21-355 **Advanced Calculus I** **Fall 2001**

Assignment 2

Due on Friday, September 28

1. Let $x_n = \dfrac{n}{n+1}\left[1+\cos\left(\dfrac{n\pi}{2}\right)\right]$ for all $n \in \mathbb{N}$. Find all cluster points of $\{x_n\}_{n=1}^{\infty}$ and find $\limsup\limits_{n\to\infty} x_n$ and $\liminf\limits_{n\to\infty} x_n$.

2. Show that the sequence defined recursively by

$$x_1 = \sqrt{2}, \quad x_{n+1} = \sqrt{2+\sqrt{x_n}}$$

for all $n \in \mathbb{N}$ is convergent.

3.* Consider the sequence $\{x_n\}_{n+1}^{\infty}$ defined recursively by $x_1 = 2,\ x_{n+1} = \dfrac{x_n}{2} + \dfrac{1}{x_n}$ for all $n \in \mathbb{N}$.

(a) Show that $\{x_n\}_{n=1}^{\infty}$ is convergent.

(b) Find $\lim\limits_{n\to\infty} x_n$.

4. Let $l, L \in \mathbb{R}$ be given and $\{x_n\}_{n=1}^{\infty},\ \{y_n\}_{n=1}^{\infty}$ be sequences. Assume that $x_n \le y_n$ for all $n \in \mathbb{N}$ and that $x_n \to l,\ y_n \to L$ as $n \to \infty$. Show that $l \le L$.

5.* Let $\{x_n\}_{n=1}^{\infty},\ \{y_n\}_{n=1}^{\infty},\ \{z_n\}_{n=1}^{\infty}$ be sequences and let $l \in \mathbb{R}$ be given. Assume that $y_n \to l$ as $n \to \infty$, $z_n \to l$ as $n \to \infty$ and that

$$y_n \le x_n \le z_n \quad \text{for all} \quad n \in \mathbb{N}.$$

Show that $x_n \to l$ as $n \to \infty$.

6.* Let $\{x_n\}_{n=1}^{\infty}$ be a bounded sequence and let $l \in \mathbb{R}$ be given. Show that $x_n \to l$ as $n \to \infty$ if and only if

$$\limsup_{n\to\infty} x_n = \liminf_{n\to\infty} x_n = l.$$

7.* Let $\{x_n\}_{n=1}^{\infty}$ be a convergent real sequence. Show that $\{x_n\}_{n=1}^{\infty}$ has a monotonic subsequence.

8. Give an example of two bounded sequences $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ such that neither sequence is convergent,

$$\limsup_{n\to\infty}(x_n + y_n) = \left(\limsup_{n\to\infty} x_n\right) + \left(\limsup_{n\to\infty} y_n\right),$$

but

$$\limsup_{n\to\infty}(x_n y_n) \neq \left(\limsup_{n\to\infty} x_n\right)\left(\limsup_{n\to\infty} y_n\right)$$

*Problems marked with an asterisk should be written up and handed in.