f)(b-a), \text{ where}$$

$$m(f) = \inf\{f(x) : x \in [a, b]\} \text{ and}$$

$$M(f) = \sup\{f(x) : x \in [a, b]\}.$$

Definition 1: Let $f \in [a, b]$ be given. The lower integral of f is defined by

$$\underline{\int_a^b} f = \sup\{L(f, P) : P \in \mathcal{P}[a, b]\}.$$

The upper integral of f is defined by

$$\overline{\int_a^b} f = \inf\{U(f, P) : P \in \mathcal{P}[a, b]\}.$$

Definition 2. Let $f \in \mathcal{B}[a, b]$ be given. We say that f is Reimann integrable if

$$\underline{\int_a^b} f = \overline{\int_a^b} f;$$

in this case we write

$$\int_a^b f = \underline{\int_a^b} f.$$

Sometimes we write $\int_a^b f(t)dt$ in place of $\int_a^b f$. The set of all Riemann integrable functions $f : [a, b] \to \mathbb{R}$ will be denoted by $\mathcal{R}[a, b]$.

Definition 3. Let $f \in \mathcal{R}[a,b]$ be given. Then we define $\int_a^a f = 0$ and $\int_b^a f = -\int_a^b f$.

Definition 4: Let $P, Q \in \mathcal{P}[a,b]$ be given. If $P \subset Q$ we say that Q is a refinement of P.

Definition 5: Let $P_1, P_2 \in \mathcal{P}[a,b]$ be given. The partition $P = P_1 \cup P_2$ is called the common refinement of $P_1 P_2$.

B. Some Key Results

VI.1 Proposition: Let $f \in \mathcal{B}[a,b]$ and $P, Q \in \mathcal{P}[a,b]$ with $P \subset Q$ be given. Then $L(f,P) \leq L(f,Q)$ and $U(f,P) \geq U(f,Q)$.

VI.2 Proposition: Let $f \in \mathcal{B}[a,b]$ be given. Then $\underline{\int_a^b} f \leq \overline{\int_a^b} f$.

VI.3 Theorem: Let $f \in \mathcal{B}[a,b]$ be given. Then $f \in \mathcal{R}[a,b]$ if and only $\forall \epsilon > 0$, $\exists P \in \mathcal{P}[a,b]$ such that

$$U(f,P) - L(f,P) < \epsilon.$$

VI.4 Theorem: Assume that $f[a,b] \to \mathbb{R}$ is monotonic. Then $f \in \mathcal{R}[a,b]$.

VI.5 Theorem: Assume that $f : [a,b] \to \mathbb{R}$ is continuous. Then $f \in \mathcal{R}[a,b]$.

VI.6 Theorem: Let $f \in \mathcal{R}[a,b]$ be given and choose $c, d, \in \mathbb{R}$ such that $c < d$ and $c \leq f(x) \leq d$ for all $x \in [a,b]$. Let $\varphi : [c,d] \to \mathbb{R}$ be given and assume that φ is continuous. Then $\varphi \circ f \in \mathcal{R}[a,b]$.

VI.7 Theorem: Let $f, g \in \mathcal{R}[a,b]$ and $\alpha \in \mathbb{R}$ be given. Then

i. $f + g \in \mathcal{R}[a,b]$ and $\int_a^b (f+g) = \int_a^b f + \int_a^b g$;

ii. $\alpha f \in \mathcal{R}[a,b]$ and $\int_a^b \alpha f = \alpha \int_a^b f$.

iii. $fg \in \mathcal{R}[a,b]$.

iv. If $f(x) \leq g(x) \quad \forall x \in [a,b]$ then $\int_a^b f \leq \int_a^b g$.

v. $|f| \in \mathcal{R}[a,b]$ and $|\int_a^b f| \leq \int_a^b |f|$.

VI.8 Theorem: Let $f \in \mathcal{R}[a,b]$ and $c, d \in \mathbb{R}$ with $a \leq c < d \leq b$ be given. Then the restriction of f to $[c,d]$ is integrable on $[c,d]$.

VI.9 Theorem: Let $f \in \mathcal{B}[a,b]$ and $c \in (a,b)$ be given. If f is integrable on $[a,c]$ and on $[c,b]$ then $f \in \mathcal{R}[a,b]$ and

$$\int_a^b f = \int_a^c f + \int_c^b f.$$

VI.10 Fundamental Lemma of Calculus: Let $f \in \mathcal{R}[a,b]$ and $c, x_0 \in (a,b)$ be given. Define $F : [a,b] \to \mathbb{R}$ by

$$F(x) := \int_c^x f(t)dt \qquad \forall x \in [a,b].$$

Then F is uniformly continuous on $[a,b]$. Moreover if f is continuous at x_0 then F is differentiable at x_0 and $F'(x_0) = f(x_0)$.

VI.11 Fundamental Theorem of Calculus: Let $f \in \mathcal{R}[a,b]$ be given and assume that f is continuous on (a,b). Let $F : [a,b] \to \mathbb{R}$ be any function that is continuous on $[a,b]$, differentiable on (a,b) and such that $F'(x) = f(x)$ for all $x \in [a,b]$. Then $\displaystyle\int_a^b f = F(b) - F(a)$.

VI.12 Mean Value Theorem for Integrals. Let $f \in \mathcal{R}[a,b]$ be given and assume that f is continuous on (a,b). Then there exists $c \in (a,b)$ such that

$$f(c) = \frac{1}{b-a}\int_a^b f.$$

C. Some Remarks.

VI.13 Remark: It is straightforward to verify that $\displaystyle\int_a^b 1 = b - a$.

VI.14 Remark: Define $f : [a,b] \to \mathbb{R}$ by

$$f(x) = \begin{cases} 0 & \forall x \in [a,b]\backslash\mathbb{Q} \\ \\ 1 & \forall x \in [a,b] \cap \mathbb{Q}. \end{cases}$$

It is straightforward to verify that

$$\underline{\int_a^b} f = 0 \quad \text{and} \quad \overline{\int_a^b} f = b - a$$

and consequently $f \notin \mathcal{R}[a,b]$.

VI.15 Remark: Let $f, g \in \mathcal{R}[a,b]$ be given and assume that $f(x) < g(x)$ for all $x \in (a,b)$. Then we have

$$\int_a^b f < \int_a^b g$$

although this seems much more difficult to prove than Theorem VI.7 (iv).

D. Some Proofs.

Proof of VI.3: Assume first that $f \in \mathcal{R}[a,b]$. Let $\epsilon > 0$ be given. Choose $P_1, P_2 \in \mathcal{P}[a,b]$ such that

$$U(f, P_1) - \overline{\int_a^b} f < \frac{\epsilon}{2} \tag{1}$$

$$\underline{\int_a^b} f - L(f, P_2) < \frac{\epsilon}{2} \tag{2}$$

and put $P = P_1 \cup P_2$. By Proposition VI.1 we have

$$U(f, P) - \overline{\int_a^b} f < \frac{\epsilon}{2} \tag{3}$$

$$\underline{\int_a^b} f - L(f, P) < \frac{\epsilon}{2}. \tag{4}$$

Since $\overline{\int_a^b} f = \underline{\int_a^b} f$ we may add (3) and (4) to obtain

$$U(f, P) - L(f, P) < \epsilon \tag{5}$$

To prove the converse implication let $\epsilon > 0$ be given and choose P such that (5) holds. Then, by Proposition VI.2 we have

$$L(f, P) \leq \underline{\int_a^b} f \leq \overline{\int_a^b} \leq U(f, P) \tag{6}$$

Combining (5) and (6) we get

$$0 \leq \overline{\int_a^{\bar{b}}} f - \underline{\int_a^b} < \epsilon. \tag{7}$$

Since $\epsilon > 0$ was arbitrary we conclude that

$$(8) \qquad \overline{\int_a^b} f - \underline{\int_a^b} f = 0$$

and $f \in \mathcal{R}[a, b]$. ■

Proof of VI.4: We treat the case when f is increasing. [The case when f is decreasing very similar.] We use Theorem VI.3. Let $\epsilon > 0$ be given. Choose $n \in \mathbb{N}$ such that

$$(9) \qquad n > \frac{(f(b) - f(a))(b - a)}{\epsilon}.$$

Let P be the uniform partition of $[a, b]$ with n sub-intervals, i.e. the partition characterized by

$$(10) \qquad x_i = a + i\left(\frac{b - a}{n}\right), \ i = 0, 1, \ldots, n.$$

Let

$$(11) \qquad \Delta x = \frac{(b - a)}{n}$$

and notice that

$$(12) \qquad x_i - x_{i-1} = \Delta x, \quad i = 1, 2, \ldots, n.$$

Since f is increasing we have

$$(13) \qquad m_i(f) = f(x_{i-1}), \ M_i(f) = f(x_i) \quad i = 1, 2, \ldots, n.$$

It follows that

$$(14) \qquad U(f, P) - L(f, P) = \sum_{i=1}^{n} [f(x_i) - f(x_{i-1})]\Delta x = \frac{(b - a)}{n}[f(b) - f(a)].$$

Combining (9) and (14) we get

$$(15) \qquad U(f, P) - L(f, P) < \epsilon. \ ■$$

Proof of VI.5. Once again, we apply Theorem VI.3. Le $\epsilon > 0$ be given. Since f is continuous on $[a, b]$ and $[a, b]$ is compact, we know that f is uniformly continuous. Therefore we may choose $\delta > 0$ so that

$$|f(t) - f(s)| < \frac{\epsilon}{b-a} \quad \forall s,\ t \in [a, b], |t - s| < \delta. \tag{16}$$

Let P be any partition of $[a, b]$ such that

$$\Delta x_i < \delta, \quad i = 1, 2, \ldots, n. \tag{17}$$

Since f is continuous, for each $i \in \{1, 2, \ldots, n\}$ we may choose $\overline{x}_i,\ x_i^* \in [x_{i-1},\ x_i]$ such that

$$f(\bar{x}_i) \leq f(x) \leq f(x_i^*) \quad \forall x \in [x_{i-1},\ x_i]. \tag{18}$$

It follows that

$$U(f, P) - L(f, P) = \sum_{i=1}^{n} [f(x_i^*) - f(\bar{x}_i)]\Delta x_i \tag{19}$$

Since $|x_i^* - \bar{x}_i| < \delta$ for all $i \in \{1, 2, \ldots, n\}$ we know that

$$f(x_i^*) - f(\bar{x}_i) < \frac{\epsilon}{b-a} \quad \forall i \in \{1, 2, \ldots, n\}. \tag{20}$$

It follows from (19) and (20) that

$$U(f, P) - L(f, P) < \sum_{i=1}^{n} \frac{\epsilon}{(b-a)}\Delta x_i = \epsilon. \ \blacksquare \tag{21}$$

Proof of VI.6: Once again, we use Theorem IV.3. Let $\epsilon > 0$ be given. Since φ is continuous on the compact set $[c, d]$ it is uniformly continuous and it is bounded. Choose $\delta > 0$ such that

$$|\varphi(t) - \varphi(s)| < \frac{\epsilon}{2(b-a)} \quad \forall s, t \in [c, d],\ |t - s| > \delta \tag{22}$$

and choose $K > 0$ such that

$$|\varphi(s)| \leq K \quad \forall s \in [c, d]. \tag{23}$$

Since $f \in \mathcal{R}[a,b]$ we may choose $P \in \mathcal{P}[a,b]$ such that

$$U(f,P) - L(f,P) < \frac{\delta\epsilon}{4K}. \tag{24}$$

Split the index $\{1,2,\ldots,n\}$ into two pieces A, B as follows:

$$i \in A \leftrightarrow M_i(f) - m_i(f) < \delta, \tag{25}$$

$$i \in B \leftrightarrow M_i(f) - m_i(f) \geq \delta. \tag{26}$$

Notice that

$$\begin{aligned} &U(\varphi \circ f,\ P) - L(\varphi \circ f,\ P) \\ &\quad = \textstyle\sum_{i \in A} [M_i(\varphi \circ f) - m_i(\varphi \circ f)] \Delta x_i \\ &\quad + \textstyle\sum_{i \in B} [M_i(\varphi \circ f) - m_i(\varphi \circ f] \Delta x_i \end{aligned} \tag{27}$$

It follows from (22) and (25) that

$$M_i(\varphi \circ f) - m_i(\varphi \circ f) \leq \frac{\epsilon}{2(b-a)} \quad \forall i \in A. \tag{28}$$

Consequently

$$\begin{aligned} &\sum_{i \in A} [M_i(\varphi \circ f) - m_i(\varphi \circ f)] \Delta x_i \\ &\leq \sum_{i \in A} \frac{\epsilon \Delta x_i}{2(b-a)} \leq \sum_{i=1}^{n} \frac{\epsilon \Delta x_i}{2(b-a)} = \frac{\epsilon}{2(b-a)}. \end{aligned} \tag{29}$$

Notice that

$$\delta \sum_{i \in B} \Delta x_i \leq \sum_{i \in B} [M_i(f) - m_i(f)] \Delta x_i \leq U(f,P) - L(f,P) < \frac{\delta\epsilon}{4k}. \tag{30}$$

It follows from (30) that

$$\sum_{i \in B} \Delta x_i < \frac{\epsilon}{4K} \tag{31}$$

For $i \in B$, we have

$$[M_k(\varphi \circ f) - m_i(\varphi \circ f)] \leq M_i(\varphi \circ f) + m_i(\varphi \circ f) \leq 2K \tag{32}$$

and consequently

$$\sum_{i \in B}[M_i(\varphi \circ f) - m_i(\varphi \circ f)]\Delta x_i \leq \sum_{i \in B} 2K\Delta x_i < 2K\left(\frac{\epsilon}{4K}\right) = \frac{\epsilon}{2} \tag{33}$$

by virture of (31) and (32). Combining (27), (29), and (33), we arrive at

$$U(\varphi \circ f,\ P) - L(\varphi \circ f,\ P) < \epsilon. \ \blacksquare \tag{34}$$

Proof of VI.7(i). Let $\epsilon > 0$ be given. Choose $P_1, P_2 \in \mathcal{P}[a, b]$ such that

$$U(f, P_1) - L(f, P_1) < \epsilon/2 \tag{35}$$

$$U(g,\ P_) - L(g,\ P_2) < \epsilon/2. \tag{36}$$

Let $P = P_1 \cup P_2$ and observe that

$$U(f, P) - L(f, P) < \epsilon/2 \tag{37}$$

$$U(g, P) - L(g, P) < \frac{\epsilon}{2} \tag{38}$$

Notice that for each $i \in \{1, 2, \ldots, n\}$ we have

$$m_i(f) \leq f(x) \leq M_i(f) \quad \forall x \in [x_{i-1}, x_i] \tag{39}$$

$$m_i(g) \leq g(x) \leq M_i(g) \quad \forall x \in [x_{i-1}, x_i]. \tag{40}$$

and consequently

$$m_i(f) + m_i(g) \leq f(x) + g(x) \leq M_i(f) + M_i(g) \quad \forall x \in [x_{i-1}, x_i]. \tag{41}$$

It follows that

$$(42)\ \ m_i(f) + m_i(g) \le m_i(f+g) \le M_i(f+g) \le M_i(f) + M_i(g) \quad \forall i \in \{1, 2, \ldots, n\}.$$

Multiplying (42) by Δx_i and summing over i we get

$$L(f, P) + L(g, P) \le L(f+g, P) \le U(f+g, P) \le U(f, P) + U(g, P). \tag{43}$$

It follows from (37), (38), and (43) that

$$U(f+g, P), L(f+g, P) - \epsilon. \tag{44}$$

We conclude that $f + g \in \mathcal{R}[a, b]$. Notice that

$$L(f, P) \le \int_a^b f \le U(f, P) \tag{45}$$

$$L(g, P) \le \int_a^b g \le U(g, P) \tag{46}$$

$$L(f+g, P) \le \int_a^b (f+g) \le U(f+g, P) \tag{45}$$

Combining (37), (38), (45), (46), and (47) in a straightforward (but perhaps tedious) fashion we arrive at

$$-\epsilon + \int_a^b f + \int_a^b g < \int_a^b (f+g) < \int_a^b f + \int_a^b g + \epsilon. \tag{48}$$

Since $\epsilon > 0$ was arbitrary we conclude that

$$\int_a^b (f+g) = \int_a^b f + \int_a^b g. \ \blacksquare \tag{49}$$

The proofs of VI.7 (ii) and VI.7 (iv) are left as exercises.

Proof of VI.7(iii). The function $t \mapsto t^2$ is continuous on $\mathbb{R}$. Therefore, by Theorem VI.6, $F^2 \in \mathcal{R}[a, b]$ for every $F \in \mathcal{R}[a, b]$. We conclude that $(f+g)^2 \in \mathcal{R}[a, b]$ and $(f-g)^2 \in \mathcal{R}[a, b]$ by virtue of Theorem VI.7 (i), (ii) and the observation above. The fact that $f, g \in \mathcal{R}[a, b]$ now follows from the equation

$$fg = \frac{1}{4}[(f+g)^2 - (f-g)^2] \tag{50}$$

and another application of Theorem VI.7(i), (ii). ■

Proof of VI.7(v): The fact that $|f| \in \mathcal{R}[a,b]$ follows from Theorem VI.6 and continuity of the function $t \mapsto |t|$ on $\mathbb{R}$. The desired inequality follows form Theorem VI.7(ii), (iv) and the observation

$$f(x) \leq |f(x)| \quad \forall x \in [a,b] \tag{51}$$

$$-f(x) \leq |f(x)| \quad \forall x \in [a,b]. \ ■ \tag{52}$$

Proof of VI.10: The uniform continuity of F is a homework problem. For $h \neq 0$ and $|h|$ small enough so that $x_0 + h \in [a,b]$ we have

$$\begin{aligned} F(x_0+h) &= \textstyle\int_c^{x_0+h} f(t)dt \\ &= \textstyle\int_c^{x_0} f(t)dt + \int_{x_0}^{x_0+h} f(t)dt \\ &= F(x_0) + \textstyle\int_{x_0}^{x_0+h} f(t)dt \end{aligned} \tag{53}$$

and consequently

$$\frac{F(x_0+h) - F(x_0)}{h} = \frac{1}{h}\int_{x_0}^{x_0+h} f(t)dt. \tag{52}$$

Let $\epsilon > 0$ be given. Since f is continuous at x_0 we may choose $\delta > 0$ such that

$$|f(t) - f(x_0)| < \frac{\epsilon}{2} \quad \forall t \in B_\delta(x_0) \cap [a,b]. \tag{53}$$

Observe that for $h \neq 0$ we have

$$f(x_0) = \frac{1}{h}\int_{x_0}^{x_0+h} f(x_0)dt. \tag{56}$$

Let $h \in B_\delta^*(0)$ be given such that $x_0 + h \in [a,b]$. Then we have

$$\frac{F(x_0+h) - F(x_0)}{h} - f(x_0) = \frac{1}{h}\int_{x_0}^{x_0+h} (f(t) - f(x_0))dt \tag{57}$$

by virtue of (54) and (56). It follows from (55) and (57) that

$$\left| \frac{F(x_0+h)-F(x_0)}{h} f(x_0) \right| \leq \frac{1}{|h|} \int_{x_0}^{x_0+h} |f(t)-f(x_0)|dt| \leq \frac{1}{|h|} \frac{\epsilon}{2} |h| < \epsilon. \tag{58}$$

We conclude that F is differentiable at x_0 and $F'(x_0) = f(x_0)$. ■

Proof of VI.II: Define $\tilde{F},\ G : [a,b] \to \mathbb{R}$ by

$$\tilde{F}(x) = \int_a^x f(t)dt \quad \forall x \in [a,b] \tag{59}$$

$$G(x) = F(x) - \tilde{F}(x) \quad \forall x \in [a,b]. \tag{60}$$

Notice that $\tilde{F}, G$ are continuous on $[a,b]$, differentiable on (a,b) and

$$G'(x) = F'(x) - \tilde{F}'(x) = f(x) - f(x) = 0 \quad \forall x \in (a,b). \tag{61}$$

We conclude that G is constant on $[a,b]$, i.e.

$$G(x) = G(a) \quad \forall x \in [a,b]; \tag{62}$$

in particular

$$G(b) = G(a). \tag{63}$$

Notice that

$$G(a) = F(a) - \tilde{F}(a) = F(a) \tag{64}$$

Combining (63) and (64) yields

$$G(b) = F(a) \tag{65}$$

Observe that

$$\int_a^b f(t)dt = \tilde{F}(b) = F(b) - G(b) = F(a) - (a) \tag{66}$$

by virtue of (59), (60), and (65). ■

Proof of VI.12: Define $F : [a, b] \to \mathbb{R}$ by

$$F(x) = \int_a^x ft)dt \quad \forall x \in [a, b] \tag{67}$$

Then F is continuous on $[a, b]$, differentialbe on (a, b) and $F'(x) = f(x)$ for all $x \in (a, b)$. By the Mean Value Theorem for derivatives we may choose $c \in (a, b)$ such that

$$\begin{aligned} f(c) &= F'(c) = \frac{F(b) - F(a)}{b - a} \\ &= \frac{1}{b-a} \left[\int_a^b f(t)dt - \int_b^a f(t)dt \right] \qquad (68) \\ &= \frac{1}{b-a} \int_a^b f(t)dt. \quad \blacksquare \end{aligned}$$