Day 9

Friday June 1, 2012

1 Containment Proofs

It is often necessary to prove that one set is contained in another. These are call containment proofs. This amount to proving

$$\forall x \in A \,.\, x \in B$$

That is, if x is in A then x is in B.

Question 1. What do you think the first line and last lines of such a proof should be?

Answer 1. Take $x \in A$ arbitrary

...

Then $x \in B$.

Recall we have defined the following operations so far:

- $A \cup B$ is the set of all elements that are in A or in B. That is:

  $$\forall x \,.\, x \in A \cup B \longleftrightarrow (x \in A \vee x \in B)$$

  Therefore, how do you think you use the information that $x \in A \cup B$?

  You do cases on whether $x \in A$ or $x \in B$.

- $A \cap B$ is the set of all elements that are in both A and B. That is:

  $$\forall x \,.\, x \in A \cap B \longleftrightarrow (x \in A \wedge x \in B)$$

  How do we use the information that $x \in A \cap B$?

  You know that $x \in A$ and $x \in B$

Example 1. For any set A and B we have

$$A \cap B \subseteq A \cup B$$

Proof. Take $x \in A \cap B$ arbitrary. So $x \in A$ and $x \in B$. In particular, $x \in A$, so $x \in A \cup B$. □

For equality remember

$$(A = B) \iff A \subseteq B \text{ and } B \subseteq A$$

Therefore, there are two directions to $A = B$; one when you take an element of A and show it is in B, and the other when you take one in B and show it's in A. We will see more example of this soon, but we need a few more ways to build sets.

Question 2. What would it take to show that $\neg(A \subseteq B)$?

Answer 2. You would have to show an x that is in A but is not in B. We will see some examples later.

1.1 Comprehension

Say we have some logical stuff, and we wish to form a new set by taking all thing things that satisfy this. For instance, maybe I want to take all the natural numbers n that are even, ie. satisfy $\exists m \in \mathbb{N} \mathbin{.} 2m = n$.

This is called **comphrension**, **seperation**, or **specification** depending on who you talk to. The notation that we use for this is called **set builder notation**. In this instance, we would write

$$\{ x \in \mathbb{N} \mid \exists m \in \mathbb{N} \mathbin{.} 2m = n \}$$

We read this as "the set of all natural numbers such that there exists an m in the naturals such that $2m$ equal n."

Note, in this case we are carving a smaller set out of a larger set. When doing this comphrension, as with relative complements, it doesn't make sense without specifying a larger set.

In general, set builder notation looks like

$$B = \{ x \in A \mid P(x) \}$$

where $P(x)$ is some formula with only x free. Then, to check membership in this set, one "loops through" the member of A,

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New Set B = \emptyset;
foreach(x in A) {
   if ( P(x) ) {
      Put x in B;
      next;
   }
   else {
      next;
   }
}
```

Question 3. If $P(x)$ is always false, what would B be?

Answer 3. The emptyset!

Example 2.

- $\{ n \in \mathbb{N} \mid \exists m \in \mathbb{N} \mathbin{.} 2m = n \}$ = even naturals
- $\{ n \in \mathbb{Z} \mid n \geq 0 \} = \mathbb{N}$
- $\{ n \in \mathbb{R} \mid n = \lceil n \rceil \} = \mathbb{Z}$
- $\{ n \in \mathbb{N} \mid (n \neq 1) \land \forall p_1, p_2 \mathbin{.} n = p_1 \cdot p_2 \to ((p_1 = 1) \lor (p_2 = 1)) \}$ = set of primes.

1.2 Relative Complements

So, one may ask the question: What is the compliment of a set, ie. the set of all elements not in that set? What is the problem with asking this?

Well, the color blue is an object, and it is not a natural number, so would the color blue be in the complement on $\mathbb{N}$? It's hard to peg down exactly the larger universe we are dealing with here, so it's not natural to take a compliment of a set.

To fix this problem we talk about the **relative complement** or **set difference**. The set difference, which we say A take away B or A minus B, is the things which are in A but not in B. We write this is

$$A \setminus B$$

(in LaTeX, this is \setminus)

Logically, we write

$$\forall x \mathbin{.} (x \in A \setminus B) \longleftrightarrow ((x \in A) \land (x \notin B))$$

Note: I used some 'slang' notation here. If $\neg(x \in B)$ we usually write $x \notin B$.

1.3 Examples of Containment Proofs

Example 3. 1. $(A \setminus B) \cup (C \setminus B) = (A \cup C) \setminus B$

Proof. $(\subseteq)$ First, take $x \in (A \setminus B) \cup (C \setminus B)$. We do cases on whether $x \in A \setminus B$ or $x \in C \setminus B$.

Case 1: $x \in A \setminus B$. then $x \in A$ and $x \notin B$. So, as $x \in A$ we know $x \in A \cup C$ as this is a larger set. So $x \in (A \cup C) \setminus B$ as $x \notin B$, which is what we want.

Case 2: $x \in C \setminus B$, then $x \in C$ and $x \notin B$. So as $x \in C$ we know $x \in A \cup C$ as this is a larger set. So $x \in (A \cup C \setminus B$ as $x \notin B$, which is what we want.

$(\supseteq)$ Take $x \in (A \cup C) \setminus B$. Then $x \in (A \cup C)$ and $x \notin B$. Do cases on whether $x \in A$ or $x \in C$.

Case 1: $x \in A$. Then $x \in A$ and $x \notin B$, so $x \in A \setminus B$. so $x \in (A \setminus B) \cup (C \setminus B)$.

Case 2: $x \in C$. Then $x \in C$ and $x \notin B$, so $x \in C \setminus B$. So $x \in (A \setminus B \cup (C \setminus B)$.

□

2. $(A \setminus B) \cap C = (A \cap C) \setminus B$.

Proof. $(\supseteq)$ First take $x \in (A \cap C) \setminus B$. Then $x \in A \cap C$ and $x \notin B$. So $x \in A$ and $x \in C$ and $x \notin B$. So, $x \in A \setminus B$, so we are done as x is also in C so $x \in (A \setminus B) \cap C$

$(\subseteq)$ Take $x \in (A \setminus B) \cap C$. then $x \in A \setminus B$ and $x \in C$. So $x \in A$ and $x \in C$ but $x \notin B$. So $x \in A \cap C$. So $x \in (A \cap C) \setminus B$ □

3. $A \setminus (B \cup C) \subseteq A \setminus B$ but equality need not hold.

Proof. Take $x \in A \setminus (B \cup C)$. Then $x \in A$ and $x \notin B \cup C$. Then $x \notin B$, as otherwise x would be in $B \cup C$. So $x \in A \setminus B$.

Equality need not hold; For instance, if $A = \{1\}$ $B = \emptyset$ and $C = A$, the lefthand side is empty, but the righthand side is $\{1\}$. □

4. $\{ n \in \mathbb{R} \mid n = \lceil n \rceil \} = \mathbb{Z}$

Proof. Take x in the left hand side. Then $x = \lceil x \rceil$. As $\lceil x \rceil$ is an integer by definition, $x \in \mathbb{Z}$.

Take x in the right hand side. x is an integer, so $x \in \lceil x \rceil$. So x is the left hand side. □

2 Indexed Families

Let Λ be a set. We can talk about a family of sets X_α where each $\alpha \in \Lambda$. That is to say, that there are a bunch of sets X_α one for each $\alpha \in \Lambda$. Then, we can take the union of all of these sets by

$$\bigcup_{\alpha \in \Lambda} X_\alpha$$

And similarly the intersection

$$\bigcap_{\alpha \in \Lambda} X_\alpha$$

Think of these a lot like summations. For instance, if $\Lambda = [5]$ then really

$$\bigcup_{\alpha \in [5]} X_\alpha = X_1 \cup X_2 \cup X_3 \cup X_4 \cup X_5$$

If you know $x \in \bigcup_{\alpha \in \Lambda} X_\alpha$ then you know there is some $\alpha \in \Lambda$ such that $x \in X_\alpha$. Similarly, if you know $x \in \bigcap_{\alpha \in \Lambda} X_\alpha$ then you know that for every $\alpha \in \Lambda$, $x \in X_\alpha$.

Does this seem familiar?

$$\forall x \,.\, ((x \in \bigcup_{\alpha\in\Lambda} X_\alpha) \longleftrightarrow (\exists \alpha \in \Lambda \,.\, x \in X_\alpha))$$

And

$$\forall x \,.\, ((x \in \bigcap_{\alpha\in\Lambda} X_\alpha) \longleftrightarrow (\forall \alpha \in \Lambda \,.\, x \in X_\alpha))$$

Example 4.

$$\text{Primes} \subseteq \bigcup p \in \text{Primes} \left\{ n \in \mathbb{N} \mid n = p^k \text{ for some } k \in \mathbb{N} \right\}$$

Proof. Take x a prime number. Then $p = p^1$. Therefore $x \in \left\{ n \in \mathbb{N} \mid n = x^k \text{ for some } k \in \mathbb{N} \right\}$ □

Example 5.

$$\{1\} = \bigcap_{i\in\mathbb{N}; i>1} \{ n \in \mathbb{N} \mid \neg(i \mid n) \}$$

Proof. ($\subseteq$) Take x in the left hand side. It must be 1. We want to show that 1 is in the right hand side; so it suffices to show for every $i \in \mathbb{N}$ $i > 1$ that $1 \in \{ n \in \mathbb{N} \mid \neg(i \mid 1) \}$. Let i be arbitrary natural larger than 1. Then i does not divide 1, so 1 is in that set.

($\supseteq$) Take $x \in \bigcap_{i\in\mathbb{N}; i>1} \{ n \in \mathbb{N} \mid \neg(i \mid n) \}$. Well, x is a natural, and we we know that for every $i \in \mathbb{N}$ where $i > 1$ that $\neg(i \mid x)$. Suppose that x is natural and not equal to 1 for contradiction. If $x = 0$ then we get an immediate contradiction because every i in $\mathbb{N}$ divides 0, in particular, 2 does.

Otherwise, $x > 1$. Then, as we know for every $i \in \mathbb{N}$ where $i > 1$ that $\neg(i \mid x)$, we know in particular $\neg(x \mid x)$. But this is a contradiction. □