(1)

Weak Solutions of SDE's

- Definition
- Tanaka's Example
- Solutions By Girsanov's Thm.

Weak Solutions.

We again consider the SDE:

$$dX_t = b(t, X_t)\,dt + \sigma(t, X_t)\,dW_t$$

where:

$b : [0,\infty) \times \mathbb{R}^d \to \mathbb{R}^d$

$\sigma : [0,\infty) \times \mathbb{R}^d \mapsto \mathbb{R}^{d \times r}$

W: standard r-dim B.M.

Let $X_0 \in \mathbb{R}^d$ $\mathcal{F}_0$ mbl be the initial value:

Recall we said X is a strong solution if X satisfies the SDE and is adapted to $\mathbb{F}$ = augmentation of the filtration generated by X_0, W.

We proved existence and uniqueness of strong solutions assuming

1) b, σ : locally Lipschitz in x, uniformly in t.

2) X does not explode in that if
$$\tau_n = \inf[t \geq 0 \mid |X_t| \geq n] \quad \text{then}$$
$$\tau_n \to \infty \quad \text{a.s.}$$

The essential feature of a strong solution is that X is $\mathbb{F}$ adapted

- X is the <u>output</u> which is mbl wrt the <u>inputs</u> X_0, W.

We now weaken this requirement that X be $\mathbb{F}^{(W, X_0)}$ adapted.

Definition (Weak Solution).

A weak solution of the SDE
$$dX_t = b(t, X_t)\,dt + \sigma(X_t, t)\,dW_t, \quad X_0 = x_0$$

(3)

is a triple $(\Omega, \mathcal{F}, P)$, $\mathbb{F}$, (X, W) where

1) $(\Omega, \mathcal{F}, P)$: prob space

2) $\mathbb{F} = \{\mathcal{F}_t\}$, $\mathcal{F}_t \subseteq \mathcal{F}$ is a filtration satisfying the usual conditions

3) X is continuous and $\mathbb{F}$ adapted

4) W is a n-dim B.M. wrt. $\mathbb{F}$.

5) $X_t = X_0 + \int_0^t b(u, X_u)\,du + \int_0^t \sigma(u, X_u)\,dW_u$ a.s. $\forall\ t \geq 0$ and integrals make sense.

KEY Difference: $\mathbb{F}$ does <u>not</u> have to be the augmentation of the filtration generated by X_0, W.

-note: i) by def $W_t \perp\!\!\!\perp \mathcal{F}_0$ so $X_0 \perp\!\!\!\perp W$.

ii) X_t is $\mathcal{F}_t$ mbl, $W_0 - W_t \perp\!\!\!\perp \mathcal{F}_t$

④

so $X_t \perp\!\!\!\perp W_\theta - W_t$ $\quad \theta > t$

(X does not depend upon future of B.M.).

<u>Now</u>

Strong Solution $\Rightarrow$ Weak Solution b/c we can take $\mathbb{F} = \mathbb{F}^{(X_0, W)}$.

<u>But</u>

Weak Solution $\not\Rightarrow$ Strong Solution.

<u>(famous) Example.</u>

$$dX_t = \text{sign}(X_t)\, dW_t \quad ; \quad X_0 = 0$$

$$\left(\text{sign}(x) = \begin{cases} 1 & x > 0 \\ -1 & x \leq 0 \end{cases} \right)$$

building a weak solution...

X : B.M. with respect to some $(\Omega, \mathcal{F}, P)$, $\mathbb{F} = \mathbb{F}^{X}$ (augmentation).

5)

Now, define

$$W_t = \int_0^t \operatorname{sign}(X_s)\,dX_s \in \mathcal{M}^{c,loc}$$

Since

$$\langle W\rangle_t = \int_0^t \operatorname{sign}^2(X_s)\,ds = t$$

we have that

W is a B.M. w.r.t. $\mathbb{F}^X$.

thus $\mathbb{F}^W \subseteq \mathbb{F}^X$. (augmented for W)

But

$$W_t = \int_0^t \operatorname{sign}(X_s)\,dX_s$$

$$\Rightarrow X_t = \int_0^t \operatorname{sign}(X_s)\,dW_s.$$

pf.

1) $X_t - \int_0^t \operatorname{sign}(X_s)\,dW_s \in \mathcal{M}^{c,loc}(\mathbb{F}^X)$

2) $\langle X - \int_0^{\cdot} \operatorname{sign}(X_u)\,dW_u\rangle_t$

$$= \langle X - \int_0^{\cdot} \cancel{\text{sing}}\ \operatorname{sign}^2(X_u)\,dX_u\rangle_t$$

$$= \langle \int_0^{\cdot} (1-\operatorname{sign}^2(X_u))\,dX_u\rangle_t$$

$$= 0$$

Thus, $(\Omega, \mathcal{F}, P)$, $\mathbb{F} = \mathbb{F}^X$, (X, W) is a weak solution of

$$dX_t = \operatorname{sign}(X_t)\, dW_t.$$

Now, we saw that $\mathbb{F}^W \subseteq \mathbb{F}^X$. In order for X to be a strong solution we need X to be $\mathbb{F}^W$ adapted, or

$$\mathbb{F}^X \subseteq \mathbb{F}^W.$$

However, we have the following claim:

$$W_t = \int_0^t \operatorname{sign}(X_s)\, dX_s$$

is $\mathbb{F}^{|W|}$ adapted: i.e. W_t is $\sigma(|W|_u, u \leq t)$ mbl $\forall$ $t \geq 0$.

So, <u>if</u> we had a strong solution then $\mathbb{F}^X \subseteq \mathbb{F}^W \subseteq \mathbb{F}^{|X|}$, but this cannot happen!

$$\{X_t \leq -1\} \neq \{|X_{t_1}| \in A_1, \ldots, |X_{t_k}| \in A_k\}$$

⑦

To rigorously show $\int_0^t \mathrm{sign}(x_0)\,dx_0$ is IF^w adopted we approximate

$$f(x) = |x|$$

by smooth φ_n s.t.

1) $\varphi_n(x) \mapsto |x|$

2) $|\dot{\varphi}_n(x)| \le 1$, $\dot{\varphi}_n(x) \to \begin{cases} 1 & x>0 \\ -1 & x<0 \end{cases}$

3) $\ddot{\varphi}_n(x) \to 0$ $x \neq 0$ and

$\ddot{\varphi}_n$ is even.

E.g.

$$p(x) = \begin{cases} c\, e^{-\frac{1}{1-x^2}} & |x|<1 \\ 0 & |x| \geq 1 \end{cases} \quad c \text{ s.t. } \int_{\mathbb{R}} p(x)\,dx = 1$$

$p_n(x) = n\,p(nx)$ $\Rightarrow$ 1) $p_n(x) = 0$ if $|x| \geq 1/n$

$\int_{\mathbb{R}} p_n(x)\,dx = 1$ 2) p_n is even.

$$\varphi_n(x) = 2\int_{-\infty}^{x}\int_{-\infty}^{y} p_n(z)\,dz\,dy - x$$

so $\dot{\varphi}_n(x) = 2\int_{-\infty}^{x} p_n(z)\,dz - 1 \to \begin{cases} 1 & x>0 \\ -1 & x<0. \end{cases}$

8

Then

$$\varphi_n(X_t) = \varphi_n(0) + \int_0^t \varphi_n'(X_u)\,dX_u + \frac{1}{2}\int_0^t \varphi_n''(X_u)\,du$$

But (up to subsequences) a.s.

$$\varphi_n(X_t) \to |X_t|$$

$$\int_0^t \varphi_n'(X_u)\,dX_u \to \int_0^t (1_{X_u>0} - 1_{X_u<0})\,dX_u$$

$$= \int_0^t \operatorname{sign}(X_u)\,dX_u \quad \text{b/c } \int_0^t 1_{X_u=0}\,dX_u \equiv 0$$

$$= W_t$$

$$\frac{1}{2}\int_0^t \varphi_n''(X_u)\,du = \frac{1}{2}\int_0^t \varphi_n''(|X_u|)\,du \quad \text{(even)}$$

now, using local time (cover later) we can show $\frac{1}{2}\int_0^t \varphi_n''(X_u)\,du$ has an a.s. limit, but clearly, whatever it is, it is $\mathbb{F}^{|W|}$ mbl.

(9)

Thus

$$|X_t| = W_t + \lim_n (Y_t^n)$$

$$Y_t^n = \frac{1}{2}\int_0^t \varphi_n(|X_u|)\,du$$

where Y^n is $\mathcal{F}_t^{|W|}$ mbl.

Take-away: $\exists$ weak solutions which are not strong solutions.

Also, If we have:

$$dX_t = b(t,X_t)\,dt + \sigma(t,X_t)\,dW_t$$

then "typically"* we can write

$$W_t = \int_0^t \sigma(u,X_u)^{-1}(dX_u - b(u,X_u)\,du)$$

* at least if $\sigma(u,x)$ is square, invertible $\forall$ (t,x)

so we see that $\mathbb{F}^W \subseteq \mathbb{F}^X$. Thus, the difficult direction is showing $\mathbb{F}^X \subseteq \mathbb{F}^W$

when obtaining strong solutions.

Weak Solutions and Uniqueness.

Q. What do we mean by a "unique" weak solution.

Form 1: Pathwise Uniqueness.

Def.

Let $(\Omega, \mathcal{F}, P)$, $\mathbb{F}$, (X, W)

$(\Omega, \mathcal{F}, P)$, $\tilde{\mathbb{F}}$, $(\tilde{X}, W)$

Be two weak solutions w/ common B.M. W. (possibly different filtrations).
and s.t. $P(X_0 = \tilde{X}_0) = 1$ (i.e. a.s. equal starting points. If

$$P(X_t = \tilde{X}_t \quad \forall\, t \geq 0) = 1$$

we say that pathwise uniqueness holds for the SDE.

11

e.g. $dX_t = \text{sign}(X_t)\, dW_t \; ; X_0 = 0$

note: if (X, W), $\mathbb{F}$, $(\Omega, \mathcal{B}, P)$ is a weak solution then so is $(-X, W)$, $\mathbb{F}$, $(\Omega, \mathcal{B}, P)$ b/c

$$-\text{sign}(X_t) = \text{sign}(-X_t)$$

except at 0 and $\int_0^t 1_{X_s = 0}\, du$

$= 0$ since we already know weak solutions are B.M.

$\to$ pathwise uniqueness can fail.

Form 2: Weakness of Law.

Def.

Let $(\Omega, \mathcal{B}, P)$, $\mathbb{F}$, (X, W)

$(\tilde{\Omega}, \tilde{\mathcal{B}}, \tilde{P})$, $\tilde{\mathbb{F}}$, $(\tilde{X}, \tilde{W})$

be two weak solutions s.t.

$$P(X_0 \in \Gamma) = \tilde{P}(\tilde{X}_0 \in \Gamma) \quad \Gamma \in \mathcal{B}(\mathbb{R}^d)$$

(12)

If $X, \tilde{X}$ have the same law then we say uniqueness in the sense of law holds.

e.g. $dX_t = \text{sign}(X_t)\, dW_t \,;\; X_0 = 0$

$\Rightarrow X$ is a B.M. so uniqueness in the sense of law holds.

so

Pathwise Uniqueness $\Rightarrow$ Uniqueness of Law

but not the other way around.

Construction Weak Solutions Using Girsanov.

-easy!

Prop.

Assume we have a weak solution to $dX_t = \sigma(t, X_t)\, dW_t \;;\; X_0 \sim \mu$

where.

(13)

- $\sigma(t,x)$ is a $d\times d$ matrix s.t. $\sigma(t,x)^{-1}$ exists $\forall\ (t,x)$ and if

$$A(t,x) = \sigma(t,x)\,\sigma(t,x)^T$$

then A is uniformly elliptic in that

$$\theta^T A(t,x)\,\theta \geq c\,\theta'\theta \quad \text{for}$$

$c>0$ and <u>all</u> $t\geq 0,\ x\in\mathbb{R}^d$.

Then, for all $b(t,x)$ bounded, mbl $\exists$ a weak solution to

$$dX_t = b(t,X_t)\,dt + \sigma(t,X_t)\,dW_t$$

$$X_0 \sim \mu$$

on $0\leq t\leq T \quad \forall\ T\geq 0$

<u>pf.</u> Let $(\Omega,\mathcal{B},P)$, $\mathbb{F}$, (X,W) be a weak solution to

$$dX_t = \sigma(t,X_t)\,dW_t \ ; \ X_0 \sim \mu.$$

(14)

Define

$$Z_t = \mathcal{E}\left(\int_0^{\cdot} Y_u^T\, dW_u\right)_t \qquad t \le T$$

where

$$Y_t = \sigma^{-1}(t, X_t)\, b(t, X_t)$$

Note:

$$\int_0^t \|Y_u\|^2\, du = \int_0^t b(u, X_u)' A^{-1}(u, X_u)\, b(u, X_u)\, du$$

$$\le \frac{1}{c} \int_0^t \|b(u, X_u)\|^2\, du \qquad \text{(u. ellipticity)}$$

$$\le \frac{K}{c}\, t \qquad \text{(b bdd).}$$

Thus, for $t \le T$, Z_t is a Martingale and we can define

$$Q(A) = E[1_A Z_T] \qquad A \in \mathcal{F}_T.$$

$$\tilde{W}_t = W_t - \int_0^t Y_u\, du \qquad t \le T.$$

so $\tilde{W}$ is a Q · B.M.

(15)

Then

$$
\begin{aligned}
X_t &= X_0 + \int_0^t \sigma(u, X_u)\, dW_u \\
&= X_0 + \int_0^t \sigma(u, X_u)\left(d\tilde{W}_u + \sigma^{-1}(u, X_u)\, b(u, X_u)\, du\right) \\
&= X_0 + \int_0^t b(u, X_u)\, du + \int_0^t \sigma(u, X_u)\, d\tilde{W}_u
\end{aligned}
$$

Also

$$
\begin{aligned}
Q(X_0 \in \Gamma) &= E[1_{X_0 \in \Gamma} Z_0] \\
&= P(X_0 \in \Gamma)
\end{aligned}
$$

since $Z_0 = 1$.

Thus, $(\Omega, \mathcal{F}, Q)$, $\mathbb{F}$, $(X, \tilde{W})$ is a weak solution w/ initial dist μ.

- much weaker conditions (esp. if $\sigma = 1_d$) for solutions than in the strong case.