Summary of Day 14

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1 Objectives

- Explore determinants as a mechanism to calculate eigenvalues.
- Prove some nice properties about determinants that will aid in their calculation.
- Learn a few more things you can read off from the determinant of a matrix.

2 Summary

- Recall: Eigenvectors are vectors that are stretched or shrunk by a linear transformation, and the factor they are stretched by is called an eigenvalue. That is, if $T : \mathbb{R}^n \to \mathbb{R}^n$ is a linear transformation, then if there is a $\mathbf{v} \neq \mathbf{0}$ and λ such that:

  $$T\mathbf{v} = \lambda\mathbf{v}$$

  then $\mathbf{v}$ is an eigenvector for the linear transformation T and λ is the corresponding eigenvalue.

  The set of all eigenvalues is called the spectrum. We will learn that the spectrum is pretty small (it is bounded by the dimension of the domain/codomain of the operator).

  Every eigenvalue of a linear operator on $\mathbb{R}^n$ is associated with infinitely many eigenvectors (that is, if the linear operator scales one thing then it scales infinitely many things). The set of eigenvectors associated with λ with the 0 vector (which is never an eigenvector) forms a subspace of $\mathbb{R}^n$ called the eigenspace.

- To find the eigenvalues of a linear transformation/matrix we realize that $\mathbf{x}$ is an eigenvector with eigenvalue λ if and only if The following:

  $$(T - \lambda I)\mathbf{x} = \mathbf{0}$$

  So, we can say that λ is an eigenvalue if and only if the null space of $T - \lambda I$ is nontrivial (which has lots of equivalent forms, like the rank isn't full, the matrix is not invertible, etc). So, we have reduced the question of asking whether a given number λ is an eigvenvalue for T to a question of whether the matrix:

  $$A = T - \lambda I$$

  has a nontrivial nullspace/is invertible.

- The two by two case is very easy then because we have an arithmetic formula to determine if a matrix is invertible.

Example Find the eigenvalues of:

$$\begin{pmatrix} 2 & 4 \\ 2 & 5 \end{pmatrix}$$

- The key idea with the 2×2 case is there is a (relatively) simple arithmetic operation that determines if the matrix is invertible (and, by the fundamental theorem of invertible matrices, it says a lot about lots of other characteristics of the matrix). We will now extend this operation to higher dimensions. The operation is called the **determinant**.

- We will first introduce some terminology to calculate the determinant. But, the basic idea is this for calculating the determinant of a $n \times n$ matrix: we will multiply every size n subset of entries from that matrix where all come from different rows and columns *in a signed way*, and then add up the result.

  So the first bit of terminology aids us in a recursive formula since we want to be able to throw away all entries on the same row and column when we select one on that row and column. This will allow us to make sure that we have multiplied all n-sized subsets. If A is a $n \times n$ matrix, then we can calculate the (i, j)-minor A by calculating the determinant of the matrix A with the ith row and jth column eliminated. If we denote the (i, j)-minor of A as A_{ij} then we can now formally define the determinant as:

$$\det(A) = \sum_{j=1}^{n} (-1)^{1+j} a_{1j} A_{1j}$$

  This is actually a pretty lousy way to calculate a determinant, but we'll do it now to illustrate

Example Calculate the determinant of:

$$\begin{pmatrix} 1 & 2 & 1 \\ 3 & 1 & -1 \\ 1 & -1 & 0 \end{pmatrix}$$

- As you can see, calculating determinants is computationally taxing. There are $n!$ terms in the sum of a determinant. That's not a good number. It gets large very fast. For $n = 3$ it is only 6, but for $n = 5$ it's 120. So, we'll learn some tricks to do it better.

  Firstly, we note that minors are all well and good, but the plus or minus term is also an important component. So we define the (i, j)-cofactor of a matrix to be:

  $$C_{ij} := (-1)^{i+j} A_{ij}$$

  where A_{ij} is the (i, j)-minor. This simplifies the presentation of our formula:

  $$\det(A) = \sum_{j=1}^{n} a_{1j} C_{1j}$$

  although obviously it doesn't change it's performance.

- We note first that if we just wanna make sure we 'caught them all' (i.e. have accounted for all the terms in the sum of the determinant) there's nothing special about cofactors along the first row. We can calculate the determinant by calculating the **cofactor expansion along the ith row**:

  $$\det(A) = \sum_{j=1}^{n} a_{ij} C_{ij}$$

  or **cofactor expansion along the jth column** if we are so daring

  $$\det(A) = \sum_{i=1}^{n} a_{ij} C_{ij}$$

Example Calculate the determinant of:

$$\begin{pmatrix} 3 & 0 & -1 & 1 \\ 1 & 1 & 6 & 9 \\ 0 & 0 & -1 & 0 \\ -3 & 0 & 9 & 1 \end{pmatrix}$$

- It helps to take a matrix and visualize it as a grid of signs to figure out whether the term is to be multiplied by 1 or -1:

$$\begin{pmatrix} + & - & + & - \\ - & + & - & + \\ + & - & + & - \\ - & + & - & + \end{pmatrix}$$

So, if you were calculating the determinant of a 4×4 matrix and doing the cofactor expansion along the second row, the signs attached to the minors would be $-$, $+$, $-$, and $+$.

- We observe from above that 0's are really awesome to have in a matrix. But, even with being able to calculate the determinant along any row or column it doesn't make the computation any less taxing. It's still at worst case $n!$ terms since there may be no 0's in the matrix at all. So, we will learn a few more tricks.

Firstly, because 0's are so awesome:

Theorem If A is an upper triangular matrix, then $\det(A)$ is the product of the diagonals.

Proof. Formally: a proof by induction. It's not hard to see though. □

Theorem Let A be a $n \times n$ matrix.

1. If A has a row (or columns) of all zeros then $\det(A) = 0$.
2. If B is obtained from interchanging two rows (or columns) of A then $\det(B) = -\det(A)$.
3. If A has two identical rows than $\det(A) = 0$.
4. If A is obtained by multiplying a row (or column) by a scalar k then $\det(B) = k\det(A)$.
5. If B is obtained from adding a multiple of one row (resp. column) of A to another row (resp. column) then $\det B = \det A$.
6. If A, B, C are identical except that the ith row (or column) of C is the sum of the ith row of A and B then $\det C = \det A + \det B$.

We won't prove these because such proofs are very arithmetic and not very enlightening. It is interesting to not however that the determinant is the unique function on matrices that satisfies all of these properties.

- Now we can do row reductions to make our live easier. If we do row reductions to get our matrix upper triangular keeping track of what changes we made to the determinant along the way, it's pretty easy to calculate: