Summary of Day 15

1 Objectives

- Do some examples of calculating determinants.
- Use determinants to calculate eigenvalues, eigenvectors, find eigenspaces.
- Define and begin to explore algebraic and geometric multiplicity.

2 Summary

- We left off yesterday by discussing the determinant and explaining that if we do operations to the row (and even columns) of A then and get a reduced matrix we can calculate easily calculate the original matrix's determinant if we do proper book keeping on which row operations we perform. Let's do an exmaple.

  Example Calculate the determinant of:

$$\begin{pmatrix} 2 & 3 & -1 \\ 0 & 5 & 3 \\ -4 & -6 & 2 \end{pmatrix}$$

  Example Calculate the determinant of:

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{pmatrix}$$

- It's also nice to note that elementary matrices are particular to easy to analyze the matrices of. Since $\det I = 1$ and all elementary matrices are either 1) a nonzero scalar times a row, 2) interchanging two rows, or 3) adding a multiple of one row to another, we know that an elementary matrix always has nonzero determinant.

Further, multiplying an elementary matrix on the left of a matrix B corresponds to doing a row operation which does whatever action was done to obtain the elementary matrix *and which called the same change to the determinant.* Therefore:

Lemma If E is a square, elementary matrix, and B is a matrix of the same size then:

$$\det(EB) = \det(E)\det(B)$$

- This lemma can be used to prove the follow *very* important purpose for determinants:

  Theorem A square matrix is invertible if and only if $\det A \neq 0$.

  Proof.

  $\square$

  Thus we can add another equivalence to the Fundamental Theorem of Invertible Matrices!

- You may ask how the determinant is affected by the operations that we typically do to matrices:

  Theorem Let A and B be $n \times n$ matrices.

  (a) $\det(AB) = \det(A)\det(B)$
  (b) $\det(kA) = k^n \det(A)$
  (c) $\det(A^T) = \det(A)$
  (d) If A is invertible then $\det(A^{-1}) = \frac{1}{\det A}$

  Proof.

  $\square$

- We can also now find eigenvalues more efficiently.

  Theorem λ is an eigenvalue of A if and only if $\det(A - \lambda I) = 0$

- Now, let's look back at the calculation of eigenvalues and eigenvectors. We will usually find the eigenvalues in the following way:

  (a) Calculate the **characteristic polynomial**: $\det(A - \lambda I)$.

  (b) The eigenvalues correspond to the zeros of this polynomial.

  (c) Calculate the null space of $A - \lambda I$ for each eigenvalue. These are the corresponding eigenvectors for each eigenvalue.

  (d) Because there are ∞-many eigenvectors associated with each eigenvalue, we will always find a basis for the eigenspace.

  Example Find the eigenvalues and a basis for each eigenspace of:

$$A = \begin{pmatrix} 1 & 3 \\ -2 & 6 \end{pmatrix}$$

  Example Find the eigenvalues and a basis for each eigenspace of:

$$B = \begin{pmatrix} 1 & 2 & 0 \\ -1 & -1 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

- It's easy to see now that there are at most n eigenvalues of a $n \times n$ matrix since a degree n polynomial could not have more than n roots. There are some conditions that would make $< n$ eigenvalues. There is a very important theorem called the *Fundamental Theorem of Algebra* which says that any degree n polynomial has n

roots *over the complex numbers*. Over the real numbers, polynomials have no such guarantee. For example, if the characteristic polynomials was $\lambda^2 + 1$ it would have no *real* eigenvalues.

- Apart from that though, the fact that a degree n polynomial has n roots counts *multiplicity*. That is, $(\lambda - 1)^2$ has only one distinct root ($\lambda = 1$) but that roots has multiplicity two.

- We define the **alegbraic multiplicity** of an eigenvalue to be it's multiplicity in the characteristic polynomial. It's **geometric multiplicity** is the dimension of it's eigenspace. We will compare these two notions soon, but they are not in general the same.

  Example Determine the algebraic and geometric multiplicity for the two matrices in the above examples.

- Question: What does it mean to have an eigenvalue of 0?

- We can read of the eigenvalues of some matrices that come from simple operations on others from the eigenvalues of the others (that wasn't the best way to say this). You'll see what I mean:

  Theorem If A is a $n \times n$ matrix with eigenvalue λ with corresponding eigenvector $\mathbf{x}$ then

  (a) λ^m is an eigenvalue of A^m with corresponding eigenvector $\mathbf{x}$.

  (b) $1/\lambda$ is an eigenvalue of A^{-1} with corresponding eigenvector $\mathbf{x}$ (assuming A is invertible)

- A useful application for eigenvalues is the aid in doing some calculations that would be otherwise infeasible to do. We'll revisit this kind of application more when we talk about diagonalization, but here's a taste:

  Example Compute:

$$\begin{pmatrix} 0 & 1 \\ 2 & 1 \end{pmatrix}^{10} \begin{pmatrix} 5 \\ 1 \end{pmatrix}$$