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Stochastic Differential Equations

- strong solutions.

<u>Idea</u>

Assign meaning to the extension of ODE's to the case where there is a driving Brownian motion:

$$dX_t = b(t, X_t)\,dt + \sigma(t, X_t)\,dW_t.$$

W: r-dim. B.M. starting at 0.

b: Borel mbl function of $[0,\infty)\times\mathbb{R}^d$ to $\mathbb{R}^d$

σ: " " " " $[0,\infty)\times\mathbb{R}^d$ to

$M^{d\times r}$: the space of $d\times r$ dim matrices.

b: "drift" of the "diffusion" X.

σ: "dispersion" matrix of X.

In integrated form, we wish to find a process X s.t.

$$X_t = X_0 + \int_0^t b(u, X_u)\,du + \int_0^t \sigma(u, X_u)\,dW_u.$$

or, component wise.

$$X_t^i = X_0^i + \int_0^t b^i(u, X_u)\,du + \sum_{j=1}^{n} \int_0^t \sigma^{ij}(u, X_u)\,dW_u^j.$$

- here, X_0 is an $\mathcal{F}_0$ mbl r.v. which can also be taken as an input to the problem.

Sometimes we can find solutions to SDE's explicitly.

e.g. $dX_t = -\gamma X_t\,dt + dW_t$. ; $X_0 = x$

$d = n = 1$, $\gamma > 0$

- Ornstein-Uhlenbeck process, which is important in Math. Finance.

- example of a <u>linear</u> sde since $b(t,x) = -\gamma x$, $\sigma(t,x) = 1$.

(3) To find X, first assume we have a solution X and compute

$$d(e^{\gamma t} X_t) = \gamma e^{\gamma t} X_t \, dt + e^{\gamma t} dX_t$$

$$= e^{\gamma t} [\gamma X_t \, dt - \gamma X_t \, dt + dW_t]$$

$$= e^{\gamma t} dW_t.$$

$$\Rightarrow \quad e^{\gamma t} X_t = x + \int_0^t e^{\gamma u} dW_u$$

or

$$X_t = e^{-\gamma t} x + e^{-\gamma t} \int_0^t e^{\gamma u} dW_u.$$

- then, go back-wards to show that this X solves the SDE.

- in the general case we will <u>not</u> be this lucky, that we have an explicit solution, so we need a general theory.

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In fact, we first have to define what we mean by a solution, as different notions of a solution are possible.

Strong Solution.

- we seek to solve

$$X_t = \xi + \int_0^t b(u, X_u)\,du + \int_0^t \sigma(u, X_u)\,dW_u.$$

$(\xi = X_0)$

- one way to view this is that our <u>inputs</u> are ξ and W and, through b, σ, our <u>output</u> is X.

- strong solutions of the SDE are those for which the output X at t only depends ~~o~~ upon the inputs ξ, $\{W_s\}_{s \le t}$.

⑤ To make this precise, let $(\Omega, \mathcal{F}, P)$ be fixed and let W be a r-dim B.M. with its own filtration $\mathbb{F}^W$.

Let $\xi \in \mathbb{R}^d$ be a r.v. $\perp\!\!\!\perp$ of $\mathcal{F}^W_\infty$

Set $\mathbb{G}$ via

$$\mathcal{A}_t = \sigma(\xi) \vee \mathcal{F}^W_t \qquad t \geq 0$$

and N via

$$\mathcal{N} = \{ N \subseteq \Omega \mid \exists\ G \in \mathcal{A}_\infty \text{ s.t. } N \subseteq G, \ P(G) = 0 \}.$$

and $\mathbb{F}$ via

$$\mathcal{F}_t = \sigma(\mathcal{A}_t \cup \mathcal{N}) \qquad \left(\mathcal{F}_\infty = \sigma\left(\bigcup_{t \geq 0} \mathcal{F}_t \right) \right)$$

- augmented filtration of that generated by ξ, W.
- satisfies usual conditions, and W is a r-dim B.M. for $\mathbb{F}$, $\xi \in \mathcal{F}_0$ mbl and $W \perp\!\!\!\perp \xi$.

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<u>Definition</u> (Strong Solution).

X is a Strong Solution of

$$* \qquad X_t = \xi + \int_0^t b(u, X_u)\,du + \int_0^t \sigma(u, X_u)\,dW_u$$

if

1) X is $\mathbb{F}$ adapted

2) $P(X_0 = \xi) = 1$

3) $P\left[\int_0^t \sum_{i=1}^d |b^i(u, X_u)|\,du + \int_0^t \sum_{\substack{i=1,\dots,d \\ j=1,\dots,r}} |\sigma^{ij}(u, X_u)|^2\,du < \infty\right]$

$= 1 \quad \forall\ t \geq 0$

4) $*$ holds a.s. $\forall\ t \geq 0$.

- Key notion above is 1), that X is $\mathbb{F}$ adapted.
  - X is a "function" of ξ, W so to speak.

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We now study existence and uniqueness of solutions to $*$.

- first, we define uniqueness.

Definition (Strong Uniqueness).

For any $(\Omega, \mathcal{F}, P)$, W, ξ $W \perp\!\!\!\perp \xi$ and augmented $\mathbb{F}$ and solutions $X, \tilde{X}$, if $P(X_t = \tilde{X}_t \ \forall t \geq 0) = 1$ then we say that strong uniqueness for (b, σ) holds.

Theorem.

Suppose b, σ are locally lipschitz in x, uniformly in t: i.e. $\forall$ integer n $\exists$ K_n s.t.

$$\sup_{t \geq 0} \ \sup_{x,y \,|\, \|x\|, \|y\| \leq n} \|b(t,x) - b(t,y)\| + \|\sigma(t,x) - \sigma(t,y)\| \leq K_n \|x-y\|$$

then strong uniqueness holds.

notes: 1) $\|\sigma(t,x)\|^2 = \sum_{\substack{i=1,\dots,d \\ j=1,\dots,n}} \sigma_{ij}(t,x)^2$

2) consider from ODE's the equation

$$X_t = \int_0^t |X_s|^\alpha \, ds$$

- for $\alpha \geq 1$ only solution is $X_t \equiv 0$.

for $0 < \alpha < 1$, $|x|^\alpha$ is not locally Lipschitz, and in fact

$$X_t \equiv 0$$

$$X_t = \frac{t^\beta}{\beta} \qquad \beta = \frac{1}{1-\alpha}$$

are two solutions

- even for ODE's we need some form of local Lipshitz behavior in the coefficients.

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pf

Recall Gronwall's Inequality:

If $0 \le g(t) \le \alpha(t) + B\int_0^t g(s)\,ds \quad t \le T$

Then $g(t) \le \alpha(t) + B\int_0^t \alpha(s) e^{B(t-s)}\,ds \quad t \le T.$

pf

$$\frac{d}{dt}\left(e^{-Bt}\int_0^t g(s)\,ds\right) = e^{-Bt}\left(g(t) - B\int_0^t g(s)\,ds\right)$$

$$\le e^{-Bt}\alpha(t)$$

$$\Rightarrow \int_0^t g(s)\,ds \le e^{Bt}\int_0^t \alpha(s)e^{-Bs}\,ds.$$

Now, set $\tau_n = \inf[t \ge 0 \mid \|X_t\| \ge n]$

$$\hat{\tau}_n = \inf[t \ge 0 \mid \|\hat{X}_t\| \ge n]$$

$$S_n = \tau_n \wedge \hat{\tau}_n \rightarrow \infty.$$

We have

$$X_{t\wedge S_n} - \hat{X}_{t\wedge S_n} = \int_0^{t\wedge S_n}(b(u,X_u) - b(u,\hat{X}_u))\,du$$

$$+ \int_0^{t\wedge S_n}(\sigma(u,X_u) - \sigma(u,\hat{X}_u))\,dW_u$$

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Write $a_u = b(u, X_u) - b(u, \tilde{X}_u)$

$\gamma_u = \sigma(u, X_u) - \sigma(u, \tilde{X}_u)$

Note that for $u \le S_n$

$$\|a_u\|^2 \le K_n^2 \|X_u - \tilde{X}_u\|^2$$

$$\|\gamma_u\|^2 \le K_n^2 \|X_u - \tilde{X}_u\|^2$$

And

$$E\left[\|X_{t \wedge S_n} - \tilde{X}_{t \wedge S_n}\|^2\right] = E\left[\left\| \int_0^{t \wedge S_n} a_u \, du + \int_0^{t \wedge S_n} \gamma_u \, dW_u \right\|^2\right]$$

$$= \sum_{i=1}^d E\left[\left(\int_0^{t \wedge S_n} a_u^i \, du + \sum_{j=1}^r \int_0^{t \wedge S_n} \gamma_u^{ij} \, dW_u^j \right)^2\right]$$

$$\le 2 \sum_{i=1}^d E\left[\left(\int_0^{t \wedge S_n} a_u^i \, du \right)^2 + \left(\sum_{j=1}^r \int_0^{t \wedge S_n} \gamma_u^{ij} \, dW_u^j \right)^2\right]$$

But

$$\sum_{i=1}^d E\left[\left(\int_0^{t \wedge S_n} a_u^i \, du \right)^2\right] \le \sum_{i=1}^d E\left[t \int_0^{t \wedge S_n} (a_u^i)^2 \, du\right]$$

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$$= E\left[t\int_0^{t\wedge S_n}\|a_u\|^2\,du\right]$$

$$\leq K_n^2 t\, E\left[\int_0^{t\wedge S_n}\|X_u-\tilde{X}_u\|^2\,du\right]$$

$$= K_n^2 t\, E\left[\int_0^{t\wedge S_n}\|X_{u\wedge S_n}-\tilde{X}_{u\wedge S_n}\|^2\,du\right]$$

$$\leq K_n^2 t\, E\left[\int_0^{t}\|X_{u\wedge S_n}-\tilde{X}_{u\wedge S_n}\|^2\,du\right]$$

so, if $t\leq T$

$$\sum_{i=1}^{d} E\left[\left(\int_0^{t\wedge S_n} a_u^i\,du\right)^2\right] \leq K_n^2 T\, E\left[\int_0^{t}\|X_{u\wedge S_n}-\tilde{X}_{u\wedge S_n}\|^2\,du\right]$$

Simlarly

$$\sum_{i=1}^{d} E\left[\left(\sum_{j=1}^{n}\int_0^{t\wedge S_n} Y_u^{ij}\,dW_u^j\right)^2\right]$$

$$= \sum_{i=1}^{d}\sum_{j,k=1}^{n} E\left[\int_0^{t\wedge S_n} Y_u^{ij}\,dW_u^j \cdot \int_0^{t\wedge S_n} Y_u^{ik}\,dW_u^k\right]$$

$$= \sum_{i=1}^{d}\sum_{j=1}^{n} E\left[\int_0^{t\wedge S_n}(Y_u^{ij})^2\,du\right] \qquad d\langle W^j,W^k\rangle_u=\delta^{jk}du$$

(12)

$$= E\left[\int_0^{t\wedge S_n} \|Y_u\|^2 du\right]$$

$$\leq K_n^2 E\left[\int_0^{t\wedge S_n} \|X_u - \tilde{X}_u\|^2 du\right]$$

$$\leq K_n^2 E\left[\int_0^{t} \|X_{u\wedge S_n} - \tilde{X}_{u\wedge S_n}\|^2 du\right]$$

Thus, for $t \leq T$

$$0 \leq g(t) = E\left[\|X_{t\wedge S_n} - \tilde{X}_{t\wedge S_n}\|^2\right]$$

$$\leq 2K_n^2(T+1)\, E\left[\int_0^t g(u) = \|X_{u\wedge S_n} - \tilde{X}_{u\wedge S_n}\|^2 du\right]$$

By Gronwall ($\alpha \equiv 0$)

$$E\left[\|X_{t\wedge S_n} - \tilde{X}_{t\wedge S_n}\|^2\right] = 0 \qquad t \leq T.$$

$\therefore\ P(X_{t\wedge S_n} = \tilde{X}_{t\wedge S_n}) = 1 \qquad \forall t \geq 0$ since T was arbitrary

By continuity:

$$P(X_{t\wedge S_n} = \tilde{X}_{t\wedge S_n}\ \forall t \geq 0) = 1.$$

Since $S_n \to \infty$ $\quad P(X_t = \tilde{X}_t \ \forall t \geq 0) = 1$.

$\therefore X, \tilde{X}$ are indistinguishable.

Having proved strong uniqueness under a <u>local</u> Lipschitz condition, we now proceed to proving existence.

Note: local Lipschitz for ODE's is <u>not</u> enough to yield global existence, even for ODE's.

<u>ex</u>

$$X_t = 1 + \int_0^t X_s^2 \, ds. \qquad (b(x) = x^2)$$

solution is $X_t = \frac{1}{1-t}$. This "explodes" as $t \nearrow 1$.

For strong existence, we assume:

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<u>Thm</u> (Strong Existence)

Suppose b, σ satisfy the "global" Lipschitz and "linear growth" conditions

i) $\|b(t,x) - b(t,y)\| + \|\sigma(t,x) - \sigma(t,y)\|$

$\leq K\|x-y\| \qquad x,y \in \mathbb{R}^d,\ t \geq 0$

ii) $\|b(t,x)\|^2 + \|\sigma(t,x)\|^2 \leq K^2(1+\|x\|^2)$

$t \geq 0,\ x \in \mathbb{R}^d$

Suppose the initial value r.v. ξ satisfies

$E[\|\xi\|^2] < \infty$.

Then $\exists$ a strong solution to the SDE. In fact, X is square integrable in that

$$E[\|X_t\|^2] \leq C(1 + E[\|\xi\|^2])\, e^{Ct} \qquad t \leq T$$

for some $C = C(T)$.

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pf

Here, we exactly mimic the ODE case, and consider the Picard-Lindelöf iterations

$$X_t^{(0)} \equiv \xi$$

$$X_t^{(n+1)} = \xi + \int_0^t b(u, X_u^{(n)})du + \int_0^t \sigma(u, X_u^{(n)})dW_u$$

$$n = 0, 1, 2, \ldots$$

$X^{(n)}$ is clearly continuous, adapted to $\mathbb{F}$.

We will use the Lipschitz condition to show that $X^{(n)}$ converge to a solution X and use the square integrability of ξ to establish the square integrability of X.

pf

similar to that of uniqueness.

$$X_t^{(n+1)} - X_t^{(n)} = B_t + M_t.$$

$$B_t = \int_0^t (b(u, X_u^{n}) - b(u, X_u^{n-1}))\,du$$

$$M_t = \int_0^t (\sigma(u, X_u^{n}) - \sigma(u, X_u^{n-1}))\,du.$$

we now claim that $\forall\ T > 0,\ \exists\ C(T)$ s.t.

$$\star \quad E[\|X_t^{(n)}\|^2] \le C(1 + E[\|\xi\|^2])\,e^{Ct} \qquad t \le T,\ n = 0, 1, \dots$$

Admitting this, M is a square integrable martingale. since

$$E[\|M_t\|^2] = E\Big[\sum_{i=1}^{d} \Big(\sum_{j=1}^{r} \int_0^t (\sigma^{ij}(u, X_u^n) - \sigma^{ij}(u, X_u^{n-1}))\,dW_u^j\Big)^2\Big]$$

$$= E\Big[\int_0^t \|\sigma(u, X_u^n) - \sigma(u, X_u^{n-1})\|^2\,du\Big]$$

$$\le K\,E\Big[\int_0^t \|X_u^n - X_u^{n-1}\|^2\,du\Big]$$

$$\le \hat{K}\,E\Big[\int_0^t (\|X_u^n\|^2 + \|X_u^{n-1}\|^2)\,du\Big]$$

<u>Thus</u>

$$E\Big[\max_{s \le t} \|M_s\|^2\Big] \le \hat{K}\, E\big[\|M_t\|^2\big] \le \hat{K}\, E\Big[\int_0^t \|X_u^{(n)} - X_u^{(n-1)}\|^2\, du\Big]$$

Also, for $s \le t \le T$

$$\|B_s\|^2 = \sum_{i=1}^{d} \Big(\int_0^s \big(b^i(u, X_u^{(n)}) - b^i(u, X_u^{(n-1)})\big)\, du \Big)^2$$

$$\le s \int_0^s \|b(u, X_u^{(n)}) - b(u, X_u^{(n-1)})\|^2\, du$$

$$\le K^2 T \int_0^t \|X_u^{(n)} - X_u^{(n-1)}\|^2\, du$$

Thus, for $t \le T$

$$E\Big[\max_{s \le t} \|X_s^{(n+1)} - X_s^{(n)}\|^2\Big] = E\Big[\max_{s \le t} \|B_s + M_s\|^2\Big]$$

$$\le 2K^2 T \int_0^t E\big[\|X_u^{(n)} - X_u^{(n-1)}\|^2\big]\, du$$

$$+ \hat{K} \int_0^t E\big[\|X_u^{(n)} - X_u^{(n-1)}\|^2\big]\, du$$

$$\triangleq L \int_0^t E\big[\|X_u^{(n)} - X_u^{(n-1)}\|^2\big]\, du.$$

At $n=1$

$$E\left[\max_{s\le t}\|X_s^{(2)}-X_s^{(1)}\|^2\right] \le LC^*t$$

$$C^* = \max_{t\le T} E\left[\|X_t^{(1)}-\xi\|^2\right] < \infty \quad \text{by}$$

At $n=2$

$$E\left[\max_{s\le t}\|X_s^{(3)}-X_s^{(2)}\|^2\right] \le L\int_0^t E\left[\|X_u^{(2)}-X_u^{(1)}\|^2\right]du$$

$$\le L^2C^*\int_0^t u\,du$$

$$= L^2C^*t^2/2$$

Continuing

$$E\left[\max_{s\le t}\|X_s^{(n+1)}-X_s^{(n)}\|^2\right]$$

$$\le C^*\,(Lt)^n/n!$$

Thus, taking $t=T$:

$$P\left(\max_{0\le s\le T}\|X_s^{(n+1)}-X_s^{(n)}\| > \frac{1}{2^{n+1}}\right)$$

$$\le (2^{n+1})^2\,C^*\,(LT)^n/n! = 4C^*\,(4LT)^n/n!$$

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Since $\sum_n (4LT)^n/n! < \infty$ by Borel-Cantelli $\exists\ \tilde{\Omega}$ with $P(\tilde{\Omega}) = 1$ s.t. for $\omega \in \tilde{\Omega}$, $\exists\ N(\omega)$ s.t. $n \geq N(\omega)$ implies

$$\max_{s \leq T} \|X_s^{(n+1)} - X_s^{(n)}\| \leq 2^{-(n+1)}$$

Thus, for $m \geq n$

$$\max_{s \leq t} \|X_s^{(n+m)} - X_s^{(n)}\|$$

$$\leq \sum_{j=1}^{m} \max_{s \leq t} \|X_s^{(n+j)} - X_s^{(n+j-1)}\|$$

$$\leq \sum_{j=1}^{m} 2^{-n-j} = 2^{-n} \sum_{j=1}^{m} 2^{-j}$$

$$\leq 2^{-n}$$

$\therefore \{X^n(\omega)\}_{n=1,\ldots}$ converges in the supremum norm to some $X(\omega)$ for $\omega \in \tilde{\Omega}$ on $[0,T]$.

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By considering $T = 1, 2, 3, \ldots$ we see that with prob. 1, $X^{(n)}(\omega)$ converge uniformly to X on compact subsets of $[0,\infty)$

i) Since the $X^{(n)}$ are adopted to $\mathbb{F}$ so ~~are~~ is X.

~~Since $P(X_0^{(n)} = \xi) = 1 \quad \forall n$, the same~~

ii) Since $X_0^{(n)}(\omega) = \xi(\omega) \quad \forall n, \omega$ we know $X_0(\omega) = \xi(\omega)$ for a.s. ω.

iii) Since

$$\sum_{\substack{i=1,\ldots,d \\ j=1,\ldots,r}} \int_0^t \left(|b^i(u,X_u)| + \sigma^{ij}(u,X_u)^2\right) du$$

$$\leq \sqrt{t \int_0^t \|b(u,X_u)\|^2} + \int_0^t \|\sigma(u,X_u)\|^2$$

$$\leq \sqrt{t K^2 \int_0^t (1+\|X_u\|^2) du} + K^2 \int_0^t (1+\|X_u\|^2) du$$

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It follows by $\bigstar$ $\to$ which holds for X that by Fatou.

$$E\left[\int_0^t |b^i(u,X_u)|\,du + \int_0^t \sigma^{ij}(u,X_u)^2\,du\right] < \infty$$

for $i=1,\dots,d$, $j=1,\dots,r$ and hence

$$P\left(\int_0^t |b^i(u,X_u)|\,du + \int_0^t \sigma^{ij}(u,X_u)^2\,du < \infty\right) = 1.$$

iv) Lastly since

$$X_t^{(n+1)} = \xi + \int_0^t b(u,X_u^{(n)})\,du + \int_0^t \sigma(u,X_u^{(n)})\,dW_u$$

$\forall n$, we have

$$X_t^{(n+1)} \to X_t \quad \text{a.s.}$$

$$\left\| \int_0^t b(u,X_u^{(n)})\,du - \int_0^t b(u,X_u)\,du \right\|^2$$

$$\leq K^2 T \int_0^T \|X_u^{(n)} - X_u\|^2$$

$$\leq K^2 T^2 \max_{u \leq T} \|X_u^{(n)} - X_u\|^2 \to 0 \quad \text{a.s.}$$

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and a similar calculation shows that $\int_0^t \cancel{\sigma(X^n)} \sigma(u, X_u^n)\, dW_u \to \int_0^t \sigma(u, X_u)\, dW_u$

a.s.

$\therefore$ X is a strong solution provided one can show A_j i.e.

$$E[\|X_t^{(n)}\|^r] \leq C(1 + E[\|\xi\|^r])e^{Ct}$$

$$t \leq T, \quad C = C(T).$$

- read solution to this problem.