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Girsanov Theorem.

Motivating Example:

Let $Z = (Z^1, \dots, Z^d)$ be a $N(0, I_d)$ r.v.

Let $\mu \in \mathbb{R}^d$. Since $E[e^{\mu' Z}] = e^{\frac{1}{2}\mu'\mu}$

we may define a probability measure

P^μ via

$$\frac{dP^\mu}{dP} = e^{\mu' Z - \frac{1}{2}\mu'\mu}$$

Then, for $\alpha \in \mathbb{R}^d$

$$E^{P^\mu}[e^{\alpha' Z}] = e^{-\frac{1}{2}\mu'\mu} E[e^{(\alpha+\mu)' Z}]$$

$$= e^{-\frac{1}{2}\mu'\mu + \frac{1}{2}\alpha'\alpha + \alpha'\mu + \mu'\mu}$$

$$= e^{\alpha'\mu + \frac{1}{2}\alpha'\alpha}$$

$\Rightarrow Z \overset{P^\mu}{\sim} N(\mu, I_d)$

or

$Z - \mu \overset{P^\mu}{\sim} N(0, 1)$.

so, if $Z \sim N(0,1)$ under P

and if $\frac{dP^\mu}{dP} = e^{\mu' Z - \frac{1}{2}\mu'\mu}$ then

$Z - \mu \sim N(0,1)$ under P^μ

Girsanov's Theorem seeks to prove an analogous result:

if W is a B.M. under P

and if $\frac{dP^X}{dP} = ???$ for some process

X (with certain properties) then

$W - X$ is a B.M. under P^X.

Notation and the Basic Result.

$(\Omega, \mathcal{F}, P)$, $\mathbb{F}$ given. Usual conditions for $\mathbb{F}$.

W: d-dim B.M. starting at 0.

X: d-dim mbl, adapted process s.t.

$$P\left[\int_0^T (X_s^i)^2 ds < \infty\right] = 1 \quad \forall\, T \geq 0$$

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so $I^{W^i}(X^i) = \int X^i_s dW^i_s$ is well defined in $M^{c,loc}$.

Define

$$Z_t = Z_t(X) = e^{\sum_{i=1}^d \int_0^t X^i_s dW^i_s - \frac{1}{2}\sum_{i=1}^d \int_0^t (X^i_s)^2 ds}$$

(-compare to $\frac{dP^\mu}{dP} = e^{\mu' Z - \frac{1}{2}\mu'\mu}$.)

By Itô:

$$dZ_t = Z_t \sum_{i=1}^d X^i_t dW^i_t \quad \Rightarrow \quad Z \in M^{c,loc}, Z_0 = 1.$$

Theorem

Assume Z is a martingale. For each T define $\tilde{P}_T$ on $\mathcal{F}_T$ via

$$\tilde{P}_T(A) = E[1_A Z_T] \quad A \in \mathcal{F}_T.$$

Next, define $\tilde{W} = (\tilde{W}^1, \dots, \tilde{W}^d)$ via

$$\tilde{W}^i_t = W^i_t - \int_0^t X^i_s ds \quad t \le T, \; i = 1, \dots, d.$$

(4)

Then $\{\tilde{W}_t\}_{t\le T}$ is an $(\mathbb{F})$ B.M. under $\tilde{P}_T$.

<u>Proof</u>

1) Basic Result (I).

$$\left.\frac{d\tilde{P}_T}{dP}\right|_{\mathcal{F}_t} = Z_t.$$

pf.

$A \in \mathcal{F}_t$. $\tilde{P}_T(A) = E[1_A Z_T]$

$$= E[1_A E[Z_T | \mathcal{F}_t]]$$

$$= E[1_A Z_t].$$

2) Basic Result (II).

$0 \le s \le t \le T$. Y : $\mathcal{F}_t$ mbl r.v. s.t. $\tilde{E}_T[|Y|] < \infty$. Then

$$\tilde{E}_T[Y | \mathcal{F}_s] = \frac{1}{Z_s} E[Y Z_t | \mathcal{F}_s].$$

— Bayes Rule.

⑤

pf

$A \in \mathcal{F}_s$.

$$\tilde{E}_T\left[1_A \frac{1}{Z_s} E[Y Z_t \mid \mathcal{F}_s]\right]$$

$$= E\left[1_A E[Y Z_t \mid \mathcal{F}_s]\right] \qquad \left.\frac{d\tilde{P}_T}{dP}\right|_{\mathcal{F}_s} = Z_s.$$

$$= E\left[E[1_A Y Z_t \mid \mathcal{F}_s]\right] \qquad A \in \mathcal{F}_s.$$

$$= E[1_A Y Z_t] \qquad \text{Tower}$$

$$= \tilde{E}_T[1_A Y] \qquad \left.\frac{d\tilde{P}_T}{dP}\right|_{\mathcal{F}_t} = Z_t. \quad \blacksquare$$

3) Corollary

$Y = \{Y_t\}_{t \le T}$ is a $\tilde{P}_T$ martingale if and only if $ZY = \{Z_t Y_t\}_{t \le T}$ is a P martingale.

1) $\tilde{E}_T[|Y_t|] = E[Z_t |Y_t|]$ ✓

2) $\tilde{E}_T[Y_t \mid \mathcal{F}_s] = \frac{1}{Z_s} E[Y_t Z_t \mid \mathcal{F}_s]$.

so $\quad E[Z_t Y_t \mid \mathcal{F}_s] = Z_s Y_s \iff \tilde{E}_T[Y_t \mid \mathcal{F}_s] = Y_s$.

Corrollary: $\tilde{W}^i \in \tilde{\mathcal{M}}_T^{c,loc} \quad i = 1, \dots, d.$

sufficies to show $\tilde{W}^i Z \in \mathcal{M}^{c,loc}$.

<u>Itô</u>

$$d(\tilde{W}^i_t Z_t) = d\left(\left(W^i_t - \int_0^t X^i_u du\right) Z_t\right)$$

$$= W^i_t dZ_t - \left(\int_0^t X^i_u du\right) dZ_t$$

$$+ Z_t dW^i_t - Z_t X^i_t dt$$

$$+ d\langle W^i, Z\rangle_t.$$

$$= W^i_t Z_t \sum_{j=1}^d X^j_t dW^j_t - \left(\int_0^t X^i_u du\right) Z_t \sum_{j=1}^d X^j_t dW^j_t$$

$$+ Z_t dW^i_t - Z_t X^i_t dt.$$

$$+ d\langle W^i, 1 + \int Z_u \sum_{j=1}^d X^j_u dW^j_u \rangle_t.$$

(7)

Now: $\int_0^{\cdot}(W_t^i - \int_0^t X_u^i du) Z_t \sum_{j=1}^d X_t^j dW_t^j \in M^{c,loc}$

$\int_0^{\cdot} Z_t dW_t^i \in M^{c,loc}$.

And

$$-Z_t X_t^i dt + d\langle \int_0^{\cdot} dW_{\theta}^i, 1 + \sum_{j=1}^d \int_0^{\cdot} Z_{\theta} X_{\theta}^j dW_{\theta}^j \rangle_t$$

$$= -Z_t X_t^i dt + Z_t X_t^i dt = 0$$

$\therefore \tilde{W}^i \in \tilde{M}_T^{c,loc}$.

By Lévy it suffices to prove that under $\tilde{P}_T$ $\langle \tilde{W}^i, \tilde{W}^j \rangle_t = \delta_{ij} t$ $t \le T$,

$i,j = 1, \ldots, d$.

But, this holds since

1) $\lim_{\|\pi\| \downarrow 0} \sum_{\ell=1}^n (W_{t_\ell}^i - W_{t_{\ell-1}}^i)(W_{t_\ell}^j - W_{t_{\ell-1}}^j) = \delta_{ij} t$

in ~~pro~~ P-prob b/c $W^{i,j}$ are continuous.

⑧

2) With $B_t^i = \int_0^t X_u^i \, du$

$$\lim_{\|\Pi\| \downarrow 0} \Big(\sum_{l=1}^{n} \left(W_{t_l}^i - B_{t_l}^i - (W_{t_{l-1}}^i - B_{t_{l-1}}^i)\right) \times \left(W_{t_l}^j - B_{t_l}^j - (W_{t_{l-1}}^j - B_{t_{l-1}}^j)\right)$$

$$- \sum_{l=1}^{n} (W_{t_l}^i - W_{t_{l-1}}^i)(W_{t_l}^j - W_{t_{l-1}}^j) \Big) = 0$$

in P prob.

~~Th~~ Thus

$$\lim_{\|\Pi\| \downarrow 0} \sum_{l=1}^{n} (\tilde{W}_{t_l}^i - \tilde{W}_{t_{l-1}}^i)(\tilde{W}_{t_l}^j - \tilde{W}_{t_{l-1}}^j) = \delta_{ij} t \quad \text{in}$$

P prob.

<u>But</u>

if $X_n \to a$ in P prob and $Q \sim P$
then $X_n \to a$ in Q prob as well.

$$Q(|X_n - a| > \varepsilon) = E\left[\frac{dQ}{dP} 1_{|X_n - a| > \varepsilon}\right]$$

$$\frac{dQ}{dP} 1_{|X_n - a| > \varepsilon} \to 0 \quad \text{in } P \text{ prob.}$$

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$$\left|\frac{dQ}{dP} 1_{|X_n - a| > \varepsilon}\right| \le \frac{dQ}{dP} \in L^1(P)$$

so result follows by the dominated convergence theorem.

$$\therefore \langle \hat{W}^i, \hat{W}^j \rangle_t = \hat{P}_T\text{-prob} \lim_{\|\pi\| \to 0} \sum_{l=1} (\hat{W}^i_{t_l} - \hat{W}^i_{t_{l-1}})(\hat{W}^j_{t_l} - \hat{W}^j_{t_{l-1}})$$

$$= \delta_{ij}\, t \qquad t \le T$$

$\therefore \hat{W}$ is a d-dim $\hat{P}_T$ B.M. ∎

- more generally above: quadratic covariations are unchanged under equivalent measure changes.

<u>Important Note</u>

The fact that this result holds for all finite $T \geq 0$ is important. The extension to $T = \infty$ is not trivial and in general does not hold.

Example: $Z_t = e^{\mu W_t - \frac{1}{2}\mu^2 t}$

This is a martingale so we can define $\tilde{P}_T$ as ~~bef~~ before $\forall\ T \geq 0$ and obtain that

$$\tilde{W}_t = W_t - \mu t \qquad t \leq T$$

is a B.M. under $P_T^\mu \triangleq \tilde{P}_T$

note: $W_t = \tilde{W}_t + \mu t$ is called a B.M. with drift μ under P^μ.

Now, the event $\{\frac{1}{t} W_t \to \mu\}$ has P probability 0 and hence is in $\mathcal{F}_0 \subseteq \mathcal{F}_T$ $\forall\ T \geq 0$ since $\mathbb{F}$ satisfies the usual conditions.

Thus, $P_T^\mu\left[\lim_{t\to\infty} \frac{1}{t} W_t = \mu\right] = 0 \quad \forall\ T.$

~~see 1~~

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But, *if* P^μ_T could be defined for $T=\infty$ on $\mathcal{F}_\infty$ and if $\hat{W}_t = W_t - \mu t$, $t \geq 0$ has a B.M. on P^μ_∞ then

$$P^\mu_\infty\left[\lim_{t\to\infty} \frac{1}{t} W_t = \mu\right]$$

$$= P^\mu_\infty\left[\lim_{t\to\infty} \frac{1}{t}\hat{W}_t + \mu = \mu\right]$$

$$= P^\mu_\infty\left[\lim_{t\to\infty} \frac{1}{t}\hat{W}_t = 0\right]$$

$$= 1 \quad \mathcal{E}.$$

thus, even extending P^μ_T to ∞ via consistency, we see that P^μ_∞ cannot agree with P: in fact they are singular.

Problem: $Z_t = e^{\mu W_t - \frac{1}{2}\mu^2 t} \to 0$ $t\to\infty$.

and is not u.i.

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so, we have that P and P^μ cannot be mutually absolutely continuous unless Z is a uniformly integrable martingale.

Note: even if we consider $\mathbb{F} = \mathbb{F}^W$ and drop the usual conditions requirement, we can still define

$$P_T^\mu(A) = E\left[1_A e^{\mu W_T - \frac{1}{2}\mu^2 T}\right] \quad A \in \mathcal{F}_T^W.$$

consistency gives a P^μ on $\mathcal{F}_\infty^W$ which agrees with P_T^μ in the case that P is Wiener measure (Kolmogorov-Daniell). In fact, one can still show $\tilde{W}_t = W_t - \mu t$ is a P^μ B.M. But we still have

$$P^\mu\left[\lim_{t\to\infty} \tfrac{1}{t} W_t = \mu\right] = 1$$

$$P\left[\lim_{t\to\infty} \tfrac{1}{t} W_t = \mu\right] = 0$$

so even though $P_T^\mu \sim P|_{\mathcal{F}_T^W}$ $\forall$ $T \geq 0$,

P^μ and P are singular on $\mathcal{F}^W_\infty$.

Novikov Condition:

Goal: give fairly general conditions on when $\{Z_t = Z(X)_t\}_{t \geq 0}$ is a martingale.

1) We know $Z(X) \in \mathcal{M}^{c,loc}$ and since $Z(X)_t \geq 0$ we know $Z(X)$ is a super-martingale.

- Martingality will follow if $E[Z(X)_t] = 1 \quad \forall\ t \geq 0$.

Novikov Condition: Thm. $M \in \mathcal{M}^{c,loc}$ (wrt $(\Omega, \mathcal{B}, P), \mathbb{F}$). Set

$$Z_t = e^{M_t - \frac{1}{2}\langle M \rangle_t} \qquad t \geq 0$$

Then $Z \in \mathcal{M}^{c,loc}$, Z a supermartingale.

(14)

If $E[e^{\frac{1}{2}\langle M\rangle_t}] < \infty$ $t \geq 0$ then $E[Z_t] = 1$ $t \geq 0$ so Z is a Martingale.

- with $M_t = \sum_{j=1}^{d} \int_0^t X_s^j dW_s^j$ we get that $Z(X)$ is a Martingale if

$$E[e^{\frac{1}{2}\int_0^t \|X_s\|^2 ds}] < \infty \quad t \geq 0.$$

pf

1) Exponential Wald Identity.

W a B.M. T a stopping time of $\mathbb{F}^W$. ~~If $P(T<\infty)=1$~~ If

$$\lim_{n\to\infty} P_n^{\mu}(T \leq n) = 1$$

then

$$E[e^{\mu W_T - \frac{1}{2}\mu^2 T}] = 1.$$

~~pf (potentially skip)~~

pf: skip b/c it is a HW problem.

2) Set $S_b = \inf[t \geq 0 \mid W_t - \mu t = b]$

Then

$$P^{\mu}_n[S_b \leq n] = P^{\mu}_n[\inf[t \geq 0 \mid \hat{W}_t = b] \leq n]$$

$$= 2 \frac{1}{\sqrt{2\pi}} \int_{|b|/\sqrt{n}}^{\infty} e^{-x^2/2} dx \to 1$$

so

$$1 = E[e^{\mu W_{S_b} - \mu^2 S_b}] = e^{\mu b} E[e^{\frac{1}{2}\mu^2 S_b}].$$

3) Set $T(s) = \inf[t \geq 0 \mid \langle M \rangle_t > s]$. By Dambins, Dubins, Schwarz ($\langle M \rangle_t \to \langle M \rangle_\infty < \infty$ OK
$t \to \langle M \rangle_t$ not strict incr OK)

theorem:

$$B_t = M_{T(t)} \quad ; \mathcal{B}_t = \mathcal{F}_{T(t)} \quad \text{is a B.M.}$$

with $S_b = \inf[t \geq 0 \mid B_t - t = b]$

we have (with $\mu = 1$)

$$E[e^{\frac{1}{2}S_b}] = e^{-b}.$$

Now: $Y_s = e^{B_s - s/2}$ is a $\mathcal{G}$ martingale as well as $N_s = Y_{s \wedge S_b}$ b/c $P(S_b < \infty) = 1$ $\left(P(S_b \le \ell) = 2 \frac{1}{\sqrt{2\pi}} \int_{|b|/\sqrt{\ell}}^{\infty} e^{-x^2/2} \, dx \right)$.

Thus

$$N_\infty = Y_{S_b} = e^{B_{S_b} - \frac{1}{2} S_b}$$

since N_∞ is a super-mart w/ last element $(N \geq 0)$ and since $E[N_\infty] = 1$ we know N is a martingale w/ last element.

$\therefore$ By OST

$$1 = E\left[e^{B_{R \wedge S_b} - \frac{1}{2} R \wedge S_b}\right] \quad \forall \, \mathcal{G} \text{ stopping } R.$$

Taking $R = \langle M \rangle_t$.

$$1 = E\left[e^{B_{\langle M \rangle_t \wedge S_b} - \frac{1}{2} \langle M \rangle_t \wedge S_b}\right]$$

$$= E\left[1_{S_b \le \langle M \rangle_t} e^{b + \frac{1}{2} S_b}\right]$$

$$+ E\left[1_{S_b > \langle M \rangle_t} e^{M_t - \frac{1}{2} \langle M \rangle_t}\right]$$

Note

1) $E[1_{S_b \le \langle M \rangle_t} e^{b + \frac{1}{2} S_b}]$

$$\le e^b E[e^{\frac{1}{2}\langle M \rangle_t}]$$

$\to 0$ as $b \downarrow -\infty$ since $E[e^{\frac{1}{2}\langle M \rangle_t}] < \infty$ by assumption:

2) $1_{S_b > \langle M \rangle_t}$ is increasing as $b \downarrow -\infty$ to 1 since for each ω

$$S_b(\omega) = \inf[t \geq 0 \mid B_t = t + b]$$

and B_t is a.s. finite.

$\therefore$ By MCT

$$1 = \lim_{b \to -\infty} E[1_{S_b \le \langle M \rangle_t} e^{b + \frac{1}{2} S_b}] + E[1_{S_b > \langle M \rangle_t} e^{M_t - \frac{1}{2}\langle M \rangle_t}]$$

$$= E[e^{M_t - \frac{1}{2}\langle M \rangle_t}] = E[Z_t] \quad \blacksquare$$

Corr: If $\exists\ t_n \nearrow \infty$ s.t.

$$E\left[e^{\frac{1}{2}\int_{t_{n-1}}^{t_n}\|X_s\|^2 ds}\right] < \infty \qquad \forall\ n \geq 1.$$

then $E[Z_t] = 1\ \ \forall t$ and Z is a martingale.

pf.

$$X^n_t = X_t \mathbb{1}_{[t_{n-1}, t_n)}(t)$$

$$\Rightarrow Z^n_t = Z(X^n)_t \text{ is a martingale.}$$

$$\Rightarrow E[Z^n_{t_n} \mid \mathcal{F}_{t_{n-1}}] = Z^n_{t_{n-1}} = 1$$

b/c
$$Z^n_t = e^{\int_0^t X^n_u dW_u - \frac{1}{2}\int_0^t \|X^n_u\|^2 du}$$
$$= e^{\int_{t\wedge t_{n-1}}^{t\wedge t_n} X_u dW_u - \frac{1}{2}\int_{t\wedge t_{n-1}}^{t\wedge t_n} \|X_u\|^2 du}$$

$$\Rightarrow E[Z_{t_n}] = E[Z_{t_{n-1}} Z^n_{t_n}] = E[Z_{t_{n-1}}]$$

$$\Rightarrow E[Z_{t_n}] = 1 \quad \forall\ n$$

so $t_n \nearrow \infty \Rightarrow E[Z_t] = 1\ \ \forall t$ since $t \mapsto E[Z_t]$ is $\searrow$.