Homework 2 Solutions

2.3.21 (a) Suppose that $\mathbf{w}$ is a linear combination of vectors $\mathbf{u}_1, \ldots, \mathbf{u}_k$ and that each $\mathbf{u_i}$ is a linear combination of vectors $\mathbf{v}_1, \ldots, \mathbf{v}_m$. Prove that $\mathbf{w}$ is a linear combination of $\mathbf{v}_1, \ldots, \mathbf{v}_m$.

Proof. Consider $\mathbf{u_i}$. We know that $\mathbf{u_i}$ is a linear combination of $\mathbf{v}_1, \ldots, \mathbf{v}_m$; therefore, we get constants $c_1^i, \ldots, c_m^i$ such that:

$$\mathbf{u_i} = c_1^i \mathbf{v_1} + \cdots + c_m^i \mathbf{v_m}$$

Further, we know that since $\mathbf{w}$ is a linear combination of $\mathbf{u}_1, \ldots, \mathbf{u}_k$ that there is $s_1, \ldots, s_k$ such that

$$\mathbf{w} = s_1 \mathbf{u_1} + \cdots + s_k \mathbf{u_k}$$

Substituting in the linear combination for $\mathbf{u_i}$ we get that

$$\mathbf{w} = s_1 \left(c_1^1 \mathbf{v_1} + \cdots c_m^1 \mathbf{v_m} \right) + \cdots + s_k \left(c_1^k \mathbf{v_1} + \cdots c_m^k \mathbf{v_m} \right)$$

Thus:

$$\mathbf{w} = \left(\sum_{i=1}^{k} s_i c_1^i \right) \mathbf{v_1} + \cdots + \left(\sum_{i=1}^{k} s_i c_m^i \right) \mathbf{v_m}$$

And so $\mathbf{w}$ is a linear combination of of $\mathbf{v}_1, \ldots, \mathbf{v}_m$. □

(b) In part (a), suppose in addition that each $\mathbf{v_j}$ is also a linear combination of $\mathbf{u}_1, \ldots, \mathbf{u}_k$. Prove that $\mathrm{span}(\mathbf{u}_1, \ldots, \mathbf{u}_k) = \mathrm{span}(\mathbf{v}_1, \ldots, \mathbf{v}_m)$.

Proof. We know that each $\mathbf{v_j}$ is a linear combination of $\mathbf{u}_1, \ldots, \mathbf{u}_k$. Therefore, invoking part (a), if we start with $\mathbf{w}$ a linear combination of $\mathbf{u}_1, \ldots, \mathbf{u}_k$ we can get $\mathbf{w}$ as a linear combination of $\mathbf{u}_1, \ldots, \mathbf{u}_k$. Thus $\mathrm{span}(\mathbf{v}_1, \ldots, \mathbf{v}_m) \subseteq \mathrm{span}(\mathbf{u}_1, \ldots, \mathbf{u}_k)$. Put in conjunction with part (a), that $\mathrm{span}(\mathbf{v}_1, \ldots, \mathbf{v}_m) = \mathrm{span}(\mathbf{u}_1, \ldots, \mathbf{u}_k)$. □

(c)

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \mathbf{e_1} \qquad \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \mathbf{e_2} \qquad \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} - \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \mathbf{e_3}$$

Thus, e_1, e_2, e_3 is a linear combination of these vectors. By (a), we have that $\mathrm{span}(\mathbf{e_1}, \mathbf{e_2}, \mathbf{e_3})$ is a subset of the span of these three vectors. As $\mathrm{span}(\mathbf{e_1}, \mathbf{e_2}, \mathbf{e_3})$ is all of $\mathbb{R}^3$, we must have that every vector in $\mathbb{R}^3$ can be written as a linear combination of these three.

2.3.24 Determine if this set of vectors is linearly dependent, and if so find a linear dependency.

$$\begin{pmatrix} 3 \\ 2 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ -6 \end{pmatrix}$$

Solution. We set up a matrix with these as the columns:

$$\begin{pmatrix} 3 & 2 & 1 \\ 2 & 1 & 2 \\ 2 & 3 & -6 \end{pmatrix}$$

We now do elementary row operations to row reduce the matrix. (steps omitted). We get this as a result:

$$\begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & -4 \\ 0 & 0 & 0 \end{pmatrix}$$

This has a free variable, and therefore there are infinitely many solutions to the homogeneous equation. Therefore, there are nontrivial solutions, and the vectors are **linearly dependent**. To find a dependency, note that th rref above tells us that:

$$\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}_1 = t \begin{pmatrix} -3 \\ 4 \\ 1 \end{pmatrix}$$

We can instantiate this at any nonzero value and get a nontrivial dependency; we'll just plug in 1. Therefore:

$$-3\begin{pmatrix}3\\2\\2\end{pmatrix}+4\begin{pmatrix}2\\1\\3\end{pmatrix}+1\begin{pmatrix}1\\2\\-6\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$$

2.3.42 (a) If the columns of a $n \times n$ matrix A are linearly independent as vectors of $\mathbb{R}^n$ then what is the rank of A?

Solution. The rank of A is n.

Proof. The matrix represents a homogeneous system of equations. If the rank was less than n there would be infinitely many solutions (by the rank theorem), and therefore nontrivial solutions. This would correspond to a nontrivial linear dependency to the column vectors of the matrix, which cannot be since the vectors are linearly independent. □

(b) If the rows of a $n \times n$ matrix are linearly independent as vectors of $\mathbb{R}^n$ then what is the rank of A?

Solution. The rank is n.

Proof. We know that the rank of a matrix is less than the number of rows if and only if the rows are linearly dependent. This is theorem 2.7 of the book. The proof idea was that you could emulate the dependency to get a row with all zeros by doing row reduction, but you could just cite this theorem. This tells us that if the vectors are linearly dependent than the rank must be exactly the number of rows, which is n. □

2.3.47 Suppose that $S = \{\mathbf{v}_1, \dots, \mathbf{v}_k, \mathbf{v}\}$ is a set of vector in $\mathbb{R}^n$ such that $\mathbf{v}$ is a linear combination of $\mathbf{v}_1, \dots, \mathbf{v}_k$. If $S' = \{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ then prove that $\text{span}(S) = \text{span}(S')$.

Proof. Clearly every element of S' can be written as an element of S since $S' \subseteq S$. Further, every element of S can be written as a linear combination of S' since for any $\mathbf{u} \in S$, either $\mathbf{u} = \mathbf{v_j}$ for some j, in which case, $\mathbf{u} \in S'$, or $\mathbf{u} = \mathbf{v}$, in which case we are assuming that $\mathbf{v}$ can be written as a linear combination of $\mathbf{v}_1, \dots, \mathbf{v}_k$. Therefore, by exercise 2.3.21b the two spans are equal. □