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Strong Solutions to SDE's

- An "explosion" result
- A counter-example
- A comparison principle

An Explosion Result.

In our existence theorem, we assumed

1) b, σ were globally Lipschitz.

2) $X_0 = \xi$ was square integrable.

We now focus on when b, σ are locally Lipschitz and $\xi = x \in \mathbb{R}^d$, providing a useful characterization of strong existence via non-explosion of the solution.

$$dX_t = b(t, X_t)\,dt + \sigma(t, X_t)\,dW_t.$$

W: r-dim B.M. $\quad b, \sigma : [0,\infty) \times \mathbb{R}^d$ to $\mathbb{R}^d, \mathbb{R}^{d \times r}$ resp.

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Assume b, σ <u>locally</u> Lipschitz in x, uniformly in t:

$$\|b(t,x)-b(t,y)\| + \|\sigma(t,x)-\sigma(t,y)\|$$

$$\leq K_n \|x-y\| \qquad \|x\|, \|y\| \leq n.$$

Now, let φ^n be a C^∞ function s.t.

1) $\varphi^n(x) = 1 \qquad \|x\| \leq n$

2) $\varphi^n(x) = 0 \qquad \|x\| \geq n+1.$

Set

$$\sigma^n(t,x) = \varphi^n(x)\sigma(t,x) + (1-\varphi^n(x)) \mathbb{1}_{d\times r}$$

$$b^n(t,x) = \varphi^n(x) b(t,x)$$

$$\mathbb{1}_{d\times r} = \begin{pmatrix} 1 & 1 & 1 & & \\ 1 & 1 & \cdot & 1 \cdots & 1 \\ 1 & 1 & \cdots & & 1 \end{pmatrix}_{d\times r}.$$

Clearly, σ^n, b^n are globally Lipschitz and

$\sigma^n = \sigma,\ b^n = b \qquad$ on $\|x\| \leq n.$

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Let X^n denote the unique strong solution with σ^n, b^n starting at x.

<u>Claim.</u>

Let $n < m$. Let $K \leq n$ and set

$$\tau_K = \inf[t \mid \|X^n_t\| \geq K] \wedge \inf[t \mid \|X^m_t\| \geq K]$$

where K is large enough so $\|x\| \leq K$.

Then $X^n = X^m$ on $[0, \tau_K]$, i.e.

$$P(X^n_{t\wedge\tau_K} = X^m_{t\wedge\tau_K} \quad \forall\, t \geq 0) = 1.$$

<u>pf</u> - essentially the same as for strong uniqueness.

$$E[\|X^n_{t\wedge\tau_K} - X^m_{t\wedge\tau_K}\|^2]$$

$$= E\Big[\Big\| \int_0^{t\wedge\tau_K} (b^n(u, X^n_u) - b^m(u, X^m_u))\,du$$

$$+ \int_0^{t\wedge\tau_K} (\sigma^n(u, X^n_u) - \sigma^m(u, X^m_u))\,dW_u \Big\|^2\Big]$$

④

But, $\sigma^n = \sigma^m = \sigma$, $b^n = b^m = b$ on $\|y\| \le k$ so, repeating the uniqueness proof.

$$E[\|X^n_{t\wedge\tau_k} - X^m_{t\wedge\tau_k}\|^2]$$

$$\le K_k^2(1+t)\, E\left[\int_0^t \|X^n_{u\wedge\tau_k} - X^m_{u\wedge\tau_k}\|^2\right]$$

so, for $t \le T$ if $g(t) = E[\|X^n_{t\wedge\tau_k} - X^m_{t\wedge\tau_k}\|^2]$

$$0 \le g(t) \le K_k^2(1+T)\int_0^t g(u)\,du$$

$$\Rightarrow \quad g(t) = 0$$

$$\Rightarrow \quad P(X^n_{t\wedge\tau_k} = X^m_{t\wedge\tau_k}) = 1 \quad \forall\, t \ge 0$$

∴ result follows by continuity. ∎

Denote by X the process taking the common value of $\{X^n\}_{n \geq k}$ on $[0, \tau_k]$

i.e.: $X_t = X_t^k$ on $t \leq \inf\{u \mid \|X_u^k\| \geq k\}$
$= \inf\{u \mid \|X_u\| \geq k\}$

then

$$X_{t \wedge \tau_k} = X + \int_0^{t \wedge \tau_k} b(X_u)\,du + \int_0^{t \wedge \tau_k} \sigma(X_u)\,dW_u.$$

So, we have the following result:

<u>Thm.</u>

Define the "explosion" time of X to be

$$\tau = \lim_{k \to \infty} \tau_k = \lim_{k \to \infty} \inf\{t \mid \|X_t\| \geq k\}$$

If $\tau_\infty = \infty$ a.s. then X is the unique strong solution of the SDE.

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pf.

$X_{t\wedge\tau_k}$ is $\mathcal{F}_{t\wedge\tau_k}$ mbl by adaptivity so by left continuity of $\mathcal{F}$ (since it is augmented version of $\mathbb{F}^W$) and $\tau_k \to \infty$ we have the result that X is $\mathbb{F}$ adapted.

Since $\tau_k \to \infty$ X satisfies the SDE and all the integrals make sense.

- very useful theorem b/c it replaces

i) globally Lipschitz

with

ii) locally Lipschitz + no explosion.

- often one can use other methods to deduce lack of explosions.

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Cantor - Example:

Consider

$$\star \quad dX_t = 3X_t^{1/3}\,dt + 3X_t^{2/3}\,dW_t.$$

$b(x) = 3x^{1/3}$, $\sigma(x) = 3x^{2/3}$

- neither function is globally Lipschitz or even locally Lipschitz.

Claim.

$\forall\ \theta \geq 0$, the process

$$X_t^{(\theta)} = \begin{cases} 0 & 0 \leq t \leq B_\theta = \inf\{s \geq \theta \mid W_s = 0\} \\ W_t^3 & B_\theta \leq t \end{cases}$$

is a strong solution of $\star$ starting at 0.

- strong existence does not imply strong uniqueness.

pf

for $0 \le t < B_\Theta$

$$X_t = 0 = \int_0^t 0\,ds + \int_0^t 0\,dW_s$$

$$= 3\int_0^t X_s^{1/3}\,ds + 3\int_0^t X_s^{2/3}\,dW_s.$$

for $t \geq B_\Theta$

$$X_t = X_t - X_{B_\Theta}$$

$$= W_t^3 - W_{B_\Theta}^3$$

$$= \int_{B_\Theta}^t 3W_s\,ds + 3\int_{B_\Theta}^t W_s^2\,dW_s$$

$$= \int_0^t 3X_s^{1/3}\,ds + 3\int_0^t X_s^{2/3}\,dW_s. \checkmark$$

so to ~~get~~ go from strong existance to strong uniqueness no need additional conditions.

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Comparison Results.

Consider the one-dimensional SDE.

$$dX_t = b(t, X_t)\,dt + \sigma(t, X_t)\,dW_t.$$

- think of b as the drift of X and σ as the noise

- since the noise heuristically pushes X in arbitrary directions, we are led to conjecture that ~~as long as~~ if the drift increases then so should the process. I.e.

$$b^1(t,x) \le b^2(t,x)$$

$$\Rightarrow X_t^1 \le X_t^2.$$

- true for ODE's and the noise should cancel out if it is common to both.

This in fact true

Thm.

Consider

$$dX_t^1 = b^1(t, X_t^1)\,dt + \sigma(t, X_t^1)\,dW_t$$

$$dX_t^2 = b^2(t, X_t^2)\,dt + \sigma(t, X_t^2)\,dW_t$$

for the same B.M. Assume

1) $b^{1,2}, \sigma$ a globally Lipschitz

- Lipschitz condition on σ can be weakened.

2) $X_0^1 = x_0^1 \le x_0^2 = X_0^2$

- can also handle X_0^1, X_0^2 r.v. if appropriate integrability holds.

3) $b^1(t,x) \le b^2(t,x)$ $\quad t \geq 0, x \in \mathbb{R}$.

Then

$$P(X_t^1 \le X_t^2 \ \forall\ t \geq 0) = 1.$$

(11)

pf

Easy - But relies on a technical fact from analysis:

$\exists$ a sequence of functions $\{\varphi_n\}$ s.t.

1) $\varphi_n \in C^2(\mathbb{R}) \ \forall n, \ \varphi_n \geq 0$

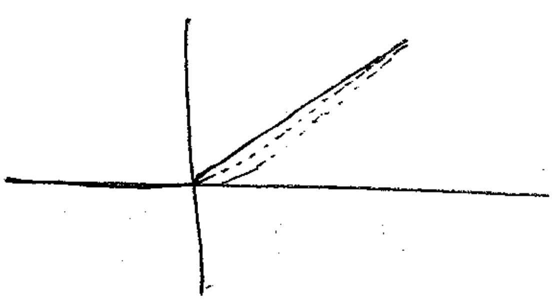
2) $\lim_{n \to \infty} \varphi_n(x) = x \, 1_{x>0}$

3) $\varphi_n(x) = 0$ for $x \leq 0 \quad \forall n$

4) $0 \leq \varphi_n'(x) \leq 1 \quad \forall n$

5) $0 \leq \varphi_n''(x) \leq \dfrac{K}{n x^2} \quad x \in \mathbb{R} \setminus \{0\}, \ \forall n.$

semi-explicit construction in pp 291-294 of the book.



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For this sequence, and assuming by localization that

$$E\left[\int_0^T \sigma(u,X_u^i)^2\,du\right] < \infty \qquad T \geq 0,\ i=1,2.$$

we have from Itô that

$$\varphi_n(X_t^1 - X_t^2)$$

$$= \int_0^t \varphi_n'(X_u^1 - X_u^2)\big(b^1(u,X_u^1) - b^2(u,X_u^2) \pm b^1(u,X_u^2)\big)\,du$$

$$+ \frac{1}{2}\int_0^t \varphi_n''(X_u^1 - X_u^2)\big(\sigma(u,X_u^1) - \sigma(u,X_u^2)\big)^2\,du$$

$$+ \int_0^t \varphi_n'(X_u^1 - X_u^2)\big(\sigma(u,X_u^1) - \sigma(u,X_u^2)\big)\,dW_u.$$

<u>Now.</u>

The local martingale is a martingale since

$$|\varphi_n'| \leq 1,\ E\left[\int_0^T \sigma^2(u,X_u^i)\,du\right] < \infty$$

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By the assumption on φ_n'', σ Lipschitz.

$$\left| \int_0^t \varphi_n''(X_u^1 - X_u^2)(\sigma(u, X_u^1) - \sigma(u, X_u^2))^2 \, du \right|$$

$$\leq \int_0^t \frac{K}{n|X_u^1 - X_u^2|^2} 1_{|X_u^1 - X_u^2| \neq 0} \tilde{K}^2 |X_u^1 - X_u^2|^2 \, du$$

$$= \frac{\tilde{K} t}{n}$$

By the assumption on φ_n', b^1, b^2:

$$\int_0^t \varphi_n'(X_u^1 - X_u^2)(b^1(u, X_u^1) - b^1(u, X_u^2) + b^1(u, X_u^2) - b^2(u, X_u^2)) \, du$$

$$\leq \int_0^t 1_{X_u^1 > X_u^2} \hat{K} |X_u^1 - X_u^2| \, du$$

$$= \hat{K} \int_0^t (X_u^1 - X_u^2)^+ \, du.$$

Thus

$$E\left[\varphi_n(X^1_t - X^2_t)\right]$$

$$\leq \frac{\tilde{K}t}{n} + \tilde{K}\int_0^t E[(X^1_u - X^2_u)^+]\,du$$

taking $n \nearrow \infty$ and using that

$$\varphi_n(x) = x^+$$

and fatou

$$E\left[(X^1_t - X^2_t)^+\right] \leq \tilde{K}\int_0^t E[(X^1_u - X^2_u)^+]\,du$$

$\Rightarrow$ By Gronwall

$$P(X^1_t > X^2_t) = 0 \qquad \forall\ t \geq 0$$

Result follows by continuity.