Summary of Day 19

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1 Objectives

- Rediscover inner products and talk about orthogonality
- Discover the uses for orthogonal sets and bases.
- Define and work with orthonormal sets.
- Defined and look at properties of orthogonal matrices.

2 Summary

- We will now talk about orthogonality (this is 5.1). We begin with revisiting the notion of the dot product.
- Recall: For vectors $\mathbf{v} = [v_1, \ldots, v_n]$ and $\mathbf{u} = [u_1, \ldots, u_n]$ of $\mathbb{R}^n$ we define the **dot product** of $\mathbf{u}$ and $\mathbf{v}$ by:

  $$\mathbf{v} \cdot \mathbf{u} = \sum_{i=1^n} v_i u_i$$

  We say they are **orthogonal** if $\mathbf{v} \cdot \mathbf{u} = 0$. We now extend this definition to a set.
- A set of vectors $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_m\}$ of $\mathbb{R}^n$ is an **orthogonal set** if the vectors in the set are pairwise orthogonal. That is:

  $$\mathbf{v_i} \cdot \mathbf{v_j} = 0 \text{ if } i \neq j$$

  Example The following three vectors form an orthogonal set:

  $$\mathbf{v_1} = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} \qquad \mathbf{v_2} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \qquad \mathbf{v_3} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$$

- Geometrically, the next theorem is fairly intuitive:

  Theorem If $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ is a orthogonal set of nonzero vectors then S is linearly independent.

  Proof.

□

- Recall that a basis is a linearly independent set that spans the space. The most used bases is the standard basis which is:

$$\mathbf{e_1} = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \dots, \mathbf{e_n} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}$$

  These vectors form an orthogonal set and a basis. Such bases are very useful, which motivates the next definition:

- A basis B is an **orthogonal basis** of a subspace W of $\mathbb{R}^n$ if it is also orthogonal.

  Example

- The standard basis is pretty useful because we can easily write vectors as a linear combination of it. For example. $[2,3] = 2[1,0]+3[0,1]$. All bases enjoy the property of being able to write every member uniquely, but most of the time it requires solving a system to find the coefficents. For the standard basis, this is not the case.

  This is a property of all orthogonal bases.

  Theorem Let W be subspace of $\mathbb{R}^n$ with orthogonal basis $\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$. Then for each $\mathbf{w} \in W$ there is a unique $c_1, \dots, c_k$ such that

$$c_1\mathbf{v_1} + \cdots + c_k\mathbf{v_k} = \mathbf{w}$$

  Moreover:

$$c_i = \frac{\mathbf{w} \cdot \mathbf{v_i}}{\mathbf{v_i} \cdot \mathbf{v_i}}$$

  Proof.

□

<u>**Example**</u> Consider again:

$$\mathbf{v_1} = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} \qquad \mathbf{v_2} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \qquad \mathbf{v_3} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$$

<u>**Remark**</u> The formula above may look familiar if you took any classes that talked about vector geometry. It is the projection of $\mathbf{w}$ onto $\mathbf{v_i}$. We will talk about this soon, no worries.

- Something else from the above formula looks familiar. Recall that we can define a **norm** of $\mathbb{R}^n$ in the following way:

  $$\|\mathbf{x}\| = \mathbf{x} \cdot \mathbf{x}$$

  We say that a vector is a **unit vector** if it has norm 1.

  <u>**Remark**</u> The standard basis consists of orthogonal unit vectors. This motivates the next definition:

- A basis is called an **orthonormal basis** if it is an orthogonal basis consisting of unit vectors.

  <u>**Remark**</u> Let $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ be such a basis. Complete the following formula:

  $$\mathbf{v_i} \cdot \mathbf{v_j} = \left\{ \right.$$

  In the event we have an orthonormal basis, the above theorem gets simpler:

  <u>**Theorem**</u> Let W be subspace of $\mathbb{R}^n$ with orthonormal basis $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$. Then for each $\mathbf{w} \in W$ there is a unique $c_1, \ldots, c_k$ such that

  $$c_1\mathbf{v_1} + \cdots + c_k\mathbf{v_k} = \mathbf{w}$$

  Moreover:

  $$c_i = \mathbf{w} \cdot \mathbf{v_i}$$

- Matrix multiplication being defined (for our purposes) in terms of the dot product means that matrix multiplication can actually be envisioned as a whole bunch of dot product. This leads to the following realization about orthonormal sets.

  <u>**Theorem**</u> The columns of $m \times n$ matrix Q form an orthonormal set if and only if $Q^T Q = I_n$.

  Proof.

- The above leads to an interesting idea. Recall that we can view the columns of a matrix as the result of where the standard basis gets sent to under viewing that matrix as a linear transformation. An interesting kind of linear transformation is one in which the standard basis gets sent to an orthonormal set (as the standard basis itself is orthonormal, the hope is that this kind of transformation will preserve a lot of geometric structure).

  We call a square $n \times n$ matrix whose columns form an orthonormal set a **orthogonal matrix**.

  Theorem Q is orthogonal if and only if $Q^T = Q^{-1}$.

  Proof.

  □

  Example Rotation matrices are orthogonal

- Geometrically, orthogonal matrices distort space is very nice ways. Namely, they preserve lengths. This property is called being an **isometry**.

  Theorem If Q is a $n \times n$ matrix then TFAE:

  1. Q is orthogonal.
  2. $(Qx) \cdot (Qy) = x\dot{y}$ for every $x, y \in \mathbb{R}^n$
  3. $\|Qx\| = \|x\|$ for every $x \in \mathbb{R}^n$.

  Proof. This proof is omitted. It is theorem 5.6 in the book.