Probability 21-325 **Homework 5 (due 28th Feb)** **TT**

1. Let X and Y be independent random variables taking values in the set $\{0, 1, \ldots\}$ with the generating functions G_X and G_Y. Let k be an integer. Show that $\mathbb{P}(X - Y = k)$ equals the coefficient at t^k in the expansion of the function $G_X(t)G_Y(1/t)$ into a formal power series.

2. Let $X_1, X_2, \ldots, X_6$ be independent identically distributed random variables uniform on the set $\{0, 1, \ldots, 9\}$. Find $\mathbb{P}(X_1 + X_2 + X_3 = X_4 + X_5 + X_6)$.

3. Let $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n$ be independent random signs. Show that $\mathrm{Var}(\sum_{i=1}^n a_i\varepsilon_i) = \sum_{i=1}^n a_i^2$ for any reals $a_1, \ldots, a_n$.

4. Let $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n$ be independent random signs. Show that for any reals $a_1, \ldots, a_n$ we have

$$\mathbb{E}\Big|\sum_{i=1}^n a_i\varepsilon_i\Big|^4 \leq 3\left(\mathbb{E}\Big|\sum_{i=1}^n a_i\varepsilon_i\Big|^2\right)^2.$$

Show that the constant 3 is best possible (in other words, is sharp), that is, if it is replaced with any smaller number, the statement is no longer true.

5. Show that

$$F(t) = \begin{cases} \frac{1}{3}e^t, & t < 0, \\ \frac{1}{2} + \frac{1}{2}(1 - e^{-t}), & t \geq 0 \end{cases}$$

is the distribution function of a random variable, say X. Compute $\mathbb{P}(X < -1)$, $\mathbb{P}(X < 0)$, $\mathbb{P}(X \leq 0)$, $\mathbb{P}(X = 0)$, $\mathbb{P}(X > 1)$ and $\mathbb{P}(X = 2)$.

6. The double exponential distribution with parameter $\lambda > 0$ has density $f(x) = \frac{\lambda}{2}e^{-\lambda|x|}$. Find its distribution function, sketch its plot, find the mean, variance and pth moment.

7. Let X be a uniform random variable on $(0, 1)$. Find the distribution function and density of $Y = -\ln X$. What is the distribution of Y called?

8. Let X be a Poisson random variable with parameter λ. Show that $\mathbb{P}(X \geq k) = \mathbb{P}(Y \leq \lambda)$, for $k = 1, 2, \ldots$, where Y is a random variable with the Gamma distribution with parameter k.

9. Let X be a random variable with continuous distribution function F. Show that $Y = F(X)$ is a random variable uniformly distributed on the interval $(0, 1)$.

10.* Let F be a distribution function and U be a uniform random variable on $(0,1)$. Define the generalised inverse of F by

$$G(y) = \inf\{x,\ F(x) \geq y\}.$$

Show that the distribution function of the random variable $G(U)$ is F.