An Explicit Formula for the Skorokhod Map on $[0, a]$

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Abstract: The Skorokhod map is a convenient tool for constructing solutions to stochastic differential equations with reflecting boundary conditions. In this work, an explicit formula for the Skorokhod map $\Gamma_{0,a}$ on $[0, a]$ for any $a > 0$ is derived. Specifically, it is shown that on

[*]Supported by the State Committee for Scientific Research of Poland Grant No. 2 PO3A 012 23

[†]Partially supported by ONR/DARPA MURI under contract N000140-01-1-0576.

[‡]Partially Supported by the National Science Foundation under Grants No. DMS-0406191, DMI-0323668-0000000965, DMS-0405343)

[§]Partially supported by the National Science Foundation under Grant No. DMS-0404682

the space $\mathcal{D}[0,\infty)$ of right-continuous functions with left limits taking values in $I\!R$, $\Gamma_{0,a} = \Lambda_a \circ \Gamma_0$, where $\Lambda_a : \mathcal{D}[0,\infty) \to \mathcal{D}[0,\infty)$ is defined by

$$\Lambda_a(\phi)(t) = \phi(t) - \sup_{s\in[0,t]} \left[(\phi(s) - a)^+ \wedge \inf_{u\in[s,t]} \phi(u) \right]$$

and $\Gamma_0 : \mathcal{D}[0,\infty) \to \mathcal{D}[0,\infty)$ is the Skorokhod map on $[0,\infty)$. In addition, properties of Λ_a are developed and comparison properties of $\Gamma_{0,a}$ are established.

AMS 2000 subject classifications: Primary 60G05, 60G17; secondary 60J60, 90B05, 90B22.
Keywords and phrases: Skorokhod map, reflection map, double-sided reflection map, comparison principle.

1. Introduction

1.1. Background

In 1961 A. V. Skorokhod [11] considered the problem of constructing solutions to stochastic differential equations on the half-line $I\!R_+$ with a reflecting boundary condition at 0. His construction implicitly used properties of a deterministic mapping on the space $\mathcal{C}[0,\infty)$ of continuous functions on $[0,\infty)$. This mapping was used more explicitly by Anderson and Orey in their study of large deviations properties of reflected diffusions on a half-space in $I\!R^N$ (see p. 194 of [1]), where they exploited the fact that the mapping, which is now called the *Skorokhod map* and is denoted here by Γ_0, has the explicit representation

$$\Gamma_0(\psi)(t) = \psi(t) + \sup_{s\in[0,t]} [-\psi(s)]^+ , \quad \psi \in \mathcal{C}[0,\infty) , \tag{1.1}$$

and is consequently Lipschitz continuous (with constant 2) with respect to the uniform norm on $\mathcal{C}[0,\infty)$. El Karoui and Chaleyat-Maurel [5] used Γ_0 in a study of local times.

Given any trajectory ψ in $\mathcal{D}[0,\infty)$, the space of right-continuous functions with left limits mapping $[0,\infty)$ into $I\!R$, Γ_0 can be extended using formula (1.1) to map ψ to a "constrained version" $\phi = \psi + \eta$ of ψ that is restricted to take values in $[0,\infty)$ by the minimal pushing term $\eta(t) \doteq \sup_{s\in[0,t]}[-\psi(s)]^+$. Minimality of η follows from the fact that η increases only at times t when $\phi(t) = 0$ (see Definition 1.1 below for a precise statement). A multidimensional extension of the Skorokhod map was introduced by Tanaka [13]. Given any right-continuous function with left limits on $[0,\infty)$ taking values in $I\!R^N$, Tanaka produced a corresponding function taking values in a given convex domain by adding a constraining term on the boundary that acts in the direction normal to the boundary. Tanaka then used the solution to this Skorokhod problem to construct solutions of stochastic differential equations with normal reflection. In general, the Skorokhod map is a convenient tool for constructing processes that are restricted to take values in a certain domain by a constraining force that can push only along specified directions at the boundary. The study of many properties of the constrained or "reflected" process then reduces to the study of corresponding properties of the associated Skorokhod map.

In this paper, we focus on the particular case when the domain is a bounded interval $[0, a]$ in $I\!R$. For functions in $\mathcal{D}[0,\infty)$, Chaleyat, et. al. [4] posed and solved a version of the Skorokhod problem, producing functions taking values in $[0, a]$. However, in [4] the treatment of jumps across the boundary is different from that of Tanaka and this paper because in [4] the constrained function really "reflects" such jumps off the boundary, taking values in the interior of $[0, a]$, rather than being "constrained" to stay at the boundary. In contrast to [4], in this paper the Skorokhod map $\Gamma_{0,a}$ maps a trajectory in $\mathcal{D}[0,\infty)$ to a trajectory $\bar{\phi}$ in $\mathcal{D}[0,\infty)$ that is constrained to take values in $[0, a]$ by a *minimal* pushing force $\bar{\eta}$ that is allowed to increase only when $\bar{\phi}$ is at the lower boundary 0 and decrease only when $\bar{\phi}$ is at the upper boundary a (see Definition 1.2 for a precise description of the Skorokhod map on $[0, a]$). Existence and uniqueness of solutions to this Skorokhod problem for continuous functions as well as step functions in $\mathcal{D}[0,\infty)$ follow directly from Tanaka [13], Lemmas 2.1, 2.3 and 2.6. In fact, it is well-known that solutions to this Skorokhod problem exist for all functions in $\mathcal{D}[0,\infty)$ (see, for example, [2]).

In contrast to the Skorokhod map (1.1) on $[0,\infty)$, prior to the present work no explicit formula for the Skorokhod map $\Gamma_{0,a}$ on $[0, a]$ (sometimes called the *two-sided reflection map*) was known. This is provided in Theorem 1.4 below. We then use this formula to establish comparison properties of $\Gamma_{0,a}$ (Theorem 1.6). This formula involves a new map, Λ_a, defined by (1.11). Properties of Λ_a are developed in Proposition 1.3 and Corollary 1.5.

The explicit formula for the Skorokhod map on $[0,\infty)$ has found application in a variety of contexts, including queueing theory and finance (see, for example, [6], [14] and [7]). More recently, it was used in [3] and [12] to derive various interesting distributional properties of quantities related to Brownian motion reflected on Brownian motion, a process that arises in the study of true self-repelling motions. In a similar fashion, the explicit formula for the Skorokhod map on $[0, a]$ is likely to have several potential applications. Already in [9] this formula plays a crucial role in the derivation of a diffusion approximation for the $GI/G/1$ queue with earliest-deadline-first service and reneging by customers who become late. In addition, in [8] the comparison properties of Theorem 1.6 are used to provide bounds on transaction costs in an optimal consumption/investment model.

The outline of the paper is as follows. In Section 1.2 we introduce notation and recall the precise definitions and basic properties of Γ_0 and $\Gamma_{0,a}$. In Section 1.3, we state the main results. Properties of Λ_a are established in Section 2. Two proofs of Theorem 1.4 are presented in Sections 3 and 4, respectively. The proof of Theorem 1.6 is given in Section 5. A technical result is relegated to the appendix.

1.2. Basic Definitions

Let $\mathcal{D}_+[0,\infty)$, $\mathcal{C}[0,\infty)$, $\mathcal{I}[0,\infty)$ and $\mathcal{BV}[0,\infty)$ denote the subspace of non-negative, continuous, non-decreasing and bounded variation functions, respectively, in $\mathcal{D}[0,\infty)$. For $f \in \mathcal{BV}[0,\infty)$, $|f|_t$ denotes the total variation of f on $[0,t]$. For $f \in \mathcal{D}[0,T]$, $||f||_T$ denotes the supremum norm of f on $[0,T]$. Let $I\!R_+$ denote the set of non-negative real numbers. Given $a, b \in I\!R$, denote $a \wedge b \doteq \min\{a,b\}$, $a \vee b \doteq \max\{a,b\}$, and $a^+ \doteq a \vee 0$. We denote by $\mathbb{I}_A$ the indicator function of a set A.

Definition 1.1. (Skorokhod map on $[0,\infty)$) *Given $\psi \in \mathcal{D}[0,\infty)$ there exists a unique pair of functions $(\phi,\eta) \in \mathcal{D}[0,\infty) \times \mathcal{I}[0,\infty)$ that satisfy the following two properties:*

1. *For every $t \in [0,\infty)$, $\phi(t) = \psi(t) + \eta(t) \in I\!R_+$;*
2. *$\eta(0-) = 0$, $\eta(0) \geq 0$, and*

$$\int_0^\infty 1_{\{\phi(s)>0\}} d\eta(s) = 0. \tag{1.2}$$

The map $\Gamma_0 \colon \mathcal{D}[0,\infty) \to \mathcal{D}_+[0,\infty)$ that takes ψ to the corresponding trajectory ϕ is referred to as the one-sided reflection map *or* Skorokhod map on $[0,\infty)$. *The pair (ϕ,η) is said to solve the* Skorokhod problem on $[0,\infty)$ *for ψ.*

Condition (1.2), often referred to as the *complementarity condition*, stipulates that the constraining term η can increase only at times t when $\phi(t) = 0$. As mentioned earlier, Γ_0, the Skorokhod map on $[0,\infty)$, has an explicit representation given by (1.1). The condition $\eta(0-) = 0$ is a convention by which we mean that $\eta(0) > 0$ implies that η has a jump at zero and, according to (1.2), we must have $\phi(0) = 0$, in which case $\eta(0) = -\psi(0)$. This can happen only if $\psi(0) < 0$. In the event that $\psi(0) \geq 0$, we have $\eta(0) = 0$. In either case,

$$\eta(0) = [-\psi(0)]^+. \tag{1.3}$$

In direct analogy with the Definition 1.1 and the explicit representation (1.1) for Γ_0, it is easy to see that $\Gamma_a : \mathcal{D}[0,\infty) \to \mathcal{D}[0,\infty)$ defined by

$$\Gamma_a(\psi)(t) \doteq \psi(t) - \sup_{s\in[0,t]} [\psi(s) - a]^+ \tag{1.4}$$

takes $\psi \in \mathcal{D}[0,\infty)$ to the unique corresponding trajectory $\phi \in \mathcal{D}[0,\infty)$ that satisfies $\phi(t) \in (-\infty, a]$ for $t \in [0,\infty)$ and is such that $\eta = \psi - \phi$ is non-decreasing and increases only at times t when $\phi(t) = a$ (i.e., such that $\int_0^\infty 1_{\{\phi(s)<a\}} d\eta(s) = 0$). Indeed, it is straightforward to verify that given $a > 0$ and $\psi \in \mathcal{D}[0,\infty)$,

$$\Gamma_a(\psi) = a - \Gamma_0(a - \psi). \tag{1.5}$$

The subject of this paper is the Skorokhod map that constrains a process in $\mathcal{D}[0,\infty)$ to remain within $[0,a]$, a map we now define.

Definition 1.2. (Skorokhod map $\Gamma_{0,a}$ on $[0,a]$) *Let $a > 0$ be given. Given $\psi \in \mathcal{D}[0,\infty)$ there exists a unique pair of functions $(\bar{\phi}, \bar{\eta}) \in \mathcal{D}[0,\infty) \times \mathcal{BV}[0,\infty)$ that satisfy the following two properties:*

1. *For every $t \in [0,\infty)$, $\bar{\phi}(t) = \psi(t) + \bar{\eta}(t) \in [0,a]$;*
2. *$\bar{\eta}(0-) = 0$ and $\bar{\eta}$ has the decomposition $\bar{\eta} = \bar{\eta}_\ell - \bar{\eta}_u$ as the difference of functions $\bar{\eta}_\ell, \bar{\eta}_u \in \mathcal{I}[0,\infty)$ satisfying*

$$\int_0^\infty 1_{\{\bar{\phi}(s)>0\}} d\bar{\eta}_\ell(s) = 0 \quad \text{and} \quad \int_0^\infty 1_{\{\bar{\phi}(s)<a\}} d\bar{\eta}_u(s) = 0. \tag{1.6}$$

We refer to the mapping $\Gamma_{0,a} \colon \mathcal{D}[0,\infty) \to \mathcal{D}[0,\infty)$ that takes ψ to the corresponding $\bar{\phi}$ as the two-sided reflection map *or the* Skorokhod map on $[0,a]$. *The pair $(\phi, \bar{\eta})$ is said to solve the* Skorokhod problem on $[0,a]$ *for ψ.*

Similarly to (1.3), the condition $\bar{\eta}(0-) = 0$ coupled with the complementarity conditions (1.6) implies that

$$\bar{\eta}(0) = [-\psi(0)]^+ - [\psi(0) - a]^+. \tag{1.7}$$

In other words, $\bar{\phi}(0) = \pi(\psi(0))$, where $\pi\colon \mathbb{R} \to [0,a]$ is the projection map

$$\pi(x) = \begin{cases} a & \text{if } x \geq a, \\ x & \text{if } 0 \leq x \leq a, \\ 0 & \text{if } x \leq 0. \end{cases} \tag{1.8}$$

Furthermore, from the explicit expressions for Γ_0 and Γ_a given in (1.1) and (1.4) respectively, it is clear (see, e.g., Section 2.3 of [6]) that $\bar{\eta}_\ell$ and $\bar{\eta}_u$ satisfy the equations

$$\bar{\eta}_\ell(t) = \sup_{s\in[0,t]} [\bar{\eta}_u(s) - \psi(s)]^+ \quad \text{and} \quad \bar{\eta}_u(t) = \sup_{s\in[0,t]} [\psi(s) + \bar{\eta}_\ell(s) - a]^+. \tag{1.9}$$

Now consider $\psi \in \mathcal{D}[0,\infty)$ and let $\bar{\eta} \doteq \Gamma_{0,a}(\psi) - \psi$, which has the decomposition $\bar{\eta} = \bar{\eta}_\ell - \bar{\eta}_u$ into the difference of processes in $\mathcal{I}[0,\infty)$ as in Definition 1.2. Denote $\tilde{\eta} \doteq \Gamma_{0,a}(a-\psi) - a + \psi$, which has the corresponding decomposition $\tilde{\eta} = \tilde{\eta}_\ell - \tilde{\eta}_u$. In a similar fashion to (1.5), it follows immediately from the definition that $\Gamma_{0,a}(\psi) = a - \Gamma_{0,a}(a-\psi)$ and, moreover, that

$$\tilde{\eta}_\ell = \bar{\eta}_u \qquad \text{and} \qquad \tilde{\eta}_u = \bar{\eta}_\ell. \tag{1.10}$$

1.3. Main Results

Our main result provides an explicit representation for the Skorokhod map $\Gamma_{0,a}$ on $[0,a]$ in terms of the mapping $\Lambda_a \colon \mathcal{D}[0,\infty) \to \mathcal{D}[0,\infty)$ defined by

$$\Lambda_a(\phi)(t) \doteq \phi(t) - \sup_{s\in[0,t]} \left[(\phi(s) - a)^+ \wedge \inf_{u\in[s,t]} \phi(u) \right]. \tag{1.11}$$

We will use the notation

$$R_t(\phi)(s) \doteq (\phi(s) - a)^+ \wedge \inf_{u\in[s,t]} \phi(u), \tag{1.12}$$

in terms of which (1.11) may be written as $\Lambda_a(\phi)(t) \doteq \phi(t) - \sup_{s\in[0,t]} R_t(\phi)(s)$. We list properties of Λ_a and then state our main result as Theorem 1.4. Proofs are relegated to later sections.

Proposition 1.3. *Λ_a maps $\mathcal{D}[0,\infty)$ into $\mathcal{D}[0,\infty)$, $\mathcal{C}[0,\infty)$ into $\mathcal{C}[0,\infty)$, $\mathcal{BV}[0,\infty)$ into $\mathcal{BV}[0,\infty)$, and absolutely continuous functions to absolutely continuous functions.*

Theorem 1.4. *Given $a > 0$, let Γ_0 and $\Gamma_{0,a}$ be the Skorokhod maps on $[0,\infty)$ and $[0,a]$ respectively. Then*

$$\Gamma_{0,a} = \Lambda_a \circ \Gamma_0. \tag{1.13}$$

Theorem 1.4 allows us to give concise proofs of the Lipschitz continuity of the map $\Gamma_{0,a}$ in the uniform, J_1 and M_1 metrics (Corollary 1.5 below). The proof of Corollary 1.5 is in Section 3. Continuity of Γ_0 and $\Gamma_{0,a}$ in the J_1 and M_1 metrics is due to [2]. For proofs of the inequalities for $\Gamma_{0,a}$ in Corollary 1.5 that are different from the proofs in this paper, see Section 14.8 of [14]. Below, d_∞ is the uniform metric on $[0,T]$, d_0 is the standard J_1 metric on $\mathcal{D}[0,T]$ (see, e.g., definition (3.2) on p. 79 of [14]), and d_1 is the standard M_1 metric on $\mathcal{D}[0,T]$ (see, e.g., definition (3.4) on p. 82 of [14]), while $\bar{d}_\infty$, $\bar{d}_0$ and $\bar{d}_1$ are the corresponding metrics on $\mathcal{D}[0,\infty)$ (see, e.g., Section 12.9 of [14]).

Corollary 1.5. *There exists a constant L such that for all $T > 0$ and $\psi_1, \psi_2 \in \mathcal{D}[0,T]$,*

$$d_i(\Lambda_a(\psi_1), \Lambda_a(\psi_2)) \le 2d_i(\psi_1,\psi_2) \quad \textit{for } i = 0,1,\infty; \tag{1.14}$$

$$d_i(\Gamma_{0,a}(\psi_1), \Gamma_{0,a}(\psi_2)) \le Ld_i(\psi_1,\psi_2) \quad \textit{for } i = 0,1,\infty. \tag{1.15}$$

Moreover, the six inequalities above continue to hold for $\psi_1, \psi_2 \in \mathcal{D}[0,\infty)$ if d_∞, d_0 and d_1 are replaced by $\bar{d}_\infty$, $\bar{d}_0$ and $\bar{d}_1$, respectively.

Lastly, in Theorem 1.6, we state comparison properties of the Skorokhod map on $[0,a]$. The proof of this result is presented in Section 5.

Theorem 1.6. *Given $a > 0$, $c_0, c_0' \in I\!\!R$ and $\psi, \psi' \in \mathcal{D}[0,\infty)$ with $\psi(0) = \psi'(0) = 0$, suppose $(\bar{\phi},\bar{\eta})$ and $(\bar{\phi}',\bar{\eta}')$ solve the Skorokhod problem on $[0,a]$ for $c_0+\psi$ and $c_0'+\psi'$, respectively. Moreover, suppose $\bar{\eta} = \bar{\eta}_\ell - \bar{\eta}_u$ is the decomposition of $\bar{\eta}$ into the difference of processes in $\mathcal{I}[0,\infty)$ satisfying (1.6) and $\bar{\eta}_\ell' - \bar{\eta}_u'$ is the corresponding decomposition of $\bar{\eta}'$. If there exists $\nu \in \mathcal{I}[0,\infty)$ such that $\psi = \psi' + \nu$, then the following four inequalities hold:*

1. $\bar{\eta}_\ell - [c_0' - c_0]^+ \le \bar{\eta}_\ell' \le \bar{\eta}_\ell + \nu + [c_0 - c_0']^+$;
2. $\bar{\eta}_u' - [c_0' - c_0]^+ \le \bar{\eta}_u \le \bar{\eta}_u' + \nu + [c_0 - c_0']^+$;
3. $\bar{\eta} - [c_0' - c_0]^+ \le \bar{\eta}' \le \bar{\eta} + \nu + [c_0 - c_0']^+$;
4. $[-[c_0 - c_0']^+ - \nu] \vee [-a] \le \bar{\phi}' - \bar{\phi} \le [c_0 - c_0']^+ \wedge a$.

2. Proof of Proposition 1.3

Let $\phi \in \mathcal{D}[0,\infty)$ be given. For each $\theta_1 \ge 0$ and $\epsilon > 0$, there exists $\theta_2 > \theta_1$ such that

$$\sup_{s,u\in[\theta_1,\theta_2)} |\phi(s) - \phi(u)| \le \epsilon. \tag{2.1}$$

Similarly, for each $\theta_2 > 0$ and $\epsilon > 0$, there exists $\theta_1 \in [0,\theta_2)$ such that (2.1) holds. It is straightforward to use this observation and the following lemma to verify that $\Lambda_a(\phi)$ is right-continuous with left-hand limits, i.e., that Λ_a maps $\mathcal{D}[0,\infty)$ into $\mathcal{D}[0,\infty)$.

Lemma 2.1. *Let $\phi \in \mathcal{D}[0,\infty)$ be given and assume that (2.1) is satisfied for given $0 \le \theta_1 < \theta_2$ and ϵ. Then $\sup_{t_1,t_2\in[\theta_1,\theta_2)} \left|\Lambda_a(\phi)(t_1) - \Lambda_a(\phi)(t_2)\right| \le 2\epsilon$.*

Proof. From the definition (1.12) of R_t, we see that for any $t \ge 0$,

$$(\phi(t)-a)^+ \wedge \phi(t) \le \sup_{s\in[0,t]} R_t(\phi)(s) \le \phi(t). \tag{2.2}$$

Let t_1, t_2 be in $[\theta_1,\theta_2)$ with $t_1 \le t_2$. Then $R_{t_2}(\phi)(s) \le R_{t_1}(\phi)(s)$. Under condition (2.1),

$$\sup_{s\in(t_1,t_2]} R_{t_2}(\phi)(s) \le \sup_{s\in(t_1,t_2]} (\phi(s)-a)^+ \le (\phi(t_1)-a)^+ + \epsilon.$$

Therefore

$$\begin{aligned}
\sup_{s\in[0,t_2]} R_{t_2}(\phi)(s) &= \sup_{s\in[0,t_1]} R_{t_2}(\phi)(s) \vee \sup_{s\in(t_1,t_2]} R_{t_2}(\phi)(s) \\
&\le \sup_{s\in[0,t_1]} R_{t_1}(\phi)(s) \vee \left[(\phi(t_1)-a)^+ + \epsilon\right] \\
&\le \sup_{s\in[0,t_1]} R_{t_1}(\phi)(s) + \epsilon,
\end{aligned}$$

which implies

$$\begin{aligned}
\Lambda_a(\phi)(t_2) &= \phi(t_2) - \sup_{s\in[0,t_2]} R_{t_2}(\phi)(s) \\
&\ge \phi(t_1) - \epsilon - \sup_{s\in[0,t_1]} R_{t_1}(\phi)(s) - \epsilon \\
&= \Lambda_a(\phi)(t_1) - 2\epsilon.
\end{aligned}$$

The second inequality in (2.2) and inequality (2.1) imply

$$\begin{aligned}
\sup_{s\in[0,t_1]} R_{t_1}(\phi)(s) - \epsilon &\le \sup_{s\in[0,t_1]} R_{t_1}(\phi)(s) \wedge (\phi(t_1)-\epsilon) \\
&\le \sup_{s\in[0,t_1]} \left[R_{t_1}(\phi)(s) \wedge \inf_{s\in(t_1,t_2]} \phi(u)\right] \\
&= \sup_{s\in[0,t_1]} R_{t_2}(\phi)(s) \\
&\le \sup_{s\in[0,t_2]} R_{t_2}(\phi)(s).
\end{aligned}$$

From this we conclude that

$$\begin{aligned}
\Lambda_a(\phi)(t_2) &= \phi(t_2) - \sup_{s\in[0,t_2]} R_{t_2}(\phi)(s) \\
&\le \phi(t_1) + \epsilon - \sup_{s\in[0,t_1]} R_{t_1}(\phi)(s) + \epsilon \\
&= \Lambda_a(\phi)(t_1) + 2\epsilon.
\end{aligned}$$

□

Remark 2.2. The proof of Lemma 2.1 shows that if in place of (2.1) we have a bound ϵ on the oscillation of ϕ over a closed interval, i.e., $\sup_{s,u\in[\theta_1,\theta_2]} |\phi(s)-\phi(u)| \leq \epsilon$, then $\sup_{t_1,t_2\in[\theta_1,\theta_2]} |\Lambda_a(\phi)(t_1) - \Lambda_a(\phi)(t_2)| \leq 2\epsilon$. Therefore, Λ_a maps $\mathcal{C}[0,\infty)$ to $\mathcal{C}[0,\infty)$.

Corollary 2.3. *Λ_a maps absolutely continuous functions to absolutely continuous functions.*

Proof. Suppose $\phi \in \mathcal{D}[0,\infty)$ is absolutely continuous. We fix an arbitrary $T > 0$. By the definition of absolutely continuity, there exists a function $v_\phi : (0,\infty) \to (0,\infty)$ such that for every $\varepsilon > 0$ and every set of non-overlapping intervals (s_j, t_j), $j = 1, \ldots, J$, contained in $[0,T]$,

$$\sum_{j=1}^{J}(t_j - s_j) < v_\phi(\varepsilon) \quad \Rightarrow \quad \sum_{j=1}^{J}|\phi(t_j) - \phi(s_j)| < \varepsilon. \tag{2.3}$$

Define the function $v_{\Lambda_a(\phi)} : (0,\infty) \to (0,\infty)$ by $v_{\Lambda_a(\phi)}(\varepsilon) \doteq v_\phi(\varepsilon/2)$ for $\varepsilon > 0$. We claim that (2.3) holds with ϕ replaced everywhere by $\Lambda_a(\phi)$, thus showing that $\Lambda_a(\phi)$ is absolutely continuous. For the proof of the claim, fix $\varepsilon > 0$ and consider any set of non-overlapping intervals (s_j, t_j), $j = 1, \ldots, J$, such that $\sum_{j=1}^{J}(t_j - s_j) < v_{\Lambda_a(\phi)}(\varepsilon)$. For $j = 1, \ldots, J$, choose $s_j \leq \bar{s}_j \leq \bar{t}_j \leq t_j$ such that $|\phi(\bar{t}_j) - \phi(\bar{s}_j)| = \max_{u,r\in[s_j,t_j]} |\phi(r) - \phi(u)|$. Remark 2.2 implies that $\Lambda_a(\phi) \in \mathcal{C}[0,\infty)$ and

$$\begin{aligned}
\sum_{j=1}^{J} |\Lambda_a(\phi)(t_j) - \Lambda_a(\phi)(s_j)| &\leq \sum_{j=1}^{J} \max_{u,r\in[s_j,t_j]} |\Lambda_a(\phi)(r) - \Lambda_a(\phi)(u)| \\
&\leq 2\sum_{j=1}^{J} \max_{u,r\in[s_j,t_j]} |\phi(r) - \phi(u)| \\
&= 2\sum_{j=1}^{J} |\phi(\bar{t}_j) - \phi(\bar{s}_j)| \\
&\leq \varepsilon,
\end{aligned}$$

where the last inequality is a consequence of (2.3) and the fact that $\sum_{j=1}^{J}(\bar{t}_j - \bar{s}_j) < v_{\Lambda(\phi)}(\varepsilon) = v_\phi(\varepsilon/2)$. □

To complete the proof of Proposition 1.3, it remains only to show that Λ_a maps $\mathcal{BV}[0,\infty)$ to $\mathcal{BV}[0,\infty)$. We do not use this fact in the present paper, and hence can use any results in the remainder of the paper to establish it. According to Theorem 4.4 below, the function C^ϕ given by (3.28) has bounded variation. If ϕ also has bounded variation, then $\Lambda_a(\phi) = \phi - C^\phi$ does as well.

3. First Proof of Theorem 1.4

The first proof of Theorem 1.4 requires Lemmas 3.1, 3.4 and 3.5 below. For these lemmas, ψ is an arbitrary element in $\mathcal{D}[0,\infty)$ and

$$\phi \doteq \Gamma_0(\psi), \quad \eta \doteq \phi - \psi, \quad \bar{\phi} \doteq \Lambda_a(\phi), \quad \bar{\eta} \doteq \bar{\phi} - \psi. \tag{3.1}$$

For simplicity, we denote $R_t(\phi)(s)$ of (1.12) simply as $R_t(s)$.

Lemma 3.1. *For $\psi \in \mathcal{D}[0,\infty)$,*

$$0 \leq (\Lambda_a \circ \Gamma_0)(\psi) \leq \Gamma_0(\psi) \wedge a. \tag{3.2}$$

In particular, for $t \in [0,\infty)$,

$$\Gamma_0(\psi)(t) = 0 \Rightarrow (\Lambda_a \circ \Gamma_0)(\psi)(t) = 0. \tag{3.3}$$

Proof. We use the notation (3.1). Inequalities (2.2) imply $\bar{\phi}(t) = \phi(t) - \sup_{s\in[0,t]} R_t(s) \geq 0$ and

$$\bar{\phi}(t) = \phi(t) - \sup_{s\in[0,t]} R_t(s) \leq \phi(t) - \left[(\phi(t)-a)^+ \wedge \phi(t)\right] = \phi(t) \wedge a.$$

□

We now derive some relations that will be used in the proofs of Lemmas 3.4 and 3.5. Indeed, from (1.11), (1.12) and the definitions of η and $\bar{\eta}$ we obtain the equalities

$$\bar{\eta}(t) - \eta(t) = \bar{\phi}(t) - \phi(t) = -\sup_{s\in[0,t]} R_t(s) \tag{3.4}$$

and

$$\bar{\eta}(t-) - \eta(t-) = -\lim_{r\uparrow t} \sup_{s\in[0,r]} R_r(s) = -\sup_{s\in[0,t)} \left[(\phi(s)-a)^+ \wedge \inf_{u\in[s,t)} \phi(u)\right]. \tag{3.5}$$

Moreover, for $\varepsilon > 0$

$$\eta(t+\varepsilon) - \bar{\eta}(t+\varepsilon) = \sup_{s\in[0,t+\varepsilon]} R_{t+\varepsilon}(s). \tag{3.6}$$

Since for $t \in [0,\infty)$, $s \in [0,t]$ and $\varepsilon > 0$ we have

$$R_{t+\varepsilon}(s) = (\phi(s)-a)^+ \wedge \inf_{u\in[s,t+\varepsilon]} \phi(u) = R_t(s) \wedge \inf_{u\in(t,t+\varepsilon]} \phi(u),$$

the right-hand side of (3.6) can be rewritten as

$$\sup_{s\in[0,t+\varepsilon]} R_{t+\varepsilon}(s) = \left[\left(\sup_{s\in[0,t]} R_t(s)\right) \wedge \inf_{u\in(t,t+\varepsilon]} \phi(u)\right] \vee \left[\sup_{s\in(t,t+\varepsilon]} R_{t+\varepsilon}(s)\right]. \tag{3.7}$$

Definition 3.2. *For a function $f \in \mathcal{D}[0,\infty)$, we say that t* is a point of increase of f *if there exists a sequence $s_n \downarrow 0$ such that $f(t+s_n) - f(t) > 0$ for every n or if $f(t) - f(t-) > 0$, and we say that t* is a point of decrease of f *if t is a point of increase of $-f$. We say that f is flat on a set $A \subset \mathbb{R}$ if for every $t \in A$, $f(t) = f(t-)$ and there exists $\varepsilon > 0$ such that $f(s) = f(t)$ for every $s \in [t, t+\varepsilon)$.*

Remark 3.3. Let $f \in \mathcal{D}[0,\infty)$ be given. It is straightforward to verify that the left-continuous function $t \mapsto f(t-)$ is nonincreasing on an interval $[\theta_1,\theta_2)$ if and only if the right-continuous function $t \mapsto f(t)$ has no point of increase in the interval, and similarly, $t \mapsto f(t-)$ is nondecreasing on $[\theta_1,\theta_2)$ if and only if $t \mapsto f(t)$ has no point of decrease in the interval. If $t \mapsto f(t)$ is flat on an interval $[\theta_1,\theta_2)$, then $t \mapsto f(t-)$ is constant there.

Lemma 3.4. *Given $t \in [0,\infty)$, suppose*

$$R_t(t) < \sup_{s\in[0,t]} R_t(s). \tag{3.8}$$

Then t is not a point of decrease of $\bar{\eta}$. Moreover, if t is a point of increase of $\bar{\eta}$, then $\bar{\phi}(t) = 0$.

Proof. Fix $t \in [0,\infty)$. Since $\phi(t) = \Gamma_0(\psi)(t) \geq 0$ by the definition of Γ_0, it follows that $R_t(t) = (\phi(t)-a)^+ \wedge \phi(t) = (\phi(t)-a)^+$. Thus condition (3.8) along with the definition (1.12) of R_t imply that

$$0 \leq R_t(t) = (\phi(t)-a)^+ < \sup_{s\in[0,t]} R_t(s) = \sup_{s\in[0,t)} \left[(\phi(s)-a)^+ \wedge \inf_{u\in[s,t)} \phi(u)\right] \wedge \phi(t), \tag{3.9}$$

which in particular implies $\phi(t) > 0$. The right continuity of ϕ then guarantees the existence of $\delta > 0$ such that

$$0 < \inf_{s\in[t,t+\delta]} \phi(s) \quad \text{and} \quad \sup_{s\in[t,t+\delta]} (\phi(s)-a)^+ < \sup_{s\in[0,t]} R_t(s).$$

The first inequality in the last display, when combined with the complementarity condition (1.2), ensures that

$$\eta(t+\varepsilon) = \eta(t) \qquad \text{for every } \varepsilon \in [0,\delta], \tag{3.10}$$

while the second inequality shows that for every $\varepsilon \in [0,\delta]$

$$\begin{array}{rcl} \sup_{s\in[t,t+\varepsilon]} R_{t+\varepsilon}(s) & = & \sup_{s\in[t,t+\varepsilon]} \left[(\phi(s)-a)^+ \wedge \inf_{u\in[s,t+\varepsilon]} \phi(u)\right] \\ & \leq & \sup_{s\in[t,t+\varepsilon]} (\phi(s)-a)^+ \\ & < & \sup_{s\in[0,t]} R_t(s). \end{array} \tag{3.11}$$

Combining the last two displays with (3.4), (3.6), (3.10) and (3.7), for $\varepsilon \in [0,\delta]$ we obtain

$$\begin{array}{l} \bar\eta(t+\varepsilon) - \bar\eta(t) \\ \quad = \eta(t+\varepsilon) - \eta(t) - \sup_{s\in[0,t+\varepsilon]} R_{t+\varepsilon}(s) + \sup_{s\in[0,t]} R_t(s) \\ \quad = \sup_{s\in[0,t]} R_t(s) - \sup_{s\in[0,t+\varepsilon]} R_{t+\varepsilon}(s) \\ \quad = \sup_{s\in[0,t]} R_t(s) \\ \qquad\quad - \left[\left(\sup_{s\in[0,t]} R_t(s)\right) \wedge \inf_{u\in[t,t+\varepsilon]} \phi(u)\right] \vee \left[\sup_{s\in[t,t+\varepsilon]} R_{t+\varepsilon}(s)\right]. \end{array} \tag{3.12}$$

When combined with (3.11), this yields the inequality

$$\bar\eta(t+\varepsilon) - \bar\eta(t) \geq \sup_{s\in[0,t]} R_t(s) - \left[\sup_{s\in[0,t]} R_t(s)\right] \vee \left[\sup_{s\in[t,t+\varepsilon]} R_{t+\varepsilon}(s)\right] = 0. \tag{3.13}$$

We now consider two cases, based on the two values that $\sup_{s\in[0,t]} R_t(s)$ can attain, as dictated by (3.9).

Case 1.

$$\sup_{s\in[0,t]} R_t(s) = \sup_{s\in[0,t)} \left[(\phi(s)-a)^+ \wedge \inf_{u\in[s,t)} \phi(u)\right] < \phi(t). \tag{3.14}$$

In this case, due to the right continuity of ϕ, by choosing $\delta > 0$ smaller if necessary we can ensure that for every $\varepsilon \in [0,\delta]$,

$$\sup_{s\in[0,t]} R_t(s) < \inf_{u\in[t,t+\varepsilon]} \phi(u).$$

Along with (3.11) and (3.12), this implies that $\bar\eta(t+\varepsilon) - \bar\eta(t) = 0$ for every $\varepsilon \in [0,\delta]$. Furthermore, (3.4) and (3.5), together with (3.14) and the fact that η is nondecreasing, dictate that

$$\bar\eta(t) - \bar\eta(t-) = \eta(t) - \eta(t-) \geq 0. \tag{3.15}$$

Thus t is not a point of decrease of $\bar{\eta}$, and the only way t can be a point of increase of $\bar{\eta}$ is for $\bar{\eta}(t) - \bar{\eta}(t-)$ to be strictly positive. But

$$\bar{\eta}(t) - \bar{\eta}(t-) > 0 \Rightarrow \eta(t) - \eta(t-) > 0 \Rightarrow \phi(t) = 0 \Rightarrow \bar{\phi}(t) = 0, \tag{3.16}$$

where the second implication uses the complementarity condition (1.2) and the third implication follows from property (3.3) of Lemma 3.1.
Case 2.

$$\sup_{s\in[0,t]} R_t(s) = \phi(t) \leq \sup_{s\in[0,t)} \left[(\phi(s) - a)^+ \wedge \inf_{u\in[s,t)} \phi(u) \right].$$

In this case, $\bar{\phi}(t) = \phi(t) - \sup_{s\in[0,t]} R_t(s) = 0$. Furthermore, (3.4), (3.5) and the fact that $\eta \in \mathcal{I}\,[0,\infty)$ show that

$$\bar{\eta}(t) - \bar{\eta}(t-) \geq \eta(t) - \eta(t-) \geq 0.$$

Thus, recalling (3.13), we have established the inequalities

$$\bar{\eta}(t) - \bar{\eta}(t-) \geq 0 \quad \text{ and } \quad \bar{\eta}(t+\varepsilon) - \bar{\eta}(t) \geq 0 \text{ for every } \varepsilon \in [0,\delta].$$

Since the equality $\bar{\phi}(t) = 0$ always holds in the second case, it also holds when t is a point of increase of $\bar{\eta}$. This concludes the proof of the lemma. □

Lemma 3.5. *Given $t \in [0,\infty)$, suppose that*

$$R_t(t) = \sup_{s\in[0,t]} R_t(s). \tag{3.17}$$

Then the following two relations are satisfied:

1. *If $\phi(t) \in [0,a)$, then t is not a point of decrease of $\bar{\eta}$. Moreover, if t is a point of increase of $\bar{\eta}$, then $\bar{\phi}(t) = 0$.*
2. *If $\phi(t) \in [a,\infty)$, then $\bar{\phi}(t) = a$ and t is not a point of increase of $\bar{\eta}$.*

Proof. Fix $t \in [0,\infty)$. Since $\phi(t) \geq 0$ we have $R_t(t) = (\phi(t) - a)^+ \wedge \phi(t) = (\phi(t) - a)^+$, and therefore condition (3.17) is equivalent to the relation $\sup_{s\in[0,t]} R_t(s) = (\phi(t) - a)^+$. In the proof of the lemma we consider the two cases separately. In both cases we will make use of the fact that

$$\begin{aligned}
&\sup\nolimits_{s\in[0,t]} R_t(s) \\
&\qquad = \sup\nolimits_{s\in[0,t)} \left[(\phi(s) - a)^+ \wedge \inf\nolimits_{u\in[s,t)} \phi(u) \wedge \phi(t) \right] \vee \left[(\phi(t) - a)^+ \wedge \phi(t) \right] \\
&\qquad = \left[\sup\nolimits_{s\in[0,t)} \left[(\phi(s) - a)^+ \wedge \inf\nolimits_{u\in[s,t)} \phi(u) \right] \vee (\phi(t) - a)^+ \right] \wedge \phi(t).
\end{aligned} \tag{3.18}$$

Case 1: $\phi(t) \in [0,a)$. In this case $\sup_{s\in[0,t]} R_t(s) = 0$ and so (3.4) and (3.5) together imply that

$$\bar{\eta}(t) - \bar{\eta}(t-) = \eta(t) - \eta(t-) + \sup_{s\in[0,t)} \left[(\phi(s)-a)^+ \wedge \inf_{u\in[s,t)} \phi(u) \right] \geq 0, \quad (3.19)$$

where the inequality follows because $\eta \in \mathcal{I}[0,\infty)$ and ϕ takes values in $[0,\infty)$. Substitute the equality $(\phi(t)-a)^+ = 0$ into (3.18) to obtain

$$\sup_{s\in[0,t]} R_t(s) = \sup_{s\in[0,t)} \left[(\phi(s)-a)^+ \wedge \inf_{u\in[s,t)} \phi(u) \right] \wedge \phi(t). \quad (3.20)$$

If $\phi(t) > 0$ then the complementarity condition (1.2) implies $\eta(t) - \eta(t-) = 0$, and furthermore the last display shows that

$$\sup_{s\in[0,t]} R_t(s) = 0 \Rightarrow \sup_{s\in[0,t)} \left[(\phi(s)-a)^+ \wedge \inf_{u\in[s,t)} \phi(u) \right] = 0.$$

When combined with (3.19), this shows that if $\phi(t) \in (0,a)$, then $\bar{\eta}(t) - \bar{\eta}(t-) = 0$. On the other hand, if $\phi(t) = 0$ then $\bar{\phi}(t) = 0$ by (3.3) of Lemma 3.1. Therefore, $\bar{\eta}(t) - \bar{\eta}(t-) > 0$ implies $\bar{\phi}(t) = 0$.

Now, due to (3.4), $\sup_{s\in[0,t]} R_t(s) = 0$ also implies that

$$\bar{\phi}(t) = \phi(t) \qquad \text{and} \qquad \bar{\eta}(t) = \eta(t). \quad (3.21)$$

Moreover since $\phi(t) \in [0,a)$, the right-continuity of ϕ guarantees the existence of $\delta > 0$ such that $\phi(s) \in [0,a)$ for all $s \in [t, t+\delta]$. This implies that for every $\varepsilon \in [0,\delta]$,

$$\sup_{s\in[t,t+\varepsilon]} R_{t+\varepsilon}(s) = \sup_{s\in[t,t+\varepsilon]} \left[(\phi(s)-a)^+ \wedge \inf_{u\in[s,t+\varepsilon]} \phi(s) \right] = 0,$$

which in turn implies that

$$0 \leq \sup_{s\in[0,t+\varepsilon]} R_{t+\varepsilon}(s) = \sup_{s\in[0,t]} R_{t+\varepsilon}(s) \leq \sup_{s\in[0,t]} R_t(s) = 0.$$

When substituted into (3.4) and (3.6), the last display along with the fact that η is non-decreasing show that for $\varepsilon \in [0,\delta]$,

$$\bar{\eta}(t+\varepsilon) - \bar{\eta}(t) = \eta(t+\varepsilon) - \eta(t) \geq 0. \quad (3.22)$$

When combined with (3.19), this shows that t is not a point of decrease of $\bar{\eta}$. Furthermore, suppose there exists a sequence $s_n \downarrow 0$ such that $\bar{\eta}(t+s_n) -$

$\bar{\eta}(t) > 0$ for every $n \in I\!N$. Then (3.22) implies that $\eta(t+s_n) - \eta(t) > 0$ (for all sufficiently large n such that $s_n < \delta$) and so the complementarity condition (1.2) implies $\phi(t) = 0$ which, due to (3.21), in turn implies that $\bar{\phi}(t) = 0$. This completes the proof of the first property of the lemma.
Case 2: $\phi(t) \in [a, \infty)$. In this case $(\phi(t) - a)^+ = \phi(t) - a < \phi(t)$ and so

$$0 \leq \sup_{s\in[0,t]} R_t(s) = R_t(t) = (\phi(t)-a)^+ \wedge \phi(t) = \phi(t) - a < \phi(t). \tag{3.23}$$

This shows that $\bar{\phi}(t) = \phi(t) - \sup_{s\in[0,t]} R_t(s) = a$. Since ϕ is right-continuous, (3.23) ensures the existence of $\delta > 0$ such that

$$0 \leq \sup_{s\in[0,t]} R_t(s) \leq \sup_{s\in[t,t+\delta]} (\phi(s)-a)^+ < \inf_{u\in[t,t+\delta]} \phi(u). \tag{3.24}$$

When combined with (3.7), the inequalities in the last display show that for every $\varepsilon \in [0, \delta]$,

$$\begin{aligned}
&\sup\nolimits_{s\in[0,t+\varepsilon]} R_{t+\varepsilon}(s) \\
&\quad = \left[\sup\nolimits_{s\in[0,t]} R_t(s)\right] \vee \left[\sup\nolimits_{s\in(t,t+\varepsilon]} R_{t+\varepsilon}(s)\right] \\
&\quad = \left[\sup\nolimits_{s\in[0,t]} R_t(s)\right] \vee \left[\sup\nolimits_{s\in(t,t+\varepsilon]} \left((\phi(s)-a)^+ \wedge \inf\nolimits_{u\in[s,t+\varepsilon]} \phi(u)\right)\right] \\
&\quad = \left[\sup\nolimits_{s\in[0,t]} R_t(s)\right] \vee \left[\sup\nolimits_{s\in(t,t+\varepsilon]} (\phi(s)-a)^+\right].
\end{aligned}$$

From the inequalities in (3.24), it also follows that $\phi(u) > 0$ for every $u \in [t, t+\delta]$. Thus by the complementarity condition (1.2) we must have

$$\eta(t+\varepsilon) = \eta(t) \qquad \text{for every } \varepsilon \in [0,\delta].$$

The last two displays along with (3.4) and (3.6) show that for every $\varepsilon \in [0, \delta]$,

$$\begin{aligned}
&\bar{\eta}(t+\varepsilon) - \bar{\eta}(t) \\
&\qquad = \sup\nolimits_{s\in[0,t]} R_t(s) - \sup\nolimits_{s\in[0,t+\varepsilon]} R_{t+\varepsilon}(s) \\
&\qquad = \sup\nolimits_{s\in[0,t]} R_t(s) - \left[\sup\nolimits_{s\in[0,t]} R_t(s)\right] \vee \left[\sup\nolimits_{s\in[t,t+\varepsilon]} (\phi(s)-a)^+\right] \\
&\qquad \leq 0.
\end{aligned} \tag{3.25}$$

Now observe that due to relation (3.2) of Lemma 3.1, $\bar{\phi}(t) = a$ implies that $\phi(t) \geq a > 0$ and hence the complementarity condition (1.2) dictates that $\eta(t) - \eta(t-) = 0$. Together with (3.4) and (3.5) this yields the equation

$$\bar{\eta}(t) - \bar{\eta}(t-) = \sup_{s\in[0,t)} \left[(\phi(s) - a)^+ \wedge \inf_{u\in[s,t)} \phi(u) \right] - \sup_{s\in[0,t]} R_t(s).$$

From the second line of (3.18), we have

$$\sup_{s\in[0,t)} \left[(\phi(s) - a)^+ \wedge \inf_{u\in[s,t)} \phi(u) \right] \wedge \phi(t) \leq \sup_{s\in[0,t]} R_t(s),$$

but since $\sup_{s\in[0,t]} R_t(s) = \phi(t) - a < \phi(t)$, we must in fact have

$$\sup_{s\in[0,t)} \left[(\phi(s) - a)^+ \wedge \inf_{u\in[s,t)} \phi(u) \right] \leq \sup_{s\in[0,t]} R_t(s).$$

This shows that

$$\bar{\eta}(t) - \bar{\eta}(t-) \leq 0.$$

Along with (3.25), this establishes the second property and the proof of the lemma is complete. □

Proof of Theorem 1.4. Using the notation (3.1), define $\overline{\tau}_0 \doteq 0$,

$$\overline{\sigma}_0 \doteq \min\{t \geq \overline{\tau}_0 | \bar{\phi}(t) = a\}, \tag{3.26}$$

and for $k = 1, 2, \ldots,$ let

$$\overline{\tau}_k \doteq \min\{t \geq \overline{\sigma}_{k-1} | \bar{\phi}(t) = 0\} \quad \text{and} \quad \overline{\sigma}_k \doteq \min\{t \geq \overline{\tau}_k | \bar{\phi} = a\}. \tag{3.27}$$

The minima in (3.26) and (3.27) are obtained (or are $+\infty$) because of the right-continuity of $\bar{\phi}$. Let $\mathcal{P} \doteq \cup_{k=0}^{\infty}[\overline{\tau}_k, \overline{\sigma}_k)$ and $\mathcal{N} \doteq \cup_{k=0}^{\infty}[\overline{\sigma}_k, \overline{\tau}_{k+1})$. Because $\bar{\phi} \in \mathcal{D}[0,\infty)$, we have $\overline{\tau}_k \uparrow \infty$ and $\overline{\sigma}_k \uparrow \infty$ as $k \to \infty$. Indeed, if this were not the case, then there would exist a number $\theta < \infty$ such that $\overline{\tau}_k \uparrow \theta$ and $\overline{\sigma}_k \uparrow \theta$, in which case $\lim_{s\uparrow\theta} \bar{\phi}(s)$ would not exist. Therefore, $\mathcal{P} \cup \mathcal{N} = [0, \infty)$.

We define two functions $\bar{\eta}_\ell$ and $\bar{\eta}_u$ in $\mathcal{D}[0,\infty)$, specifying their left-continuous versions by the formulas

$$\begin{aligned}
\bar{\eta}_\ell(t-) &\doteq \sum_{k=0}^{\infty} \left[\bar{\eta}(t \wedge \overline{\sigma}_k-) - \bar{\eta}(t \wedge \overline{\tau}_k-) \right], \\
\bar{\eta}_u(t-) &\doteq -\sum_{k=1}^{\infty} \left[\bar{\eta}(t \wedge \overline{\tau}_k-) - \bar{\eta}(t \wedge \overline{\sigma}_{k-1}-) \right].
\end{aligned}$$

Then the function $t \mapsto \bar{\eta}_\ell(t-)$, and hence also the function $t \mapsto \bar{\eta}_\ell(t)$, captures the jumps of $\bar{\eta}$ at the times $\tau_1, \tau_2, \ldots$ but not the jumps of $\bar{\eta}$ at $\sigma_0, \sigma_1, \ldots$. Furthermore, $\bar{\eta}_\ell(0) = 0$ if $\sigma_0 = \tau_0$, whereas $\bar{\eta}_\ell(0) = \bar{\eta}(0)$ if $\sigma_0 > \tau_0$. In fact, $\bar{\eta}_\ell$ is flat on $\mathcal{N}$. Similarly, $\bar{\eta}_u$ is flat on $\mathcal{P}$. Finally, $\bar{\eta}(t-) = \bar{\eta}_\ell(t-) - \bar{\eta}_u(t-)$, and hence $\bar{\eta} = \bar{\eta}_\ell - \bar{\eta}_u$.

For $t \in [0,\infty)$, Lemmas 3.4 and 3.5 together imply that t is a point of increase of $\bar{\eta}$ only if $\bar{\phi}(t) = 0$ and t is a point of decrease of $\bar{\eta}$ only if $\bar{\phi}(t) = a$. The fact that $\bar{\phi}(t) < a$ for $t \in \mathcal{P}$ implies that $\bar{\eta}$ has no point of decrease in each of the intervals $[\overline{\tau}_k, \overline{\sigma}_k)$. According to Remark 3.3, $\bar{\eta}(\cdot-)$ and hence also $\bar{\eta}_\ell$, is nondecreasing on each of the intervals $[\overline{\tau}_k, \overline{\sigma}_k)$. Being flat on $\mathcal{N}$, $\bar{\eta}_\ell$ must in fact be nondecreasing on all of $[0,\infty)$. Similarly, since $\bar{\phi}(t) > 0$ for $t \in \mathcal{N}$, it follows that $\bar{\eta}(\cdot-)$, and therefore $-\bar{\eta}_u$, is nonincreasing on the components of $\mathcal{N}$. Being flat on $\mathcal{P}$, $\bar{\eta}_u$ must be nondecreasing on all of $[0,\infty)$.

The first equation in (1.6) follows from the fact that the points of increase t of $\bar{\eta}_\ell$ are the points of increase of $\bar{\eta}$, and $\bar{\phi}(t) = 0$ at all of these. Similarly, the points of increase t of $\bar{\eta}_u$ are the points of decrease of $\bar{\eta}$, and $\bar{\phi}(t) = a$ at all of these. Therefore, the second equation in (1.6) also holds. ■

We now present the proof of Corollary 1.5. Recall the definition of $R_t(\phi)$ given in (1.12). For $\phi \in \mathcal{D}[0,\infty)$, it will be convenient to introduce the function $C^\phi \in \mathcal{D}[0,\infty)$ defined by

$$C^\phi(t) \doteq \sup_{s\in[0,t]} [R_t(\phi)(s)] = \sup_{s\in[0,t]} \left[(\phi(s)-a)^+ \wedge \inf_{u\in[s,t]} \phi(u) \right] \qquad \text{for } t \in [0,\infty). \tag{3.28}$$

Note that then $\Lambda(\phi) = \phi - C^\phi$ for every $\phi \in \mathcal{D}[0,\infty)$.

Proof of Corollary 1.5. We first prove (1.14) for $i = \infty$. For $\phi_1, \phi_2 \in \mathcal{D}[0,T]$, we have

$$||\Lambda_a(\phi_1) - \Lambda_a(\phi_2)||_T \le ||\phi_1 - \phi_2||_T + ||C^{\phi_1} - C^{\phi_2}||_T. \tag{3.29}$$

For $t \in [0,T]$, because $(a_1 \wedge b_1) - (a_2 \wedge b_2) \le (a_1 - a_2) \vee (b_1 - b_2)$, we have

$$\begin{aligned}
&C^{\phi_1}(t) - C^{\phi_2}(t) \\
&\le \sup_{s\in[0,t]} [R_t(\phi_1)(s) - R_t(\phi_2)(s)] \\
&\le \sup_{s\in[0,t]} \left[|(\phi_1(s)-a)^+ - (\phi_2(s)-a)^+| \vee \left| \inf_{u\in[s,t]} \phi_1(u) - \inf_{u\in[s,t]} \phi_2(u) \right| \right] \\
&\le \sup_{s\in[0,t]} \left[|\phi_1(s) - \phi_2(s)| \vee \sup_{u\in[s,t]} |\phi_1(u) - \phi_2(u)| \right] \\
&\le ||\phi_1 - \phi_2||_T.
\end{aligned}$$

Taking the supremum over $t \in [0,T]$ and interchanging ϕ_1 and ϕ_2, we get

$$\left|\left|C^{\phi_1} - C^{\phi_2}\right|\right|_T \leq ||\phi_1 - \phi_2||_T. \tag{3.30}$$

From (3.29) and (3.30), we obtain (1.14) for $i = \infty$.

Now let $\mathcal{M}$ be the class of strictly increasing continuous functions λ of $[0,T]$ onto itself. Then for any $\lambda \in \mathcal{M}$, the scaling property

$$\Lambda_a(\phi \circ \lambda) = \Lambda_a(\phi) \circ \lambda \tag{3.31}$$

is easily deduced directly from the definition of Λ_a. Moreover, by the definition of d_0, given any $\phi_1, \phi_2 \in \mathcal{D}[0,T]$, $\phi_1 \neq \phi_2$, for every $\delta > 0$ there exists $\lambda \in \mathcal{M}$ (possibly depending on δ) such that

$$\sup_{t\in[0,T]} |\lambda(t) - t| \leq d_0(\phi_1,\phi_2) + \delta[1 \wedge d_0(\phi_1,\phi_2)]$$

and

$$\sup_{t\in[0,T]} |\phi_1(t) - \phi_2(\lambda(t))| \leq d_0(\phi_1,\phi_2) + \delta[1 \wedge d_0(\phi_1,\phi_2)].$$

The scaling property (3.31) along with (1.14) for $i = \infty$ implies that

$$\sup_{t\in[0,T]} |\Lambda_a(\phi_1)(t) - \Lambda_a(\phi_2)(\lambda t)| \leq 2(d_0(\phi_1,\phi_2) + \delta[1 \wedge d_0(\phi_1,\phi_2)]).$$

Since this is true for all $\delta > 0$, by the definition of d_0 this implies that

$$d_0(\Lambda_a(\phi_1), \Lambda_a(\phi_2)) \leq 2d_0(\phi_1,\phi_2),$$

which is the inequality (1.14) for $i = 0$. Clearly, (1.14) holds also in the case $i = 0$, $\phi_1 = \phi_2 \in \mathcal{D}[0,T]$.

We now prove (1.14) for $i = 1$. For a given $\phi \in \mathcal{D}[0,T]$, let $\phi(0-) \doteq \phi(0)$ and let

$$G_\phi = \{(t,z) \in [0,T] \times I\!\!R : z \in [\phi(t-) \wedge \phi(t), \phi(t-) \vee \phi(t)]\}$$

be the graph of ϕ ordered by the following relation: $(t_1, z_1) \leq (t_2, z_2)$ if either $t_1 < t_2$ or $t_1 = t_2$ and $|\phi(t_1-) - z_1| \leq |\phi(t_1-) - z_2|$. Let $\Pi(\phi)$ be the set of all parametric representations of G_ϕ, i.e., continuous nondecreasing (in the order relation just defined) functions (r,g) mapping $[0,1]$ onto G_ϕ. For $\phi_1, \phi_2 \in \mathcal{D}[0,T]$,

$$d_1(\phi_1,\phi_2) \doteq \inf\{||r_1 - r_2||_T \vee ||g_1 - g_2||_T : (r_i, g_i) \in \Pi(\phi_i), i = 1,2\}.$$

We show in Lemma A.1 in Appendix A that if $(r, g) \in \Pi(\phi)$, then $(r, \Lambda_a(g)) \in \Pi(\Lambda_a(\phi))$. Therefore,

$$\begin{aligned} &d_1(\Lambda_a(\phi_1), \Lambda_a(\phi_2)) \\ &\quad \le \inf\{||r_1 - r_2||_T \vee ||\Lambda_a(g_1) - \Lambda_a(g_2)||_T \colon (r_i, g_i) \in \Pi(\phi_i), i = 1, 2\} \\ &\quad \le 2d_1(\phi_1, \phi_2), \end{aligned}$$

where the last inequality follows from (1.14) for $i = \infty$. We have proved (1.14).

It is well-known (see, for example, Lemma 13.5.1 and Theorem 13.5.1 of [14]) that for any $T < \infty$ and $\psi_1, \psi_2 \in \mathcal{D}[0, T]$

$$d_i(\Gamma_0(\psi_1), \Gamma_0(\psi_2)) \le 2d_i(\psi_1, \psi_2) \tag{3.32}$$

for $i = 0, 1, \infty$. The representation $\Gamma_{0,a} = \Lambda_a \circ \Gamma_0$ stated in (1.13), along with (1.14) and (3.32) then implies that (1.15) holds with $L = 4$.

By the argument in Theorem 12.9.4 in [14], the validity of (1.14) and (1.15) on $\mathcal{D}[0, T]$ for every $T > 0$ implies the same bound on $\mathcal{D}[0, \infty)$. ■

Remarks. Example 13.5.1 in [14] shows that the bound in (3.32) with $i = \infty$ is tight. Similarly, the bound (1.14) for $i = \infty$ is tight. To see this, let us consider $\phi_1, \phi_2 \in \mathcal{D}[0, 1]$ defined by $\phi_1 = 2I_{[0,1]}$, $\phi_2 = 3I_{[0,1/2)} + I_{[1/2,1]}$. With $a = 2$ we have $\Lambda_a(\phi_1) = \phi_1$, $\Lambda_a(\phi_2) = 2I_{[0,1/2)}$, $||\phi_1 - \phi_2||_1 = 1$ and $||\Lambda_a(\phi_1) - \Lambda_a(\phi_2)||_1 = 2$. However, Theorem 14.8.1 in [14] shows that (1.15) for $i = \infty$ (and thus also for $i = 0, 1$) actually holds with $L = 2$. Clearly, the bound (1.15) with $L = 2$ is tight, because the bound (3.32) is tight.

4. Second Proof of Theorem 1.4

A more complex, but perhaps more intuitive way of constructing $\bar{\phi} \doteq \Lambda_a(\phi)$ from $\phi \doteq \Gamma_0(\psi)$ is to first create two increasing sequences of times $\{\sigma_k\}_{k=0}^{\infty}$ and $\{\tau_k\}_{k=1}^{\infty}$ so that on each interval of the form $[\sigma_{k-1}, \tau_k)$, there is only pushing of ϕ from above and on each interval of the form $[\tau_k, \sigma_k)$, there is only pushing of ϕ from below. In this section we execute that construction and thereby obtain a decomposition (4.24) below of the bounded variation process C^{ϕ} defined by (3.28) into the difference of two nondecreasing processes. We provide this alternate proof of Theorem 1.4 in order to illuminate the nature of the process C^{ϕ}. For this construction, we assume that ϕ is in $\mathcal{D}_+[0, \infty)$. We have in mind that $\phi = \Gamma_0(\psi)$ for some $\psi \in \mathcal{D}[0, \infty)$.

For $\phi \in \mathcal{D}_+[0, \infty)$ and $a > 0$, we set $\tau_0 \doteq 0$,

$$\sigma_0 \doteq \min\{t \ge 0 | \phi(t) - a \ge 0\}, \tag{4.1}$$

and for $k \geq 1$, we set

$$\tau_k \doteq \min\left\{ t \geq \sigma_{k-1} \,\middle|\, \sup_{s\in[\sigma_{k-1},t]} \phi(s) - a \geq \phi(t) \right\}, \tag{4.2}$$

$$\sigma_k \doteq \min\left\{ t \geq \tau_k \,\middle|\, \phi(t) - a \geq \inf_{u\in[\tau_k,t]} \phi(u) \right\}. \tag{4.3}$$

When ϕ and $\bar{\phi}$ are related by (3.1), the stopping times σ_k and τ_k, $k = 0, 1, \ldots$ coincide with the stopping times $\overline{\sigma}_k$ and $\overline{\tau}_k$, $k = 0, 1, \ldots$ of (3.26) and (3.27); see Remark 4.5 below. The minima in (4.1)–(4.3) over t are obtained (or are $+\infty$) because of the right-continuity of ϕ. In particular, for $k \geq 1$,

$$\sup_{s\in[\sigma_{k-1},u]} \phi(s) - a < \phi(u) \ \forall u \in [\sigma_{k-1}, \tau_k), \tag{4.4}$$

$$\sup_{s\in[\sigma_{k-1},\tau_k]} \phi(s) - a \geq \phi(\tau_k), \tag{4.5}$$

$$\phi(s) - a < \inf_{u\in[\tau_k,s]} \phi(u) \ \forall s \in [\tau_k, \sigma_k), \tag{4.6}$$

$$\phi(\sigma_k) - a \geq \inf_{u\in[\tau_k,\sigma_k]} \phi(u). \tag{4.7}$$

Furthermore,

$$\phi(\sigma_0) - a \geq 0. \tag{4.8}$$

We have $0 = \tau_0 \leq \sigma_0 < \tau_1 < \sigma_1 < \tau_2 < \sigma_2 < \cdots$.

Proposition 4.1. *As $k \to \infty$, we have $\tau_k \uparrow \infty$ and $\sigma_k \uparrow \infty$.*

Proof. Assume the proposition is false. Then there is a number $\theta < \infty$ such that $\tau_k \uparrow \theta$ and $\sigma_k \uparrow \theta$. Relation (4.5) implies the existence of $\rho_k \in [\sigma_{k-1}, \tau_k]$ such that $\phi(\rho_k) \geq \phi(\tau_k) + \frac{a}{2}$. Since $\rho_k \uparrow \theta$, ϕ does not have a left-hand limit at θ. This contradicts the membership of ϕ in $\mathcal{D}_+[0,\infty)$. □

Proposition 4.2. *For $k \geq 1$, $C^\phi(t) = \sup_{s\in[\sigma_{k-1},t]} (\phi(s) - a)^+$ for all $t \in [\sigma_{k-1}, \tau_k)$.*

Proof. Let $t \in (\sigma_{k-1}, \tau_k)$ and $\rho \in (\sigma_{k-1}, t]$ be given. Let $\{\rho_n\}_{n=1}^\infty$ be a sequence in (σ_{k-1}, ρ) satisfying $\rho_n \uparrow \rho$. By definition, $C^\phi(t) \geq (\phi(\rho_n) - a)^+ \wedge \inf_{u\in[\rho_n,t]} \phi(u)$, and letting $n \to \infty$, we obtain

$$C^\phi(t) \geq (\phi(\rho-) - a)^+ \wedge \phi(\rho-) \wedge \inf_{u\in[\rho,t]} \phi(u), \quad \sigma_{k-1} < \rho \leq t < \tau_k. \tag{4.9}$$

Now let $t \in [\sigma_{k-1}, \tau_k)$ be given. Then there exists ρ_t such that either

$$\rho_t \in [\sigma_{k-1}, t] \ \text{ and } \sup_{s\in[\sigma_{k-1},t]} \phi(s) = \phi(\rho_t), \tag{4.10}$$

or else

$$\rho_t \in (\sigma_{k-1}, t] \quad \text{and} \quad \sup_{s\in[\sigma_{k-1},t]} \phi(s) = \phi(\rho_t-). \tag{4.11}$$

If (4.11) is the case, which can happen only if $t > \sigma_{k-1}$, then for $u \in [\rho_t, t]$, (4.4) implies

$$\phi(\rho_t-) - a = (\phi(\rho_t-) - a) \wedge \sup_{s\in[\sigma_{k-1},u]} (\phi(s) - a) \leq \phi(\rho_t-) \wedge \phi(u),$$

which yields $\phi(\rho_t-) - a \leq \phi(\rho_t-) \wedge \inf_{u\in[\rho_t,t]} \phi(u)$. This inequality together with (4.9) and (4.11) shows that

$$C^\phi(t) \geq (\phi(\rho_t-) - a)^+ = \sup_{s\in[\sigma_{k-1},t]} (\phi(s) - a)^+. \tag{4.12}$$

If, on the other hand, (4.10) is the case, then (4.4) implies

$$\phi(\rho_t) - a = \sup_{s\in[\sigma_{k-1},u]} \phi(s) - a < \phi(u) \ \forall u \in [\rho_t, t],$$

and hence $\phi(\rho_t) - a \leq \inf_{u\in[\rho_t,t]} \phi(u)$. This shows that

$$C^\phi(t) \geq (\phi(\rho_t) - a)^+ \wedge \inf_{u\in[\rho_t,t]} \phi(u) = (\phi(\rho_t) - a)^+ = \sup_{s\in[\sigma_{k-1},t]} (\phi(s) - a)^+.$$

We again have the lower bound (4.12).

To obtain the reverse of inequality (4.12), we consider separately the cases $k = 1$ and $k \geq 2$. If $k = 1$, then $(\phi(s) - a)^+ = 0$ for $s \in [0, \sigma_0)$ and

$$C^\phi(t) \doteq \sup_{s\in[0,t]} \left[(\phi(s) - a)^+ \wedge \inf_{u\in[s,t]} \phi(u) \right] \leq \sup_{s\in[\sigma_0,t]} (\phi(s) - a)^+,$$

as desired. If $k \geq 2$, we may write $C^\phi(t) = S_1 \vee S_2 \vee S_3$, where

$$S_1 = \sup_{s\in[0,\tau_{k-1}]} \left[(\phi(s) - a)^+ \wedge \inf_{u\in[s,t]} \phi(u) \right], \tag{4.13}$$

$$S_2 = \sup_{s\in(\tau_{k-1},\sigma_{k-1})} \left[(\phi(s) - a)^+ \wedge \inf_{u\in[s,t]} \phi(u) \right], \tag{4.14}$$

$$S_3 = \sup_{s\in[\sigma_{k-1},t]} \left[(\phi(s) - a)^+ \wedge \inf_{u\in[s,t]} \phi(u) \right]. \tag{4.15}$$

We show that each of the terms S_i is dominated by $\sup_{s\in[\sigma_{k-1},t]}(\phi(s)-a)^+$. For S_3, this is obvious. For S_1, we use (4.7) and the fact that $t \geq \sigma_{k-1}$ to write

$$S_1 \leq \sup_{s\in[0,\tau_{k-1}]} \inf_{u\in[s,t]} \phi(u) \leq \inf_{u\in[\tau_{k-1},\sigma_{k-1}]} \phi(u) \leq \phi(\sigma_{k-1})-a \leq \sup_{s\in[\sigma_{k-1},t]} (\phi(s)-a)^+. \tag{4.16}$$

Finally, for $s \in (\tau_{k-1}, \sigma_{k-1})$, (4.6) implies $\phi(s) - a < \inf_{u\in[\tau_{k-1},s]} \phi(u)$, and hence

$$S_2 \leq \sup_{s\in(\tau_{k-1},\sigma_{k-1})} \left[\inf_{u\in[\tau_{k-1},s]} \phi(u) \wedge \inf_{u\in[s,t]} \phi(u) \right] = \inf_{u\in[\tau_{k-1},t]} \phi(u) \leq \inf_{u\in[\tau_{k-1},\sigma_{k-1}]} \phi(u).$$

We conclude as in (4.16). □

Proposition 4.3. *We have $C^\phi(t) = 0$ for $t \in [0, \sigma_0)$. For $k \geq 1$, $C^\phi(t) = \inf_{u\in[\tau_k,t]} \phi(u)$ for all $t \in [\tau_k, \sigma_k)$.*

Proof. It is obvious from (3.28) that $C^\phi(t) = 0$ for $t \in [0, \sigma_0)$. Now let $k \geq 1$ and $t \in [\tau_k, \sigma_k)$ be given. By definition,

$$C^\phi(t) = \sup_{s\in[0,\tau_k]} \left[(\phi(s)-a)^+ \wedge \inf_{u\in[s,t]} \phi(u) \right] \vee \sup_{s\in[\tau_k,t]} \left[(\phi(s)-a)^+ \wedge \inf_{u\in[s,t]} \phi(u) \right]. \tag{4.17}$$

It is obvious that

$$\sup_{s\in[0,\tau_k]} \left[(\phi(s)-a)^+ \wedge \inf_{u\in[s,t]} \phi(u) \right] \leq \sup_{s\in[0,\tau_k]} \inf_{u\in[s,t]} \phi(u) = \inf_{u\in[\tau_k,t]} \phi(u).$$

In addition, (4.6) implies

$$\begin{aligned} &\sup\nolimits_{s\in[\tau_k,t]} \left[(\phi(s)-a)^+ \wedge \inf\nolimits_{u\in[s,t]} \phi(u) \right] \\ &\qquad \leq \sup\nolimits_{s\in[\tau_k,t]} \left[\inf\nolimits_{u\in[\tau_k,s]} \phi(u) \wedge \inf\nolimits_{u\in[s,t]} \phi(u) \right] \\ &\qquad = \inf\nolimits_{u\in[\tau_k,t]} \phi(u). \end{aligned}$$

We have obtained the upper bound

$$C^\phi(t) \leq \inf_{u\in[\tau_k,t]} \phi(u). \tag{4.18}$$

For the reverse inequality, we observe that there exists ρ such that either

$$\rho \in [\sigma_{k-1}, \tau_k] \quad \text{and} \quad \sup_{s\in[\sigma_{k-1},\tau_k]} \phi(s) = \phi(\rho), \tag{4.19}$$

or else

$$\rho \in (\sigma_{k-1}, \tau_k] \text{ and } \sup_{s\in[\sigma_{k-1},\tau_k]} \phi(s) = \phi(\rho-). \tag{4.20}$$

In either case, we have from (4.4) that for $u \in [\rho, \tau_k)$,

$$\phi(u) > \sup_{s\in[\sigma_{k-1},u]} \phi(s) - a = \sup_{s\in[\sigma_{k-1},\tau_k]} \phi(s) - a,$$

and hence, by (4.5),

$$\inf_{u\in[\rho,\tau_k)} \phi(u) \geq \sup_{s\in[\sigma_{k-1},\tau_k]} \phi(s) - a \geq \phi(\tau_k). \tag{4.21}$$

In the case (4.19), we write

$$C^{\phi}(t) \geq (\phi(\rho) - a)^+ \wedge \inf_{u\in[\rho,\tau_k)} \phi(u) \wedge \inf_{u\in[\tau_k,t]} \phi(u)$$

and use (4.19), (4.5), and (4.21) to conclude that

$$C^{\phi}(t) \geq \inf_{u\in[\tau_k,t]} \phi(u). \tag{4.22}$$

In the case (4.20), we choose a sequence $\{\rho_n\}_{n=1}^{\infty}$ in (σ_{k-1}, ρ) with $\rho_n \uparrow \rho$ and write

$$C^{\phi}(t) \geq (\phi(\rho_n) - a)^+ \wedge \inf_{u\in[\rho_n,\tau_k)} \phi(u) \wedge \inf_{u\in[\tau_k,t]} \phi(u). \tag{4.23}$$

Letting $n \to \infty$, we obtain

$$\begin{aligned} C^{\phi}(t) &\geq (\phi(\rho-) - a)^+ \wedge \phi(\rho-) \wedge \inf_{u\in[\rho,\tau_k)} \phi(u) \wedge \inf_{u\in[\tau_k,t]} \phi(u) \\ &\geq (\phi(\rho-) - a)^+ \wedge \inf_{u\in[\rho,\tau_k)} \phi(u) \wedge \inf_{u\in[\tau_k,t]} \phi(u). \end{aligned}$$

We now use (4.20), (4.5), and (4.21) to conclude (4.22). □

In summary, Propositions 4.2 and 4.3 imply that $C^{\phi}(t)$ given by (3.28) has the form

$$C^{\phi}(t) = \begin{cases} 0, & 0 \leq t < \sigma_0, \\ \sup_{s\in[\sigma_{k-1},t]} (\phi(s) - a)^+, & \sigma_{k-1} \leq t < \tau_k,\ k \geq 1, \\ \inf_{u\in[\tau_k,t]} \phi(u), & \tau_k \leq t < \sigma_k,\ k \geq 1. \end{cases} \tag{4.24}$$

From (4.24), (4.5), and (4.7), we see that for $k \geq 1$,

$$C^\phi(\tau_k-) = \sup_{s\in[\sigma_{k-1},\tau_k)} (\phi(s)-a)^+ \geq \phi(\tau_k) = C^\phi(\tau_k), \tag{4.25}$$

$$C^\phi(\sigma_k-) = \inf_{u\in[\tau_k,\sigma_k)} \phi(u) \leq \phi(\sigma_k) - a = C^\phi(\sigma_k). \tag{4.26}$$

We define $C^\phi(0-) = 0$ and we have

$$C^\phi(\sigma_0-) = 0 \leq C^\phi(\sigma_0) = \phi(\sigma_0) - a. \tag{4.27}$$

In particular, C^ϕ is increasing on each interval $[\sigma_{k-1}, \tau_k)$, with a possible upward jump at σ_{k-1}, and C^ϕ is decreasing on each interval $[\tau_k, \sigma_k)$, with a possible downward jump at τ_k.

Theorem 4.4. *Let $\phi \in \mathcal{D}_+[0,\infty)$ be given, define C^ϕ by (3.28), and set $\bar{\phi} = \phi - C^\phi$. Then $C^\phi \in \mathcal{BV}[0,\infty)$, $\bar{\phi} \in \mathcal{D}[0,\infty)$, and $\bar{\phi}$ takes values only in $[0,a]$. Furthermore,*

$$|C^\phi|(t) = \int_0^t 1_{\{\bar{\phi}(s)=0 \text{ or } \bar{\phi}(s)=a\}} d|C^\phi|(s), \tag{4.28}$$

$$C^\phi(t) = -\int_0^t 1_{\{\bar{\phi}(s)=0\}} d|C^\phi|(s) + \int_0^t 1_{\{\bar{\phi}(s)=a\}} d|C^\phi|(s). \tag{4.29}$$

Proof. From (4.24) we see that $C^\phi \in \mathcal{BV}[0,\infty)$. From its definition (3.28), we see that C^ϕ further satisfies $(\phi - a)^+ \leq C^\phi \leq \phi$, and hence

$$0 \leq \bar{\phi} \leq a \wedge \phi. \tag{4.30}$$

Furthermore, (4.25)–(4.27) show that

$$\bar{\phi}(\tau_k) = 0, \quad \bar{\phi}(\sigma_{k-1}) = a, \quad k \geq 1. \tag{4.31}$$

Since $C^\phi = 0$ on $[0, \sigma_0)$, we only need to consider $t \geq \sigma_0$ in what follows. Define the set

$$A \doteq \{t \geq \sigma_0 : \bar{\phi}(t) \in (0,a)\}. \tag{4.32}$$

We show below that $\int_A d|C^\phi| = 0$, so that (4.28) holds. We further show that for $t \geq \sigma_0$,

$$\bar{\phi}(t) = 0 \ \Rightarrow t \in [\tau_k, \sigma_k) \text{ for some } k, \tag{4.33}$$

whereas

$$\bar{\phi}(t) = a \ \Rightarrow t \in [\sigma_{k-1}, \tau_k) \text{ for some } k. \tag{4.34}$$

We can then conclude that C^ϕ does not increase on $\{t \geq 0 | \bar{\phi}(t) = 0\}$ (the positive variation of C^ϕ assigns zero measure to this set) and C^ϕ does not decrease on the set $\{t \geq 0 | \bar{\phi}(t) = a\}$ (the negative variation of C^ϕ assigns zero measure to this set). This together with (4.28) will imply (4.29).

We first establish (4.33) and (4.34). Suppose $t \in [\sigma_{k-1}, \tau_k)$ for some k. Then (4.4) and either (4.7) or (4.8) imply

$$\phi(t) > \sup_{s \in [\sigma_{k-1}, t]} \phi(s) - a \geq \phi(\sigma_{k-1}) - a \geq 0.$$

From this and (4.24) we have

$$C^\phi(t) = \sup_{s \in [\sigma_{k-1}, t]} \left(\phi(s) - a\right)^+ = \sup_{s \in [\sigma_{k-1}, t]} \phi(s) - a < \phi(t).$$

Therefore, $\bar{\phi}(t) = \phi(t) - C^\phi(t) > 0$. This is the contrapositive of (4.33). Similarly, suppose $t \in [\tau_k, \sigma_k)$ for some k. Then (4.24) and (4.6) imply $C^\phi(t) = \inf_{u \in [\tau_k, t]} \phi(u) > \phi(t) - a$, so that $\bar{\phi}(t) = \phi(t) - C^\phi(t) < a$. This is the contrapositive of (4.34).

We next show that $\int_A d|C^\phi| = 0$. For $t \in A$, define

$$\alpha(t) \doteq \inf\left\{s \in [\sigma_0, t] \big| (s, t] \subset A\right\}, \quad \beta(t) \doteq \sup\left\{s \in [t, \infty) \big| [t, s) \in A\right\}.$$

Because of the right-continuity of $\bar{\phi}$, we have $\beta(t) \notin A$, whereas $\alpha(t)$ might or might not be in A. We also have $\alpha(t) \leq t < \beta(t)$, and so the open interval $(\alpha(t), \beta(t))$ is nonempty. It follows that A is the countable union of such disjoint open intervals together with a countable set of left endpoints, i.e.,

$$A = \left(\cup_{i \in I} (\alpha_i, \beta_i)\right) \cup \{\alpha_j | j \in J\},$$

where I is a countable index set and $J \subset I$.

As a first step in showing $\int_A d|C^\phi| = 0$, we show that if $j \in J$, so $\alpha_j \in A$, then C^ϕ is continuous at α_j. From (4.31) we see that α_j is in the interior of an interval of the form (τ_k, σ_k) or of the form (σ_{k-1}, τ_k). By the definition of α_j, there is a sequence of points $\{\gamma_n\}_{n=1}^\infty$ in $(0, \alpha_j) \cap A^c$ such that $\gamma_n \uparrow \alpha_j$.

We consider first the case that $\bar{\phi}(\gamma_n) = a$, or equivalently, $C^\phi(\gamma_n) = \phi(\gamma_n) - a$, for infinitely many values of n. From (4.34), we see that $\gamma_n \in [\sigma_{k-1}, \tau_k)$ for some k. By choosing n sufficiently large, we may assume that k does not depend on n and $\alpha_j \in (\sigma_{k-1}, \tau_k)$. We have

$$\begin{array}{rclcl} a & = & \phi(\gamma_n) - C^\phi(\gamma_n) & = & \phi(\gamma_n) - \sup_{s \in [\sigma_{k-1}, \gamma_n]} \left(\phi(s) - a\right)^+ \\ & \leq & \phi(\gamma_n) - \left(\phi(\gamma_n) - a\right)^+ & = & \phi(\gamma_n) \wedge a \leq a. \end{array}$$

Therefore, the above inequalities must be equalities and we conclude that

$$0 \leq \phi(\gamma_n) - a = C^\phi(\gamma_n) = \sup_{s\in[\sigma_{k-1},\gamma_n]} (\phi(s) - a)^+.$$

Letting $n \to \infty$, we see that

$$0 \leq \phi(\alpha_j-) - a = C^\phi(\alpha_j-) = \sup_{s\in[\sigma_{k-1},\alpha_j)} (\phi(s) - a)^+.$$

On the other hand, $C^\phi(\alpha_j) = \sup_{s\in[\sigma_{k-1},\alpha_j]} (\phi(s) - a)^+$. This shows that $C^\phi(\alpha_j) \geq C^\phi(\alpha_j-)$. Furthermore, $C^\phi(\alpha_j) > C^\phi(\alpha_j-)$ implies $C^\phi(\alpha_j) = \phi(\alpha_j) - a$. But in this case, $\bar\phi(\alpha_j) = a$. This contradicts the membership of α_j in A and establishes the continuity of C^ϕ at α_j.

If $\bar\phi(\gamma_n) = a$ does not hold for infinitely many values of n, then $\bar\phi(\gamma_n) = 0$, or equivalently, $C^\phi(\gamma_n) = \phi(\gamma_n)$, must hold for infinitely many values of n. From (4.33), we see that $\gamma_n \in [\tau_k, \sigma_k)$ for some k. By choosing n sufficiently large, we may assume that k does not depend on n and $\alpha_j \in (\tau_k, \sigma_k)$. We have

$$0 = \phi(\gamma_n) - C^\phi(\gamma_n) = \phi(\gamma_n) - \inf_{u\in[\tau_k,\gamma_n]} \phi(u) \geq 0.$$

Therefore, the above inequality must be an equality and we conclude that

$$\phi(\gamma_n) = C^\phi(\gamma_n) = \inf_{u\in[\tau_k,\gamma_n]} \phi(u).$$

Letting $n \to \infty$, we see that

$$\phi(\alpha_j-) = C^\phi(\alpha_j-) = \inf_{u\in[\tau_k,\alpha_j)} \phi(u).$$

On the other hand, $C^\phi(\alpha_j) = \inf_{u\in[\tau_k,\alpha_j]} \phi(u)$. This shows that $C^\phi(\alpha_j) \leq C^\phi(\alpha_j-)$. Furthermore, $C^\phi(\alpha_j) < C^\phi(\alpha_j-)$ implies $C^\phi(\alpha_j) = \phi(\alpha_j)$. But in this case, $\bar\phi(\alpha_j) = 0$. This contradicts the membership of α_j in A, which establishes the continuity of C^ϕ at α_j.

To establish $\int_A d|C^\phi| = 0$, it remains only to show that $\int_{(\alpha_i,\beta_i)} d|C^\phi| = 0$ for every $i \in I$. Because ϕ is strictly between 0 and a on (α_i, β_i), (4.31) shows that (α_i, β_i) must be entirely contained in an interval of the form (τ_k, σ_k) or of the form (σ_{k-1}, τ_k). We consider the latter case; the former case is analogous. It suffices to show that C^ϕ is constant on $[a_i, b_i]$ whenever $\alpha_i < a_i < b_i < \beta_i$, where

$$C^\phi(t) = \sup_{s\in[\sigma_{k-1},t]} (\phi(s) - a)^+ \;\forall t \in (\alpha_i, \beta_i).$$

Define

$$\rho = \inf\{t \in [a_i, b_i] | C^\phi(t) > C^\phi(a_i)\}.$$

Assume $\rho < \infty$. Because C^ϕ is right-continuous, we must have $C^\phi(t) = C^\phi(a_i)$ for all $t \in [a_i, \rho)$ and either $C^\phi(\rho) = \phi(\rho) - a > C(a_i)$ or else $C^\phi(\rho) = \phi(\rho) - a = C^\phi(a_i)$. In either case, $\bar{\phi}(\rho) = a$, contradicting the definition of A. Therefore, $\rho = \infty$ and C^ϕ is constant on $[a_i, b_i]$. □

Remark 4.5. The stopping times σ_k, τ_k, $k = 0, 1, \ldots$, of (4.1)–(4.2) coincide with the stopping times $\overline{\sigma}_k, \overline{\tau}_k$, $k = 0, 1, \ldots$, of (3.26) and (3.27). Indeed, for $k \geq 1$, the contrapositive of (4.33) dictates that $\bar{\phi}(t) > 0$ for $t \in [\sigma_{k-1}, \tau_k)$. Moreover, according to (4.25), $\bar{\phi}(\tau_k) = \phi(\tau_k) - C^\phi(\tau_k) = 0$, and so

$$\tau_k = \min\{t \geq \sigma_{k-1} | \bar{\phi}(t) = 0\}. \tag{4.35}$$

Similarly, the contrapositive of (4.34) dictates that $\bar{\phi}(t) < a$ for $t \in [\tau_k, \sigma_k)$, and according to (4.26), $\bar{\phi}(\sigma_k) = \phi(\sigma_k) - C^\phi(\sigma_k) = a$, so

$$\sigma_k = \min\{t \geq \tau_k | \bar{\phi}(t) = a\}. \tag{4.36}$$

Finally, for $0 \leq t < \sigma_0$, (4.24) implies $\bar{\phi}(t) = \phi(t)$, so $\bar{\phi}(t) < a$. According to (4.27), $\bar{\phi}(\sigma_0) = \phi(\sigma) - C^\phi(\sigma_0) = a$, so

$$\sigma_0 = \min\{t \geq 0 | \bar{\phi}(t) = a\}. \tag{4.37}$$

Equations (4.35)–(4.37) coincide with (3.26), (3.27).

Second proof of Theorem 1.4. Let $\psi \in \mathcal{D}[0, \infty)$ be given and define $\phi = \Gamma_0(\psi)$. Then $\eta \doteq \phi - \psi \in \mathcal{I}[0, \infty)$ satisfies (see (1.2))

$$\eta(t) = \int_0^t 1_{\{\phi(s)=0\}} d\eta(s), \quad \int_0^t 1_{\{\phi(s)>0\}} d\eta(s) = 0 \quad \forall t \geq 0. \tag{4.38}$$

With C^ϕ defined by (3.28), set

$$\bar{\phi} = \Lambda_a(\phi) = \phi - C^\phi = \psi + \eta - C^\phi.$$

Theorem 4.4 implies $\eta - C^\phi \in \mathcal{BV}[0, \infty)$, $\bar{\phi} \in \mathcal{D}[0, \infty)$, and $\bar{\phi}$ takes values only in $[0, a]$. It remains to show that for all $t \geq 0$,

$$|\eta - C^\phi|(t) = \int_0^t 1_{\{\bar{\phi}(s)=0 \text{ or } \bar{\phi}(s)=a\}} d|\eta - C^\phi|(s), \tag{4.39}$$

$$\eta(t) - C^\phi(t) = \int_0^t 1_{\{\bar{\phi}(s)=0\}} d|\eta - C^\phi|(s) - \int_0^t 1_{\{\bar{\phi}(s)=a\}} d|\eta - C^\phi|(s). \tag{4.40}$$

Because $\{s|\phi(s) = 0\} \subset \{s|\bar{\phi}(s) = 0\}$ (see (4.30)) and C^ϕ is decreasing on this set (see (4.29)), (4.38) implies $|\eta - C^\phi| = \eta + |C^\phi|$. Equations (4.39) and (4.40) follow from (4.38), (4.28), and (4.29). ■

5. Comparison Properties of the Double Reflection Map

In this section we present the proof of Theorem 1.6. We first establish some preliminary results that may be of independent interest. In the proofs we make repeated use of the elementary inequalities $[b_1 + b_2]^+ \leq b_1^+ + b_2^+$ and $[b_1 - b_2]^+ \geq b_1^+ - b_2^+$ for $b_1, b_2 \in I\!R$, without explicit reference.

Lemma 5.1. *Given $c_0, c_0' \in I\!R$ and $\psi, \psi' \in \mathcal{D}[0,\infty)$ with $\psi(0) = \psi'(0) = 0$, suppose (ϕ, η) and (ϕ', η') solve the Skorokhod problem on $[0,\infty)$ for $c_0 + \psi$ and $c_0' + \psi'$, respectively. If there exists $\nu \in \mathcal{I}[0,\infty)$ such that $\psi' \leq \psi \leq \psi' + \nu$, then the following two properties are satisfied:*

1. $\eta - [c_0' - c_0]^+ \leq \eta' \leq \eta + \nu + [c_0 - c_0']^+$;
2. $\phi' - \nu - [c_0' - c_0]^+ \leq \phi \leq \phi' + \nu + [c_0 - c_0']^+$.

Moreover, if $\psi = \psi' + \nu$ then

$$\phi' - [c_0' - c_0]^+ \leq \phi \leq \phi' + \nu + [c_0 - c_0']^+. \tag{5.1}$$

Proof. Using the explicit representations for η and η' that follow from (1.1), along with the fact that $\nu \in \mathcal{I}[0,\infty)$ and $\psi \leq \psi' + \nu$, we see that for every $t \in [0,\infty)$,

$$\begin{array}{rcl} \eta(t) = \sup_{s\in[0,t]}[-c_0 - \psi(s)]^+ & \geq & \sup_{s\in[0,t]}[-c_0' - \psi'(s) - \nu(s) - c_0 + c_0']^+. \\ & \geq & \sup_{s\in[0,t]}[-c_0' - \psi'(s) - \nu(t) - c_0 + c_0']^+. \\ & \geq & \sup_{s\in[0,t]}[-c_0' - \psi'(s)]^+ - [\nu(t) + c_0 - c_0']^+ \\ & \geq & \eta'(t) - \nu(t) - [c_0 - c_0']^+. \end{array}$$

Likewise, (1.1) and the fact that $\psi \geq \psi'$ shows that for every $t \in [0,\infty)$,

$$\begin{array}{rcl} \eta'(t) = \sup_{s\in[0,t]}[-c_0' - \psi'(s)]^+ & \geq & \sup_{s\in[0,t]}[-c_0 - \psi(s) - (c_0' - c_0)]^+ \\ & \geq & \sup_{s\in[0,t]}[-c_0 - \psi(s)]^+ - [c_0' - c_0]^+ \\ & = & \eta(t) - [c_0' - c_0]^+. \end{array} \tag{5.2}$$

When combined, the last two relations establish property 1. Moreover, the first relation and the fact that $\eta' = -c_0' - \psi' + \phi'$ also implies that

$$\phi = \psi + c_0 + \eta \geq \psi + c_0 - c_0' - \psi' + \phi' - \nu - [c_0 - c_0']^+ = \phi' + \psi - \psi' - \nu - [c_0' - c_0]^+,$$

which is no less than $\phi' - \nu - [c_0' - c_0]^+$ if $\psi' \leq \psi \leq \psi' + \nu$ and is no less than $\phi' - [c_0' - c_0]^+$ if $\psi = \psi' + \nu$. On the other hand the second relation, (5.2), shows that

$$\phi = c_0 + \psi + \eta \leq c_0' + \psi' + \eta' + c_0 - c_0' + [c_0' - c_0]^+ + \psi - \psi' = \phi' + [c_0 - c_0']^+ + \psi - \psi'.$$

Together, the last two displays establish property 2 and (5.1). □

The representation (1.13) for $\Gamma_{0,a}$ as the composition of Λ_a and Γ_0, allows us to easily deduce the following corollary from Lemma 5.1.

Corollary 5.2. *Given $a > 0$, $c_0, c_0' \in I\!R$ and $\psi, \psi' \in \mathcal{D}[0,\infty)$ with $\psi(0) = \psi'(0) = 0$, suppose $(\bar\phi, \bar\eta)$ and $(\bar\phi', \bar\eta')$ solve the Skorokhod problem on $[0,a]$ for $c_0 + \psi$ and $c_0' + \psi'$, respectively. If $\psi = \psi' + \nu$, where $\nu \in \mathcal{I}[0,\infty)$, then the following two properties hold:*

1. $\bar\eta - 2[c_0' - c_0]^+ \le \bar\eta' \le \bar\eta + 2\nu + 2[c_0 - c_0']^+$;
2. $[-|c_0' - c_0| - \nu] \vee [-a] \le \bar\phi' - \bar\phi \le [|c_0' - c_0| + \nu] \wedge a$.

Proof. Let $C = C^\phi$ be the function defined in (3.28) and let $C' = C^{\phi'}$. From the first inequality in (5.1) of Lemma 5.1, it follows that

$$\begin{aligned} C'(t) &= \sup_{s\in[0,t]}[(\phi'(s) - a)^+ \wedge \inf_{u\in[s,t]} \phi'(u)] \\ &\le \sup_{s\in[0,t]}[(\phi(s) - a + [c_0' - c_0]^+)^+ \wedge \inf_{u\in[s,t]} (\phi(u) + [c_0' - c_0]^+)] \\ &\le \sup_{s\in[0,t]}[(\phi(s) - a)^+ \wedge \inf_{u\in[s,t]} \phi(u)] + [c_0' - c_0]^+ \\ &= C(t) + [c_0' - c_0]^+. \end{aligned}$$

Similarly, the second inequality in (5.1) along with the fact that ν is non-decreasing implies that $C'(t)$ is equal to

$$\begin{aligned} & \sup_{s\in[0,t]}[(\phi'(s) - a)^+ \wedge \inf_{u\in[s,t]} \phi'(u)] \\ \ge\ & \sup_{s\in[0,t]}[(\phi(s) - a - \nu(t) - [c_0 - c_0']^+)^+ \wedge \inf_{u\in[s,t]}(\phi(u) - \nu(t) - [c_0 - c_0']^+)] \\ \ge\ & \sup_{s\in[0,t]}[(\phi(s) - a)^+ \wedge \inf_{u\in[s,t]} \phi(u)] - \nu(t) - [c_0 - c_0']^+ \\ =\ & C(t) - \nu(t) - [c_0 - c_0']^+. \end{aligned}$$

Let $\eta = \Gamma_0(c_0 + \psi) - c_0 - \psi$ and, likewise, let $\eta' = \Gamma_0(c_0' + \psi') - c_0' - \psi'$, and note that due to the representation for $\Gamma_{0,a}$ in (1.13), the definition (1.11) of Λ_a and the definitions of C, C', we can write $\bar\eta = \eta - C$ and $\bar\eta' = \eta' - C'$. The last two displays, together with property 1 of Lemma 5.1, then show that

$$\bar\eta = \eta - C \le \eta' + [c_0' - c_0]^+ - C' + [c_0' - c_0]^+ = \bar\eta' + 2[c_0' - c_0]^+$$

and

$$\bar\eta = \eta - C \ge \eta' - \nu - [c_0 - c_0']^+ - C' - \nu - [c_0 - c_0']^+ = \bar\eta' - 2\nu - 2[c_0 - c_0']^+,$$

which establishes the first property of the corollary. The second property follows from the first property, the fact that $\bar\phi', \bar\phi \in [0,a]$ and the relation

$$\bar\phi' - \bar\phi = c_0' + \psi' + \bar\eta' - c_0 - \psi - \bar\eta = c_0' - c_0 - \nu + \bar\eta' - \bar\eta. \tag{5.3}$$

□

We introduce the family of shift operators $T_r : \mathcal{D}[0,\infty) \to \mathcal{D}[0,\infty)$, $r \in [0,\infty)$, defined by

$$[T_r f](t) = f(r+t) - f(r) \quad \text{for } t \in [0,\infty).$$

We shall also make use of the well-known (and easily verified) fact that if $\phi = \Gamma(\psi)$, where Γ is either the one-sided reflection map at zero or a, or the two-sided reflection map on $[0,a]$, then for every $\alpha > 0$,

$$\phi(\alpha + s) = \Gamma(\phi(\alpha) + T_\alpha \psi)(s). \tag{5.4}$$

Remark 5.3. The first and second inequalities in Corollary 5.2 can be strengthened to the inequalities

$$-[c_0' - c_0]^+ \leq \bar{\eta}' - \bar{\eta} \leq [c_0 - c_0']^+ + \nu \tag{5.5}$$

and

$$[-[c_0 - c_0']^+ - \nu] \vee [-a] \leq \bar{\phi}' - \bar{\phi} \leq [c_0' - c_0]^+ \wedge a, \tag{5.6}$$

which are both easily seen to be tight. Since $\bar{\phi}(t), \bar{\phi}'(t) \in [0,a]$ for all $t \in [0,\infty)$, in order to establish (5.6), it suffices to show that

$$-[c_0 - c_0']^+ - \nu(t) \leq \bar{\phi}'(t) - \bar{\phi}(t) \leq [c_0' - c_0]^+ \qquad \text{for } t \in [0,\infty). \tag{5.7}$$

We use the projection operator π of (1.8). This operator is monotone and Lipschitz with Lipschitz constant 1.

First suppose $c_0 \geq c_0'$. Then, due to the monotonicity property of the projection operator π and the Lipschitz continuity of $\Gamma_{0,a}$, Lemma 4.2 of [10] shows that the upper bound $\bar{\phi}' - \bar{\phi} \leq 0 = [c_0' - c_0]^+$ in (5.7) holds, while the lower bound in (5.7) follows from the first inequality in part 2 of Corollary 5.2.

Now suppose $c_0 < c_0'$. Define

$$\tau \doteq \inf\{t \geq 0 : \bar{\phi}(t) \geq \bar{\phi}'(t)\}.$$

The fact that $\bar{\phi}(0) = \pi(c_0) \leq \pi(c_0') = \bar{\phi}'(0)$ and $\bar{\phi}(t), \bar{\phi}'(t) \in [0,a]$ imply $\bar{\phi}(t) < a$ and $\bar{\phi}'(t) > 0$ for $t \in [0,\tau)$. (It could happen that $\pi(c_0) = \pi(c_0')$, and then $\tau = 0$ and all assertions concerning $t \in [0,\tau)$ are vacuously true.) Definitions 1.1, 1.2 and relation (1.4) then show that for $t \in [0,\tau)$, $\bar{\phi}(t) = \Gamma_0(c_0 + \psi)(t)$ and $\bar{\phi}'(t) = \Gamma_a(c_0' + \psi')(t)$. Therefore for $t \in [0,\tau)$, $c_0 + \psi(t) \leq \bar{\phi}(t) < \bar{\phi}'(t) \leq c_0' + \psi'(t)$, which in turn implies that

$$-\nu(t) \leq 0 \leq \bar{\phi}'(t) - \bar{\phi}(t) \leq c_0' - c_0 + \psi'(t) - \psi(t) \leq c_0' - c_0 \quad \text{for } t \in [0,\tau). \tag{5.8}$$

This shows that (5.7) is satisfied for $t \in [0,\tau)$. In particular, this implies that $\bar{\phi}'(\tau-) \geq \bar{\phi}(\tau-) - \nu(\tau-)$. By the monotonicity property of the projection operator π, we have

$$\begin{array}{rcl} \bar{\phi}'(\tau) & = & \pi\left(\bar{\phi}'(\tau-) + \psi'(\tau) - \psi'(\tau-)\right) \\ & \geq & \pi\left(\bar{\phi}(\tau-) - \nu(\tau-) + \psi(\tau) - \psi(\tau-) - (\nu(\tau) - \nu(\tau-))\right) \\ & \geq & \pi\left(\bar{\phi}(\tau-) + \psi(\tau) - \psi(\tau-)\right) - \nu(\tau) \\ & = & \bar{\phi}(\tau) - \nu(\tau), \end{array} \tag{5.9}$$

where the explicit definition of π is used to obtain the second inequality. Now for $s \in [0,\infty)$, $\bar{\phi}(\tau+s) = \Gamma_{0,a}(\bar{\phi}(\tau) + T_\tau\psi)(s)$ and, likewise, $\bar{\phi}'(\tau+s) = \Gamma_{0,a}(\bar{\phi}'(\tau) + T_\tau\psi')(s)$. Since $\bar{\phi}(\tau) \geq \bar{\phi}'(\tau)$ due to the right-continuity of $\bar{\phi}, \bar{\phi}'$, we can apply (5.7) (with c_0, c_0', ψ, ψ' and ν replaced by $\bar{\phi}(\tau), \bar{\phi}'(\tau)$, $T_\tau\psi$, $T_\tau\psi'$ and $T_\tau\nu$), and use (5.9) to obtain for $s \in [0,\infty)$,

$$-\nu(\tau+s) \leq -[\bar{\phi}(\tau)-\bar{\phi}'(\tau)]^+ - T_\tau\nu(s) \leq \bar{\phi}'(\tau+s)-\bar{\phi}(\tau+s) \leq [\bar{\phi}'(\tau)-\bar{\phi}(\tau)]^+ = 0,$$

which shows that (5.7) also holds for $t \in [\tau,\infty)$.

We have established (5.7), and hence (5.6). The inequality (5.5) can be deduced from (5.6) using the basic relation

$$\bar{\eta}'-\bar{\eta} = \bar{\phi}'-\bar{\phi}-(c_0'-c_0)-(\psi'-\psi) = \bar{\phi}'-\bar{\phi}-(c_0'-c_0)+\nu. \qquad \blacksquare$$

Although Corollary 5.2 provides bounds on the difference between the net constraining terms $\bar{\eta}$ and $\bar{\eta}'$, it is often desirable to compare the individual constraining terms at the upper and lower barriers. Such bounds are provided in Theorem 1.6. To establish these bounds, we recall that if $(\bar{\phi},\bar{\eta})$ solves the Skorokhod problem on $[0,a]$ for $\psi \in \mathcal{D}[0,\infty)$, and if $\bar{\eta}$ admits the decomposition $\bar{\eta} = \bar{\eta}_\ell - \bar{\eta}_u$ that satisfies (1.9), then for any $t \in [0,\infty)$,

$$\begin{array}{rcl} \bar{\eta}_\ell(t) - \bar{\eta}_\ell(t-) & = & \sup_{s\in[0,t]} [\bar{\eta}_u(s) - \psi(s)]^+ - \sup_{s\in[0,t)} [\bar{\eta}_u(s) - \psi(s)]^+ \\ & = & [\bar{\eta}_u(t) - \psi(t) - \bar{\eta}_\ell(t-)]^+ \\ & = & [-\bar{\phi}(t-) - \psi(t) + \psi(t-) + \bar{\eta}_u(t) - \bar{\eta}_u(t-)]^+. \end{array} \tag{5.10}$$

Proof of Theorem 1.6. Define

$$\alpha \doteq \inf\left\{t > 0 : \bar{\eta}_\ell(t) + \nu(t) + [c_0 - c_0']^+ < \bar{\eta}_\ell'(t) \text{ or } \bar{\eta}_u(t) + [c_0' - c_0]^+ < \bar{\eta}_u'(t)\right\},$$

with $\alpha \doteq \infty$ if the infimum is over the empty set. Then the definition of α dictates that the following two relations are satisfied for $s \in [0,\alpha)$:

$$\bar{\eta}_\ell'(s) \leq \bar{\eta}_\ell(s) + \nu(s) + [c_0 - c_0']^+; \tag{5.11}$$

$$\bar{\eta}'_u(s) \leq \bar{\eta}_u(s) + [c'_0 - c_0]^+. \tag{5.12}$$

Suppose $\alpha < \infty$. Then we claim (and prove below) that it is also true that

$$\bar{\eta}'_\ell(\alpha) \leq \bar{\eta}_\ell(\alpha) + \nu(\alpha) + [c_0 - c'_0]^+ \tag{5.13}$$

and

$$\bar{\eta}'_u(\alpha) \leq \bar{\eta}_u(\alpha) + [c'_0 - c_0]^+. \tag{5.14}$$

To see why the claim is true, first note that since ν, $\bar{\eta}_\ell$ and $\bar{\eta}_u$ are non-decreasing, it is clear from (5.11) that if $\bar{\eta}'_\ell$ is continuous at α, then (5.13) holds. Likewise, if $\bar{\eta}'_u$ is continuous at α, then (5.12) implies that (5.14) is satisfied. Now suppose $\bar{\eta}'_\ell(\alpha) - \bar{\eta}'_\ell(\alpha-) > 0$. Then the complementarity conditions in (1.6) show that $\bar{\phi}'(\alpha) = 0$ and $\bar{\eta}'_u(\alpha-) = \bar{\eta}'_u(\alpha)$. Hence, (5.10) applied to $\bar{\eta}'_\ell$ implies that

$$\bar{\eta}'_\ell(\alpha) = \bar{\eta}'_\ell(\alpha-) - \bar{\phi}'(\alpha-) - \psi'(\alpha) + \psi'(\alpha-).$$

Making the further substitutions $\bar{\eta}'_\ell(\alpha-) - \bar{\phi}'(\alpha-) + \psi'(\alpha-) = -c'_0 + \bar{\eta}'_u(\alpha-)$, $\psi = \psi' + \nu$ and then $\bar{\eta}_u(\alpha-) = c_0 + \psi(\alpha-) + \bar{\eta}_\ell(\alpha-) - \bar{\phi}(\alpha-)$ into the last display, we obtain

$$\begin{aligned}
\bar{\eta}'_\ell(\alpha) &= -c'_0 + \bar{\eta}'_u(\alpha-) - \psi'(\alpha) \\
&= -c'_0 + \bar{\eta}'_u(\alpha-) - \psi(\alpha) + \nu(\alpha) \\
&= -c'_0 + c_0 + \psi(\alpha-) + \bar{\eta}_\ell(\alpha-) - \bar{\phi}(\alpha-) - \psi(\alpha) + \nu(\alpha) + \bar{\eta}'_u(\alpha-) - \bar{\eta}_u(\alpha-).
\end{aligned}$$

Taking limits as $s \uparrow \alpha$ in (5.12) yields the inequality $\bar{\eta}'_u(\alpha-) - \bar{\eta}_u(\alpha-) \leq [c'_0 - c_0]^+$. When substituted into the last display, this shows that

$$\begin{aligned}
\bar{\eta}'_\ell(\alpha) &\leq -c'_0 + c_0 + \psi(\alpha-) + \bar{\eta}_\ell(\alpha-) - \bar{\phi}(\alpha-) - \psi(\alpha) + \nu(\alpha) + [c'_0 - c_0]^+ \\
&= \psi(\alpha-) + \bar{\eta}_\ell(\alpha-) - \bar{\phi}(\alpha-) - \psi(\alpha) + \nu(\alpha) + [c_0 - c'_0]^+.
\end{aligned} \tag{5.15}$$

Since $\bar{\eta}_u(\alpha) - \bar{\eta}_u(\alpha-) \geq 0$, (5.10) implies that

$$\begin{aligned}
\bar{\eta}_\ell(\alpha) &= \bar{\eta}_\ell(\alpha-) + [-\bar{\phi}(\alpha-) - \psi(\alpha) + \psi(\alpha-) + \bar{\eta}_u(\alpha) - \bar{\eta}_u(\alpha-)]^+ \\
&\geq \bar{\eta}_\ell(\alpha-) - \bar{\phi}(\alpha-) - \psi(\alpha) + \psi(\alpha-).
\end{aligned}$$

When substituted into (5.15) this yields (5.13). The proof of the remaining fact that (5.14) continues to hold even if $\bar{\eta}'_u(\alpha) - \bar{\eta}'_u(\alpha-) > 0$ is exactly analogous and is thus omitted.

Having established (5.13) and (5.14), we note from the definition of α that there must exist a sequence $\{s_n\}$ with $s_n \downarrow 0$ as $n \to \infty$ such that one of the following two cases holds:

$$(i) \quad \bar{\eta}'_\ell(\alpha + s_n) > \bar{\eta}_\ell(\alpha + s_n) + \nu(\alpha + s_n) + [c_0 - c'_0]^+ \quad \forall n \in I\!N; \tag{5.16}$$

$$(ii) \quad \bar{\eta}'_u(\alpha + s_n) > \bar{\eta}_u(\alpha + s_n) + [c'_0 - c_0]^+ \qquad \forall n \in I\!N. \quad (5.17)$$

First, suppose that Case (i) holds. Then due to (5.16), the fact that $s_n \downarrow 0$ and the right continuity of $\bar{\eta}'_\ell, \bar{\eta}_\ell$ and ν, it follows that $\bar{\eta}'_\ell(\alpha) \geq \bar{\eta}_\ell(\alpha)+\nu(\alpha)+[c_0 - c'_0]^+$. When combined with (5.13), this yields the equality

$$\bar{\eta}'_\ell(\alpha) = \bar{\eta}_\ell(\alpha) + \nu(\alpha) + [c_0 - c'_0]^+. \quad (5.18)$$

We now show that in this case $\bar{\phi}(\alpha) = \bar{\phi}'(\alpha) = 0$. First, combining (5.18), (5.16) and the fact that $\bar{\eta}_\ell + \nu$ is non-decreasing, we have $\bar{\eta}'_\ell(\alpha+s_n) > \bar{\eta}'_\ell(\alpha)$ for every $n \in I\!N$. Since $s_n \downarrow 0$, the first complementarity condition in (1.6) ensures that $\bar{\phi}'(\alpha) = 0$. Along with (5.14), (5.18) and the relations $\bar{\phi}'(\alpha) = 0$ and $\psi = \psi' + \nu$, this implies that

$$\begin{aligned} \bar{\phi}(\alpha) = \bar{\phi}(\alpha) - \bar{\phi}'(\alpha) &= c_0 - c'_0 + \nu(\alpha) + \bar{\eta}_\ell(\alpha) - \bar{\eta}'_\ell(\alpha) + \bar{\eta}'_u(\alpha) - \bar{\eta}_u(\alpha) \\ &\leq c_0 - c'_0 - [c_0 - c'_0]^+ + [c'_0 - c_0]^+ \\ &= 0. \end{aligned}$$

Since $\bar{\phi} \in [0, a]$, this implies $\bar{\phi}(\alpha) = 0$.

The right continuity of $\bar{\phi}$ and $\bar{\phi}'$ then ensures the existence of $\varepsilon > 0$ such that for every $s \in [0, \varepsilon]$, $\bar{\phi}(\alpha + s) < a$ and $\bar{\phi}'(\alpha + s) < a$. Hence, due to the complementarity conditions (1.6), property (5.4) and the definitions of Γ_0 and $\Gamma_{0,a}$, for $s \in [0, \varepsilon]$ we can write

$$\begin{aligned} \bar{\phi}(\alpha + s) &= \Gamma_0(\bar{\phi}(\alpha) + T_\alpha\psi)(s) &&= \Gamma_0(T_\alpha\psi)(s); \\ \bar{\phi}'(\alpha + s) &= \Gamma_0(\bar{\phi}'(\alpha) + T_\alpha\psi')(s) &&= \Gamma_0(T_\alpha\psi')(s); \\ T_\alpha\bar{\eta}_\ell(s) &= T_\alpha\bar{\eta}(s) &&= \Gamma_0(T_\alpha\psi)(s) - T_\alpha\psi(s); \\ T_\alpha\bar{\eta}'_\ell(s) &= T_\alpha\bar{\eta}'(s) &&= \Gamma_0(T_\alpha\psi')(s) - T_\alpha\psi'(s). \end{aligned}$$

Since $T_\alpha\psi = T_\alpha\psi' + T_\alpha\nu$ and $\bar{\phi}(\alpha) = \bar{\phi}'(\alpha) = 0$, property 1 of Lemma 5.1 (replacing c_0 and c'_0 by 0 and ψ' and ψ by $T_\alpha\psi'$ and $T_\alpha\psi$, respectively) shows that for every $s \in [0, \varepsilon]$,

$$\bar{\eta}'_\ell(\alpha+s)-\bar{\eta}'_\ell(\alpha) = T_\alpha\bar{\eta}'_\ell(s) \leq T_\alpha\bar{\eta}_\ell(s)+T_\alpha\nu(s) = \bar{\eta}_\ell(\alpha+s)-\bar{\eta}_\ell(\alpha)+\nu(\alpha+s)-\nu(\alpha).$$

When combined with (5.18) this yields the inequality

$$\bar{\eta}'_\ell(\alpha + s) \leq \bar{\eta}_\ell(\alpha + s) + \nu(\alpha + s) + [c_0 - c'_0]^+ \quad \text{for } s \in [0, \varepsilon],$$

which contradicts (5.16) and so Case (i) does not hold.

Thus we have shown that there does not exist any sequence $\{s_n\}$ with $s_n \downarrow 0$ that satisfies (5.16). Together with (5.11) and (5.13), this means that there must exist $\delta > 0$ such that

$$\bar{\eta}'_\ell(s) \leq \bar{\eta}_\ell(s) + \nu(s) + [c_0 - c'_0]^+ \quad \text{for } s \in [0, \alpha + \delta].$$

Combining (1.9) with the above inequality we then obtain for $t \in [0, \alpha + \delta]$,

$$\begin{aligned} \bar{\eta}'_u(t) &= \sup_{s\in[0,t]} [c'_0 + \psi'(s) + \bar{\eta}'_\ell(s) - a]^+ \\ &= \sup_{s\in[0,t]} [c'_0 + \psi(s) - \nu(s) + \bar{\eta}'_\ell(s) - a]^+ \\ &\leq \sup_{s\in[0,t]} [c_0 + \psi(s) + \bar{\eta}_\ell(s) - a + c'_0 - c_0 + [c_0 - c'_0]^+]^+ \\ &\leq \sup_{s\in[0,t]} [c_0 + \psi(s) + \bar{\eta}_\ell(s) - a]^+ + [c'_0 - c_0]^+. \\ &= \bar{\eta}_u(t) + [c'_0 - c_0]^+. \end{aligned}$$

However this contradicts (5.17) and so we conclude that neither Case (i) nor Case (ii) holds, which in turn contradicts the fact that $\alpha < \infty$. Thus $\alpha = \infty$ or, in other words, the second inequality in property 1 and the first equality in property 2 of the theorem are satisfied.

Applying the result just proved above with ψ, ψ', c_0, c'_0 replaced by $-\psi', -\psi, a - c'_0, a - c_0$ respectively, and invoking (1.10), it follows that $\beta = \infty$, where

$$\beta \doteq \inf \{t > 0 : \bar{\eta}'_u(t) + \nu(t) + [c_0 - c'_0]^+ < \bar{\eta}_u(t) \text{ or } \bar{\eta}'_\ell(t) + [c'_0 - c_0]^+ < \bar{\eta}_\ell(t)\} .$$

This completes the proof of the first two properties of the theorem. The third and fourth properties are the content of Remark 5.3. ■

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Appendix A: Transformation of graph parametrizations under Λ_a

Lemma A.1. *Let $\phi \in \mathcal{D}[0,T]$ be given. For $(r,g) \in \Pi(\phi)$, we have $(r, \Lambda_a(g)) \in \Pi(\Lambda_a(\phi))$.*

Proof. Since the mapping (r,g) is continuous, by Proposition 1.3 the map $(r, \Lambda_a(g))$ is also continuous. We will show that for every $s \in [0,1]$, $(r(s), \Lambda_a(g(s))) \in G_{\Lambda_a(\phi)}$. Fix $t \in [0,T]$. We consider two cases.

Case 1. $\phi(t) = \phi(t-)$.
Consider $s \in [0,1]$ such that $r(s) = t$. We want to show that eq1

$$\Lambda_a(g)(s) = \Lambda_a(\phi)(t), \tag{A.1}$$

which clearly implies $(r(s), \Lambda_a(g)(s)) \in G_{\Lambda_a(\phi)}$. In the case under consideration, u-phi

$$g(s) = \phi(t) \tag{A.2}$$

and (A.1) is equivalent to eq2

$$\sup_{s' \in [0,s]} \left[(g(s') - a)^+ \wedge \inf_{s'' \in [s',s]} g(s'') \right] = \sup_{t' \in [0,t]} \left[(\phi(t') - a)^+ \wedge \inf_{t'' \in [t',t]} \phi(t'') \right]. \tag{A.3}$$

The inequality eq2.5

$$\sup_{s' \in [0,s]} \left[(g(s') - a)^+ \wedge \inf_{s'' \in [s',s]} g(s'') \right] \geq \sup_{t' \in [0,t]} \left[(\phi(t') - a)^+ \wedge \inf_{t'' \in [t',t]} \phi(t'') \right] \tag{A.4}$$

follows from (A.2) and the monotonicity of (r, g), together with the fact that the graph of $(r(s'), g(s'))$, $s' \in [0, s]$, consists of the graph of $(t', \phi(t'))$, $t' \in [0, t]$, and the vertical segments $t' \times [\phi(t'-) \wedge \phi(t'), \phi(t'-) \vee \phi(t')]$, $t' \in [0, t]$. To prove the opposite inequality, let $s_0 \in [0, s]$ attain the supremum on the left-hand side of (A.3). Let $t_0 = r(s_0)$ and let $[b, c] = r^{-1}(t_0)$. We want to show that s_0 may be chosen to be either b or c (in other words, that the supremum is attained at one of the endpoints of $[b, c]$). This is obvious if $\phi(t_0) = \phi(t_0-)$, since then $g \equiv \phi(t_0)$ on $[b, c]$. If $\phi(t_0-) < \phi(t_0)$, then by the case assumption, $t_0 < t$ and $s_0 \leq c < s$. In this case, g increases on $[b, c]$ and the supremum on the left-hand side of (A.3) is attained at c. Thus, if $\phi(t_0-) \leq \phi(t_0)$, we have

$$\begin{aligned} \sup_{s' \in [0,s]} \left[(g(s') - a)^+ \wedge \inf_{s'' \in [s',s]} g(s'') \right] &= (g(c) - a)^+ \wedge \inf_{s'' \in [c,s]} g(s'') \\ &= (\phi(t_0) - a)^+ \wedge \inf_{t'' \in [t_0,t]} \phi(t'') \\ &\leq \sup_{t' \in [0,t]} \left[(\phi(t') - a)^+ \wedge \inf_{t'' \in [t',t]} \phi(t'') \right]. \end{aligned}$$

On the other hand, if $\phi(t_0-) > \phi(t_0)$, we again have $t_0 < t$ and $s_0 \leq c < s$, but now g decreases on $[b, c]$ and the supremum on the left-hand side of

(A.3) is attained at b. In this case

$$\begin{aligned}\sup_{s'\in[0,s]}\left[(g(s')-a)^+\wedge\inf_{s''\in[s',s]}g(s'')\right] &= (g(b)-a)^+\wedge\inf_{s''\in[b,s]}g(s'')\\ &= (\phi(t_0-)-a)^+\wedge\phi(t_0-)\wedge\inf_{t''\in[t_0,t]}\phi(t'')\\ &\le \sup_{t'\in[0,t]}\left[(\phi(t')-a)^+\wedge\inf_{t''\in[t',t]}\phi(t'')\right].\end{aligned}$$

Thus, regardless of the relationship between $\phi(t_0)$ and $\phi(t_0-)$, (A.3) holds.

Case 2. $\phi(t)\neq\phi(t-)$.

Let $[b,c]=r^{-1}(t)$, $\phi'=\phi-(\phi(t)-\phi(t-))\mathbb{I}_{[t,T]}$, $g'(s)=g(s)-\big(g(s\wedge c)-g(s\wedge b)\big)$. Then $g'=g$ on $[0,b]$, $\phi'=\phi$ on $[0,t)$ and $\phi'(t)=\phi(t-)$. This in turn shows that $\Lambda_a(g')(b)=\Lambda_a(g)(b)$, $\Lambda_a(\phi')(t)=\Lambda_a(\phi)(t-)$ and $(r,g')\in\Pi(\phi')$ on $[0,t]$. Since $\phi'(t)=\phi'(t-)$, we can apply (A.1) to conclude that $\Lambda_a(g)(b)=\Lambda_a(g')(b)=\Lambda_a(\phi')(t)=\Lambda_a(\phi)(t-)$. For $t'>t$ such that $\phi(t')=\phi(t'-)$ and $s'\in[0,1]$ such that $r(s')=t'$ we have, again by (A.1), $\Lambda_a(g)(s')=\Lambda_a(\phi)(t')$. Taking $t'\downarrow t$, we get $\Lambda_a(g)(c)=\Lambda_a(\phi)(t)$. Finally, $\Lambda_a(g)(s)$ moves continuously and monotonically from $\Lambda_a(g)(b)$ to $\Lambda_a(g)(c)$ as s increases over $[b,c]$. Hence, for $s\in[b,c]$, $(r(s),\Lambda_a(g)(s))=(t,\Lambda_a(g)(s))\in G_{\Lambda_a(\phi)}$.

Conclusion.

We have shown that the map $(r,\Lambda_a(g))$ takes values in $G_{\Lambda_a(\phi)}$. If $\Lambda_a(\phi)$ is discontinuous at $t\in(0,T]$, then ϕ is also discontinuous at t. The Case 2 analysis shows that when ϕ is discontinuous at t, the function $\Lambda_a(g)$ traverses the the vertical segment $t\times[\phi(t-)\wedge\phi(t),\phi(t-)\vee\phi(t)]$ in the direction from $\phi(t-)$ to $\phi(t)$, which means that (r,g) is nondecreasing in the order relation on the graph of $G_{\Lambda_a(\phi)}$ on the interval $r^{-1}(t)$. For values of t for which $\Gamma_a(\phi)$ is continuous, we use the fact r is nondecreasing to again conclude that (r,g) is nondecreasing. □