Local Time of Brownian Motion

- Definition
- Tanaka's formula
- Itô's formula for convex functions.

Idea and Definition.

W: Brownian motion (1 dim)

we are interested in the amount of time W spends near x.

Now, with

$$\mathcal{L}_\omega(x) \triangleq \text{leb}[t \geq 0 \mid W_t(\omega) = x]$$

we have $\mathcal{L}_\cdot(x)$ is $\mathcal{F}$ mbl and

$$E[\mathcal{L}_\omega(x)] = E\left[\int_0^\infty 1_{W_t = x}\, dt\right]$$

$$= \int_0^\infty P(W_t = x)\, dt$$

$$= 0$$

$\Rightarrow \mathcal{L}_\cdot(x) = 0$ a.s. $\forall\ x \in \mathbb{R}^d$

- 0 time spent at any particular point.

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But, to get a more useful measure of how long W spends near x, Lévy introduced the ~~quat~~ quantity

$$L_t(x)(\omega) = \lim_{\epsilon \downarrow 0} \frac{1}{2\epsilon} \, \mathrm{Leb}[0 \le s \le t \mid |W_s - x| \le \epsilon]$$

$t > 0$, $x \in \mathbb{R}$.

- $L_t(\omega)$ is the normalized percent of time W spends near x on $[0,t]$

- either $L_t(x)$ or $\frac{1}{2} L_t(x)$ is called the local time of W near x.

We use this definition and prove the following result, known as Tanaka's formula.

Thm (Tanaka)

$$L_t(a) = |W_t - a| - |z - a| - \int_0^t \mathrm{sign}(W_s - a)\, dW_s$$

a.s. on $(0,\infty)$ where the measure is P^z, i.e. the B.M. starts at z.

(3) We want to formally apply Itô's rule to the function $f(x) = |x-a|$

If we could

$$\hat{f}(x) = \begin{matrix} 1 & x > a \\ -1 & x < a \end{matrix} = \text{sign}(x-a)$$ if we ignore a and set $f'(a) = -1$.

$$\ddot{f}(x) = \delta_a = \text{dirac at } a.$$

Then, under P^z

well defined b/c $\text{sign}^2 = 1$

$$|W_t - a| = |z-a| + \int_0^t \text{sign}(W_s - a)\, dW_s$$

$$+ \frac{1}{2} \text{Leb}[0 \le s \le t \mid W_s = a]$$

Making the approximation

$$\frac{1}{2} \text{Leb}[0 \le s \le t \mid W_s = a]$$

$$\approx \frac{1}{2\epsilon} \text{Leb}[0 \le s \le t \mid |W_s - a| \le \epsilon]$$

small ϵ

we get.

(4)

$$|W_t - a| = |z - a| + \int_0^t \operatorname{sign}(W_s - a)\, dW_s + L_t(a)$$

- desired result.

Workaround

Set $g_\varepsilon(x) = \begin{cases} |x-a| & |x-a| > \varepsilon \\ \frac{1}{2}\left(\varepsilon + \frac{(x-a)^2}{\varepsilon}\right) & |x-a| \le \varepsilon. \end{cases}$

Then, $g_\varepsilon \in C^1(\mathbb{R})$ with

$$\dot{g}_\varepsilon(x) = \begin{cases} 1 & x > a + \varepsilon \\ -1 & x < a - \varepsilon \\ \frac{(x-a)}{\varepsilon} & a - \varepsilon \le x \le a + \varepsilon. \end{cases}$$

Also, $g_\varepsilon \in C^2(\mathbb{R} \setminus \{a\})$ with

$$\ddot{g}_\varepsilon(x) = \begin{cases} 0 & x > a+\varepsilon,\ x < a - \varepsilon \\ \frac{1}{\varepsilon} & a - \varepsilon < x < a + \varepsilon. \end{cases}$$

Now, using mollification, we have the following:
- Oksendal Exercise 4.8 b), Appendix D.

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If $g_\varepsilon \in C'$, $\ddot{g}_\varepsilon$ exists and is continuous for all but finitely many x and $|\ddot{g}_\varepsilon(x)| \le M$ $\forall x$ then we can use Itô:

$$g_\varepsilon(W_t) = g_\varepsilon(W_0) + \int_0^t \dot{g}_\varepsilon(W_s)\,dW_s + \frac{1}{2}\int_0^t \ddot{g}_\varepsilon(W_s)\,ds.$$

So, for our g_ε

$$g_\varepsilon(W_t) = g_\varepsilon(z) + \int_0^t \dot{g}_\varepsilon(W_s)\,dW_s$$

$$+ \frac{1}{2\varepsilon} \text{Leb}[0 \le s \le t \mid |W_s - a| < \varepsilon]$$

$$\text{Leb}[0 \le s \le t \mid |W_s - a| \le \varepsilon]$$

$$= g_\varepsilon(z) + \int_0^t \dot{g}_\varepsilon(W_s)\,dW_s$$

$$+ \frac{1}{2\varepsilon} \text{Leb}[0 \le s \le t, |W_s - a| \le \varepsilon].$$

Now, as $\varepsilon \to 0$

1) $g_\varepsilon(z) \cancel{\quad} = \begin{cases} |z-a| & \text{if } |z-a| > \varepsilon \\ \frac{1}{2}\left(\varepsilon + \frac{(z-a)^2}{\varepsilon}\right) & \text{if } |z-a| \le \varepsilon \end{cases} \to |z-a|$

⑥

Similarly, a.s. $g_\varepsilon(W_t) \to |W_t - a|$.

• note: $\int_0^t |\dot g_\varepsilon(W_s)|^2 \, ds \le t$

Now, for $t \le T$

$$\int_0^t |\dot g_\varepsilon(W_s) - \operatorname{sign}(W_s - a)|^2 \, ds$$

$$\le \int_0^T 1_{|W_s - a| \le \varepsilon} \left(\frac{W_s - a}{\varepsilon} - \operatorname{sign}(W_s - a) \right)^2 ds$$

$$\le 4 \int_0^T 1_{|W_s - a| \le \varepsilon} \, ds.$$

so

$$E\left[\int_0^t |\dot g_\varepsilon(W_s) - \operatorname{sign}(W_s - a)|^2 \, ds \right]$$

$$\le 4 \int_0^T P(|W_s - a| \le \varepsilon) \, ds \to 0 \text{ (easy to check).}$$

so $\int_0^{\cdot} \dot g_\varepsilon(W_s) \, dW_s \xrightarrow{L^2[\Omega \times [0,T]]} \int_0^{\cdot} \operatorname{sign}(W_s - a) \, dW_s$

and $\exists$ a sub-~~se~~ sequence s.t. the convergence holds a.s. $P \times \mathrm{Leb}[0,T]$.

Thus, along this subsequence, a.s. P^z

$$\lim_{a\to 0} \frac{1}{2a} \operatorname{Leb}[0 \le s \le t \mid |W_s - a| \le a]$$

exists and the limit $L_t(a)$ satisfies

$$L_t(a) = |W_t - a| - |z - a| - \int_0^t \operatorname{sign}(W_s - a)\, dW_s$$

∎

- note: we also have that $L_t(a)$ is $\mathcal{F}_t$ mbl.

Next, let $t, \tau > 0$, $a, b \in \mathbb{R}$. Under P^z, a.s.

$$\begin{aligned} |L_t(a) - L_\tau(b)| &\le \big| |W_t - a| - |W_\tau - b| \big| + \big| |z-a| - |z-b| \big| \\ &\quad + \left| \int_0^t \operatorname{sign}(W_s - a)\, dW_s - \int_0^\tau \operatorname{sign}(W_s - b)\, dW_s \right| \end{aligned}$$

Using this, one can show (Kolmogorov–Centsov, K+S pg 207–209) that $\exists$ a modification of $L_t(a)$ which is a.s. (P^z) jointly continuous in (t, a) and still satisfies

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Tanaka's equation (which we now use to <u>define</u> the local time). In fact we can define it in such a way that it is jointly continuous in (t, a) a.s. (single Ω^*) $\forall\ P^z$, $z \in \mathbb{R}$.

The local time can ~~be~~ be used to evaluate functionals of the Brownian Path, as the following shows.

<u>Thm.</u>

For all Borel mbl $f : \mathbb{R} \mapsto [0, \infty)$ we have a.s. P^z $\forall z$. and $t \geq 0$

$$\int_0^t f(W_s)\, ds = \int_{\mathbb{R}} f(x) L_t(x)\, dx.$$

<u>pf.</u>

we will consider $f(x) = \mathbb{1}_{(a,b)}(x)$ $a < b \in \mathbb{Q}$ as the rest will follow ~~b~~ via an approximation.

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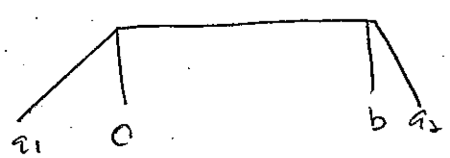
Now, set

$$h(x) = \begin{cases} 0 & x \le a_1 < 0 \\ \dfrac{x - a_1}{-a_1} & a_1 \le x \le 0 \\ 1 & 0 \le x \le b \\ \dfrac{a_2 - x}{a_2 - b} & b \le x \le a_2 \quad a_2 > b \\ 0 & x \geq a_2. \end{cases}$$



and

$$H(x) = \int_{\mathbb{R}} h(u)(x-u)^+ \, du = \int_{-\infty}^{x} \int_{-\infty}^{y} h(u) \, du \, dy$$

so $H \in C^2$ with

$$H'(x) = \int_{-\infty}^{\infty} h(u) 1_{(u,\infty)}(x) \, du = \int_{-\infty}^{x} h(u) \, du$$

$$H''(x) = h(x).$$

Thus, under P^z

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$$\frac{1}{2}\int_0^t h(W_s)\,ds = H(W_t) - H(z) - \int_0^t \dot{H}(W_s)\,dW_s.$$

$$= \int_{\mathbb{R}} h(a)(W_t - a)^+\,da - \int_{\mathbb{R}} h(a)(z-a)^+\,da$$

$$- \int_0^t \left(\int_{\mathbb{R}} h(a)\,1_{(a,\infty)}(W_s)\,da \right) dW_s.$$

$$= \frac{1}{2}\int_{\mathbb{R}} h(a)\,L_t(a)\,da + \int_{\mathbb{R}} h(a)\left(\int_0^t 1_{(a,\infty)}(W_s)\,dW_s\right) da$$

$$- \int_0^t \left(\int_{\mathbb{R}} h(a)\,1_{(a,\infty)}(W_s)\,da \right) dW_s.$$

- Here, we have used

$$\frac{1}{2}L_t(a) = (W_t - a)^+ - (z-a)^+ - \int_0^t 1_{(a,\infty)}(W_s)\,dW_s$$

which follows similarly to Tanaka's formula using $f(x) = (x-a)^+$ which is approximated by

$$g_\varepsilon(x) = \begin{cases} (x-a) & x \geq a+\varepsilon \\ \frac{1}{2}\left(\varepsilon + (x-a)^2/\varepsilon\right) & a \leq x \leq a+\varepsilon \\ \varepsilon/2 & x < a \end{cases}$$

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Now, for h continuous, compactly supported one can show (K+S pg 209)

$$\int_{\mathbb{R}} h(a)\left(\int_0^t 1_{(a,\infty)}(W_s)\,dW_s\right)da$$

$$= \int_0^t\left(\int_{\mathbb{R}} h(a)\,1_{(a,\infty)}(W_s)\,da\right)dW_s.$$

- consider simple process approximations for both stochastic integrals.

$\Rightarrow$

$$\frac{1}{2}\int_0^t h(W_s)\,ds = \frac{1}{2}\int_{\mathbb{R}} L_t(a)h(a)\,da$$

$$+ \int_{\mathbb{R}} h(a)\left(\cancelto{0}{\int_0^t \left(1_{(a,\infty)}(W_s) - 1_{(a,\infty)}(W_s)\right)dW_s}\right)da$$

$\blacksquare$.

So, we have shown

$$\int_0^t h(W_s)\,ds = \int_{\mathbb{R}} L_t(a)h(a)\,da \qquad h : \text{Borel mbl}, \geq 0$$

We now use this to relax the C^2 requirement in Itô's formula, replacing it with the assumption that the function is convex.

Convex functions.

We say $f : \mathbb{R} \mapsto \mathbb{R}$ is convex if

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda) f(y) \quad *.$$

$\forall \; x<y, \; 0 \leq \lambda \leq 1.$

Setting $z = \lambda x + (1-\lambda)y$ we obtain

$$f(z) \leq \frac{z-y}{x-y} f(x) + \frac{x-z}{x-y} f(y) \quad **$$

$$y < z < x.$$

Facts

1) f convex $\Rightarrow$ f continuous

- take $z \nearrow x$, $z \downarrow y$, $x \downarrow z$ and $y \nearrow z$ in $**$.

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2) f convex implies

$$h \mapsto \frac{f(x+h) - f(x)}{h} \qquad h \neq 0$$

is non-decreasing in h, bounded above by $f(x+1) - f(x)$ $(0<h<1)$ and below by $f(x-1) - f(x)$ $(-1<h<0)$. Thus

$$D^{\pm} f(x) = \lim_{h \to 0^{\pm}} \frac{1}{h}\big(f(x+h) - f(x)\big) \qquad (A)$$

exists and is finite. Furthermore,

$$D^{+} f(x) \leq D^{-} f(y) \leq D^{+} f(y) \qquad x < y \; (B)$$

Lastly, $D^{+} f$ is right-continuous and $D^{-} f$ is left continuous.

Pf
<u>Sketch</u> - use **

$0<h<1$: ~~[illegible]~~ $y = x,\ z = x+h,\ x = x+\varepsilon$ $\quad 0<h<\varepsilon \leq 1$

Non-Decr. $\quad \varepsilon = 1$ gives bound

$-1<h<0$: $\quad y = x,\ z = x-h,\ x = x-\varepsilon$, $\quad -1 \geq \varepsilon > h > 0$

$\varepsilon = -1$ gives bound.

- thus (A) holds

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for (B) take $y = x - \varepsilon,\ z = x,\ x = x + \delta$

to get $\dfrac{f(x) - f(x-\varepsilon)}{\varepsilon} \le \dfrac{f(x+\delta) - f(x)}{\delta}$

$\Rightarrow D^- f(x) \le D^+ f(x).$

Also, take $\varepsilon = x - y < 0$, $\varepsilon < h < 0$ for $x < y$

to get

~~$f(y-x) - f(y)$~~

$$\frac{f(y) - f(x)}{y - x} = \frac{f(y-\varepsilon) - f(y)}{-\varepsilon}$$

$$\le \frac{f(y-h) - f(y)}{-h} \to D^- f(y)$$

$$\Rightarrow \quad f(y) - f(x) \le (y-x) D^- f(y)$$

But for $\varepsilon = y - x > 0$ $\quad 0 < h < \varepsilon$

$$\frac{f(x+h) - f(x)}{h} \le \frac{f(x+\varepsilon) - f(x)}{\varepsilon}$$

$$\downarrow \qquad\qquad \|$$

$$D^+ f(x) \le \frac{f(y) - f(x)}{y - x}.$$

$$\Rightarrow \quad D^+ f(x)(y-x) \le (y-x) D^- f(y).$$

$$(\le (f(y) - f(x)) \le (y-x) D^- f(y)$$

Now, as $y \downarrow x$ we have

$$D^+f(x) \leq \lim_{y \downarrow x} D^+f(y).$$

But for $z > x$

$$\frac{f(z) - f(x)}{z - x} = \lim_{y \to x} \frac{f(y) - f(x)}{y - x} \geq D^+f(x)$$

so $z \downarrow x$ gives result.

With all this, we define the measure on $\mathbb{R}$ via

$$\mu((a,b)) = D^-f(b) - D^-f(a) \quad -\infty < a < b < \infty.$$

By integration by ~~parth~~ parts, if $g \in C^1$ and compactly supported then

$$\int_{\mathbb{R}} g(x)\mu(dx) = -\int_{\mathbb{R}} g'(x) D^-f(x)\, dx.$$

-note: we use $\mu([a,b))$ since D^- is left-continuous.

With all this notation, we have the following version of Itô's formula for convex functions:

Thm

If f is convex then $\forall\, z \in \mathbb{R}$, under P^z

$$f(W_t) = f(z) + \int_0^t D^- f(W_s)\,dW_s + \frac{1}{2}\int_{\mathbb{R}} L_t(x)\,\mu(dx).$$

pf

By localization we can assume $D^- f$ is uniformly bounded. With the ρ_n mollifier of before*, set

* slight change: $\rho_n(x) = n\rho(nx)$; $\rho(y) = \begin{cases} c e^{\frac{1}{(y-1)^2-1}} & 0<y<2 \\ 0 & \text{else.} \end{cases}$

$$f_n(x) = \int_{-\infty}^{\infty} \rho_n(x-y) f(y)\,dy = \int_{-\infty}^{\infty} \rho(z) f(x - z/n)\,dz$$

to obtain

1) $\dot{f}_n$ is uniformly bounded (in n) on compacts

2) $f_n(x) \to f(x)$

3) $\dot{f}_n(x) \to D^- f(x)$.

4) If $g \in C^1(\mathbb{R})$ with compact support

$$\int_{\mathbb{R}} g(x) f_n''(x)\,dx = -\int_{\mathbb{R}} g'(x) f_n'(x)\,dx$$

$$\xrightarrow[n\to\infty]{} -\int_{\mathbb{R}} g'(x) D^- f(x)\,dx$$

$$= \int_{\mathbb{R}} g(x)\mu(dx).$$

5) the same calculation in 4 holds for $g \in C(\mathbb{R})$ with compact support since g can be uniformly approximated by C^1 functions.

Thus, we have

$$\begin{array}{ccccccc}
f_n(W_t) & = & f_n(z) & + & \int_0^t f_n'(W_s)\,dW_s & + & \frac{1}{2}\int_0^t f_n''(W_s)\,ds \\
\Big\downarrow{\scriptstyle a.s.} & & \Big\downarrow & & \Big\downarrow{\scriptstyle L^2} & & \\
f(W_t) & & f(\cancel{z}) & & \int_0^t D^- f(W_s)\,dW_s & &
\end{array}$$

Also

$$\int_0^t f_n''(W_s)\,ds = \int_{\mathbb{R}} f_n''(x) L_t(x)\,dx$$

$$\to \int_{\mathbb{R}} L_t(x)\,\mu(dx) \qquad \text{a.s.}$$

since for each ω, $x \mapsto L_t(x)$ is continuous in x and supported in $[\min_{s \le t} W_s, \max_{s \le t} W_s]$

$\therefore$ up to sub-sequences we have a.s. convergence so for each t

$$f(W_t) = f(z) + \int_0^t D^- f(X_s)\,dW_s + \frac{1}{2}\int_{\mathbb{R}} L_t(x)\,\mu(dx)$$

By a.s. continuity in t the same holds $\forall t$ simultaneously.