Assignment 4

Friday June 8, 2012

1 Set Theory

This material is intended to be completed Tuesday June 5th. Here will fully cement our knowledge of basic operations on sets.

Problem 1. Prove or disprove the following:

1. $(A \setminus C) \cup (B \setminus C) = (A \cup B) \setminus C$
2. $(A \setminus B) \cap C = A \setminus (B \cap C)$

Problem 2.

1. Prove:
$$\bigcap_{\alpha \in (A \cup B)} A_\alpha \subseteq \Big(\bigcap_{\alpha \in A} A_\alpha\Big) \cup \Big(\bigcap_{\alpha \in B} A_\alpha\Big)$$
2. Show equality need not hold.

Problem 3. Let A and B be sets.

1. Prove
$$\wp(A) \cup \wp(B) \subseteq \wp(A \cup B)$$
2. Show that equality need not hold.

2 Relations

2.1 Basics

This material is intended to be completed on Wednesday June 6th.

Problem 4. For each of the following relations, determine if they are reflexive, symmetric, and transitive; justify your answers

1. Define R on $\mathbb{R}$ such that aRb if and only if $ab = 0$.
2. Define S on $\mathbb{R}$ such that aSb if and only if $ab \neq 0$.
3. Define d on $\mathbb{R}$ such that $d(a, b)$ if and only if $|a - b| < \frac{1}{2}$.
4. Define T on $\mathbb{Z}$ such that nTm if and only if $n - m$ is even.

2.2 Equivalence Relations

This section is intended to be completed on Wednesday June 6th. Here we will take a look at equivalence relations.

Recall that we proved that a equivalence relation yields a partition of **equivalence classes**. That is, if $\sim$ is an equivalence relation on S, then for every element $a \in S$ we have a corresponding equivalence class

$$[a]_\sim = \{ b \in S \mid a \sim b \}$$

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Problem 5. In this problem we prove the converse of the above; that is, we prove that given a partition of a set we can "read off" an equivalence relation.

Let S be a set, and P a partition of S. Recall that P partitions S if and only if

- $A \in P \implies A \in \wp(S)$ ie. P is a set of subsets of A.
- $A, B \in P \implies A \cap B = \emptyset$, ie. every two members of the partition are disjoint.
- $x \in S \implies \exists A \in P \,\textbf{.}\, x \in P$, ie. every member of S is in some element of the partition.

1. Give a partition of the set $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$.
2. Fix S and P as above. Define a relation $\sim$ on S by $a \sim b$ if and only if there is a $A \in P$ such that $a, b \in A$. Prove that the relation is an equivalence relation.
3. Consider the partition of $\mathbb{N}$ into the following 3 sets:

$$\{0, 3, 6, 9, 12, 15, \ldots\}$$
$$\{1, 4, 7, 10, 13, 16, \ldots\}$$
$$\{2, 5, 8, 11, 14, 17, \ldots\}$$

   Take $\sim$ to be the corresponding relation that you get from this partition (from part 2). Give a necessary and sufficient condition for deciding if $a \sim b$ that does not refer to the partition. (ie. $a \sim b$ if and only if ...)

Problem 6. Prove that each of the following are equivalence relations:

1. Define $\sim$ on $\mathbb{Z}^+ \times \mathbb{Z}^+$ by $(a, b) \sim (c, d)$ if and only if $ad = bc$.
2. Define $\cong$ on $\mathbb{R}$ by $a \cong b$ if and only if $\frac{a}{b} \in \mathbb{Q}$.

2.3 Preview of Modular Arithmetic

This material is intended to be completed on Thursday June 7th. This is a preview section. Here, we will preview Modular Arithmetic, which we will talk about in great detail on Friday.

Problem 7. Prove

$$\forall n, d \in \mathbb{N} \,\textbf{.}\, \exists r, q \in \mathbb{N} \,\textbf{.}\, (n = dq + r) \land (0 \leq r < n)$$

Hint: Take d arbitrary and do induction on n; consider cases: what if d is smaller than n? What if it is larger?

Tomorrow in class we will prove that r and d are **unique**

Problem 8. Find the r and q for the given n and d.

1. $n = 5$, $d = 3$.
2. $n = 123$, $d = 6$.
3. $n = 10^{10}$, $d = 2$.

Definition 1. For a fixed n, the q in the theorem above is call the **quotient**, the d the **divisor**, and r the **remainder**. So, this says that any given a number and a divisor d, there is a unique quotient and remainder.

Problem 9. Fix $n \in \mathbb{N}$. Define a relation $\sim$ where $p \sim q$ if and only if p and q have the same remainder when you divide by n. Prove this is an equivalence relation. Moreover, determine how many equivalence classes does this relation have (this will depend on n of course) and prove it.

Problem 10. What are the equivalence classes in the above defined relation when $n = 2$?