Homework 1 Solutions

2.2.1-8 We will determine if the matrix is in row echelon form, and if so decide whether it's in reduced row echelon form. Note: No work or explanation was required for these problems, but I'll explain what conditions they violate when applicable.

1. $$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 3 \\ 0 & 1 & 0 \end{pmatrix}$$

   This matrix **is not** in row echelon form because the leading entry of the second row is to the right of the leading entry of the 3rd row.

2. $$\begin{pmatrix} 7 & 0 & 1 & 0 \\ 0 & 1 & -1 & 4 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

   This matrix **is** in row echelon, but it **is not** in reduced row echelon form as the leading coefficent of the first row is not 1.

3. $$\begin{pmatrix} 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

   This matrix **is** in reduced row echelon form.

4. $$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

   This matrix **is** in reduced row echelon form.

5. $$\begin{pmatrix} 1 & 0 & 3 & -4 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 5 & 0 & 1 \end{pmatrix}$$

   This matrix **is not** in row echelon form because all the 0 rows are not at the bottom of the matrix.

6. $$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

   This matrix **is not** in row echelon form because the leading entry of the first row is to the right of the leading entry of the second row (among other problems).

7. $$\begin{pmatrix} 1 & 2 & 3 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

   This matrix **is not** in row echelon form because the leading entry of the first row is not to the left of the leading entry of the second row.

8. $$\begin{pmatrix} 2 & 1 & 3 & 5 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

   This matrix **is** in row echelon form but is not in reduced row echelon form since the leading entry of the first (and third) row is not 1.

2.2.16 In general, what is the elementary row operation that 'undoes' each of the three elementary row operations $R_i \leftrightarrow R_j$, kR_i, and $R_i + kR_j$?

Soln:

- The operation which undoes $R_i \leftrightarrow R_j$ is $R_j \leftrightarrow R_i$.
- The operation which undoes kR_i is $\frac{1}{k}R_i$; recall that k must be non-zero.
- The operation which undoes $R_i + kR_j$ is $R_i - kR_j$; recall that R_j stays the same after the operation.

2.2.19 What is wrong with the following 'proof' that every matrix with at least two rows is row equivalent to a matrix with a zero row?

> Perform $R_2 + R_1$ and $R_1 + R_2$. Now rows 1 and 2 are identical, so perform $R_2 - R_1$ and now the second row is a row with all zeros.

Soln: The problem is the sentence that says 'now rows 1 and 2 are identical.' After the operation $R_2 + R_1$ row two has changed to $R_2' = R_2 + R_1$. And then in the operation $R_1 + R_2$ you are actually using the modified row R_2' and therefore you get :

$$\begin{pmatrix} 2R_1 + R_2 \\ R_2 + R_1 \end{pmatrix}$$

which, in general, are different quantities, so will not yield a zero row when subtracted.

2.2.35-38 For each of the following, we will determine if each of the linear systems has a unique solution, infinitely many, or no solutions.

35.

$$\left(\begin{array}{ccc|c} 0 & 0 & 1 & 2 \\ 0 & 1 & 3 & 1 \\ 1 & 0 & 1 & 1 \end{array}\right)$$

This matrix has a unique solution, as switching rows 1 and 3 yields a matrix in row echelon form where the rank is full (i.e. there is a leading entry in each column) meaning, by the Rank Theorem, there is no free variables.

36.

$$\left(\begin{array}{cccc|c} 3 & -2 & 0 & 1 & 1 \\ 1 & 2 & -3 & 1 & -1 \\ 2 & 4 & -6 & 2 & 0 \end{array}\right)$$

This matrix has no solutions, because the second row of the coefficient row is a multiple of the 3rd row of the coefficient matrix, but the constant terms of each are not the same multiple of each other, meaning if you subtracted twice the second row from the third you'd have an obviously inconsistent row.

37.

$$\left(\begin{array}{cccc|c} 1 & 2 & 3 & 4 & 0 \\ 5 & 6 & 7 & 8 & 0 \\ 9 & 10 & 11 & 12 & 0 \end{array}\right)$$

This matrix has infinitely many solution. By inspection, it has at least one solution (namely $\mathbf{0}$ is a solution), so the system is not inconsistent. By the Rank Theorem, the number of free variables is at least 1 since the rank of the matrix is at most 3 (since there are only 3 rows, and hence only the possibility for 3 leading coefficents) but there are 4 columns. Therefore, there are infinitely many solutions

38.

$$\left(\begin{array}{ccccc|c} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 5 & 4 & 3 & 2 & 1 \\ 7 & 7 & 7 & 7 & 7 & 7 \end{array}\right)$$

Since the number of columns exceeds rows, by the Rank Theorem, we know if the system is consistent then there are free variables. By inspection the system is consistent, since setting the first variable to 1 the rest to 0 satisfies the equations. Therefore, there is infinitely many solutions.

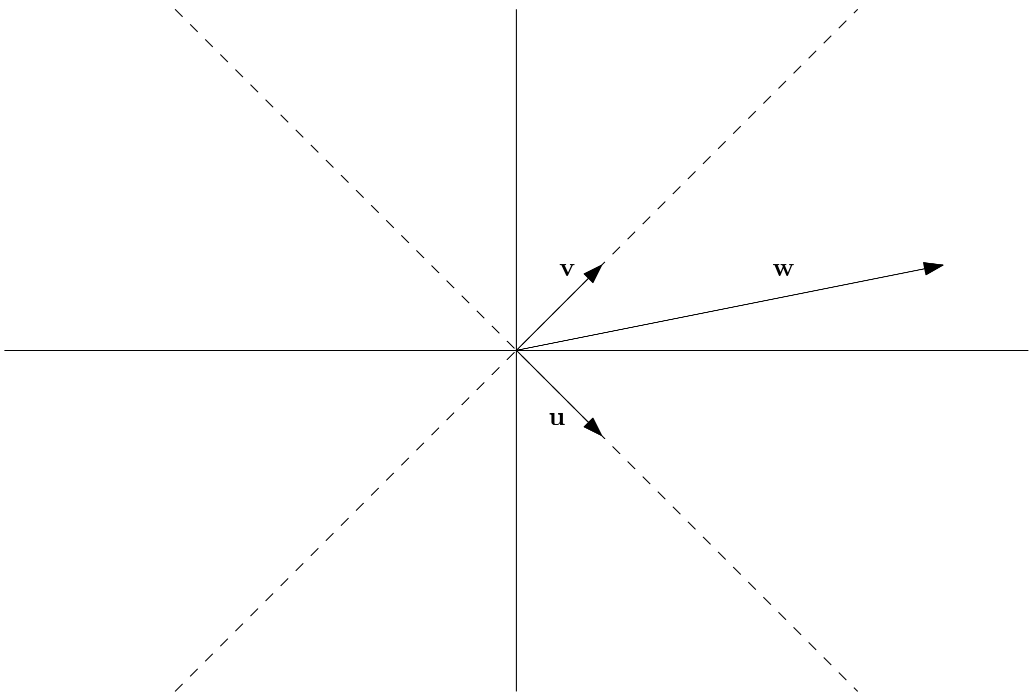
1.1.17 Solve the for the vector $\mathbf{x}$ in terms of $\mathbf{a}$ and $\mathbf{b}$ in

$$\mathbf{x} - \mathbf{a} = 2(\mathbf{x} - 2\mathbf{a})$$

Soln: For this problem we didn't need to state the theorems we were using to do algebra, but I will for the sake of completeness.

$$\begin{aligned} & \mathbf{x} - \mathbf{a} = 2(\mathbf{x} - 2\mathbf{a}) \\ \Longrightarrow\ & \mathbf{x} - \mathbf{a} = 2\mathbf{x} + 2(-2\mathbf{a}) \\ \Longrightarrow\ & \mathbf{x} - \mathbf{a} = 2\mathbf{x} - 4\mathbf{a} \\ \Longrightarrow\ & \quad -\mathbf{x} = -3\mathbf{a} \\ \Longrightarrow\ & \quad \mathbf{x} = 3\mathbf{a} \end{aligned}$$

1.1.19 Draw the coordinate axes relative to $\mathbf{u} = [1, -1]$ and $\mathbf{v} = [1, 1]$ and locate $\mathbf{w} = 2\mathbf{u} + 3\mathbf{v}$



2.3.7 Determine if $\mathbf{b} = [5, 6]$ is in the span of the columns of the A:

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$$

Soln. We set up the augmented matrix:

$$\left(\begin{array}{cc|c} 1 & 2 & 5 \\ 3 & 4 & 6 \end{array}\right) \xrightarrow{R_2 - 3R_1} \left(\begin{array}{cc|c} 1 & 2 & 5 \\ 0 & -2 & -9 \end{array}\right)$$

This matrix has full rank and is consistent, and therefore has a unique solution. Therefore, $\mathbf{b}$ is in the span of the columns. We can go farther and determine what linear combination of the columns produce $\mathbf{b}$ by getting it in reduced row echelon form.

$$\left(\begin{array}{cc|c} 1 & 2 & 5 \\ 0 & -2 & -9 \end{array}\right) \xrightarrow{R_1 + R_2} \left(\begin{array}{cc|c} 1 & 0 & -4 \\ 0 & -2 & -9 \end{array}\right) \xrightarrow{-1/2 R_2} \left(\begin{array}{cc|c} 1 & 0 & -4 \\ 0 & 1 & 9/2 \end{array}\right)$$

So, $-4[1, 3] + 9/2[2, 4] = [5, 6]$

2.3.19 Prove that $\mathbf{u}, \mathbf{v}$, and $\mathbf{w}$ are all in $\text{span}(\mathbf{u}, \mathbf{u}+\mathbf{v}, \mathbf{u}+\mathbf{v}+\mathbf{w})$

Soln. By inspection:

- $\mathbf{u} = 1\mathbf{u} + 0(\mathbf{u}+\mathbf{v}) + 0(\mathbf{u}+\mathbf{v}+\mathbf{w})$
- $\mathbf{v} = -1\mathbf{u} + 1(\mathbf{u}+\mathbf{v}) + 0(\mathbf{u}+\mathbf{v}+\mathbf{w})$
- $\mathbf{w} = 0\mathbf{u} - 1(\mathbf{u}+\mathbf{v}) + 1(\mathbf{u}+\mathbf{v}+\mathbf{w})$

If you were unable to inspect this, you could have write down a linear combination of these vectors:

$$c_1\mathbf{u} + c_2(\mathbf{u}+\mathbf{v}) + c_3(\mathbf{u}+\mathbf{v}+\mathbf{w})$$

Which you can write as:

$$(c_1 + c_2 + c_3)\mathbf{u} + (c_2 + c_3)\mathbf{v} + c_3\mathbf{w}$$

Then, in the case when you want to find what linear combination is $\mathbf{u}$ you want to solve the following system:

$$\begin{aligned} c_1 + c_2 + c_3 &= 1 \\ c_2 + c_3 &= 0 \\ c_3 &= 0 \end{aligned}$$

Similarly for the other two cases.