Properties of Brownian Motion II:
The Strong Markov Property.

~~The Augmented filtration of~~

Motivating Example: The Reflection Principle.

B: 1 dim. standard B.M. under P_0, $\mathbb{F}$.

Let $b>0$, and define

$$T_b = \inf[t \mid B_t \geq b]$$

as the hitting time of B to b. T_b is a stopping time, called the "passage time" to b.

Goal: compute the distribution of T_b:

$$P_0(T_b<t) = P_0(T_b<t, B_t>b) + P_0(T_b<t, B_t<b)$$

$$= P(B_t>b) + P_0(T_b<t, B_t<b)$$

- since $B_t>b \Rightarrow T_b<t$.

Now, let ω be s.t. $T_b(\omega)<t$, $B_t(\omega)<b$.

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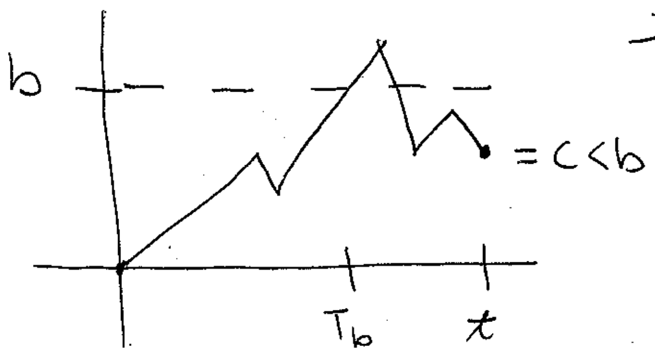
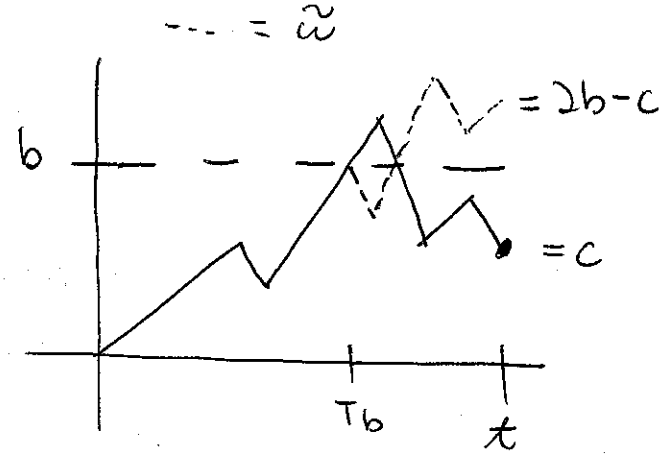
(Simplified Picture)



→ path got up to b then went below to $c < b$ at t.

"Reflection Principle": heuristic conjecture that $\forall$ path as above, there is a "shadow" path $\tilde{\omega}$ s.t. $\tilde{\omega} = \omega$ up to T_b, but $\tilde{\omega}$ is the mirror image of ω after T_b reflected around b.



- conjecture is motivated by the symmetry of B.M.

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If we believe this then $\forall$ path s.t. $T_b < t$, $B_t < b$, $\exists$ a path s.t. $T_b < t$, $B_t > b$ so

$$P(T_b < t, B_t < b) = P(T_b < t, B_t > b)$$
$$= P(B_t > b)$$

Thus

$$P(T_b < t) = P(T_b < t, B_t > b) + P(T_b < t, B_t < b)$$
$$= 2 P(B_t > b)$$
$$= \frac{2}{\sqrt{2\pi t}} \int_b^\infty e^{-\frac{z^2}{2t}} dz$$
$$= \frac{2}{\sqrt{2\pi}} \int_{b/\sqrt{t}}^\infty e^{-x^2/2} dx.$$

So, T_b has a density

$$P(T_b \in dt) = \frac{b}{\sqrt{2\pi t^3}} e^{-b^2/2t} \qquad t > 0.$$

The main idea behind this argument is that B "starts anew" at T_b:

i.e.

$$W_t = B_{T_b+t} - B_{T_b} = B_{T_b+t} - b$$

is a B.M. $\perp\!\!\!\perp$ of T_b.

Indeed, if so

$$P(T_b<t, B_t<b) = P(T_b<t, W_{t-T_b}<0)$$

$$= \int_{(0,t)} P(W_{t-r}<0 \mid T_b=r)\, P(T_b\in dr)$$

$$= \frac{1}{2} P(T_b<t)$$

so

$$P(T_b<t) = P(B_t>b) + \frac{1}{2} P(T_b<t).$$

Now, as we have shown, if T is a <u>bounded</u> stopping time then

$$W_t = B_{T+t} - B_T \quad ; \tilde{\mathcal{F}}_t = \mathcal{F}_{T+t}$$

is a ~~B.M.~~ martingale,

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To obtain this for general stopping times we introduce the notion of a Strong Markov process.

Definition (Strong Markov Process)

$(\Omega, \mathcal{F}, P^\mu)$, $\mathbb{F}$ given. X: d-dim. adapted, progressively measurable* (e.g. right-continuous) process. X is a <u>Strong Markov</u> process with initial distribution μ if

i) $P^\mu[X_0 \in \Gamma] = \mu(\Gamma) \qquad \Gamma \in \mathcal{B}(\mathbb{R}^d)$

ii) for optional times S of $\mathbb{F}$, $t \geq 0$ and $\Gamma \in \mathcal{B}(\mathbb{R}^d)$

$$P^\mu[X_{S+t} \in \Gamma \mid \mathcal{F}_{S+}] = P^\mu[X_{S+t} \in \Gamma \mid X_S]$$

a.s. P^μ on $\{S < \infty\}$.

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Notes (*)

i) $X_{S(\omega)} \triangleq X_{S(\omega)}(\omega)$ on $\{S<\infty\}$.

ii) If S is a stopping time then X_S is ~~mbl if~~ $\mathcal{F}_S$ mbl if X is progressively mbl.

- why we need prog. mbl. X.

iii) $\{X_{S+t} \in \Gamma\} \triangleq \{X_{S+t} \in \Gamma, S<\infty\}$.

iv) $\sigma(X_S)$ = collection of sets $\{X_S \in A\}$ and $\{X_S \in A\} \cup \{S=\infty\}$, where $\{X_S \in A\}$ implicitly assumes $\{S<\infty\}$ for $A \in \mathcal{B}(\mathbb{R}^d)$

$$\Rightarrow P^\mu[X_{S+t} \in \Gamma \mid \mathcal{F}_{S+}] = 0$$
$$= P^\mu[X_{S+t} \in \Gamma \mid X_S] \text{ on } \{S=\infty\}.$$

We have a similar definition for Strong Markov families:

Definition (Strong Markov family). $(\Omega, \mathcal{F})$, $\mathbb{F}$ given. X: d-dim, adapted progressively mbl process. $\{P^x\}_{x \in \mathbb{R}^d}$: family of measures on $(\Omega, \mathcal{F})$. Then, X, $\{P^x\}_{x \in \mathbb{R}^d}$ is a strong Markov family if:

i) $x \mapsto P^x(F)$ is Borel (universally) mbl $\forall F \in \mathcal{F}$.

ii) $P^x[X_0 = x] = 1 \quad \forall x \in \mathbb{R}^d$

iii) $x \in \mathbb{R}^d, \Gamma \in \mathcal{B}(\mathbb{R}^d), t \geq 0$, S optional imply

$$P^x[X_{S+t} \in \Gamma \mid \mathcal{F}_{S+}] = P^x[X_{S+t} \in \Gamma \mid X_S]$$

P^x a.s. on $\{S < \infty\}$.

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iv) $x \in \mathbb{R}^d$, $\Gamma \in \mathcal{B}(\mathbb{R}^d)$, $t \geq 0$, S optional imply

$$P^x[X_{S+t} \in \Gamma \mid X_S = y]$$

$$= P^y[X_t \in \Gamma]$$

$P^x X_S^{-1}$ a.s. $y \in \mathbb{R}^d$, where

$$P^x X_S^{-1}(A) \triangleq P^x[X_S \in A, S < \infty].$$

Notes: If S is a stopping time

$$P^x[X_{S+t} \in \Gamma \mid \mathcal{F}_S] = P^x[X_{S+t} \in \Gamma \mid X_S]$$

P^x a.e.

so, if $S \equiv \Delta \geq 0$ we have

$$P^x[X_{\Delta+t} \in \Gamma \mid \mathcal{F}_\Delta] = P^x[X_{\Delta+t} \in \Gamma \mid X_\Delta]$$

$\Rightarrow$ Strong Markov implies Markov.

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Equivalent formulation:

As with Markov processes we can replace iii), iv) above with

v) $P^x[X_{s+t} \in \Gamma \mid \mathcal{F}_{s+}] = U_t 1_\Gamma(X_s)$

P^x a.s. on $X_s < \infty$

Now, this can also be expressed as

v') $\cancel{P^x}$ $E^x[f(X_{s+t}) \mid \mathcal{F}_{s+}] = U_t f(X_s)$

$\triangleq g(X_s)$

f : bdd Borel mbl.

But, it actually is OK to test with f continuous and bounded so if we have i), ii) and

v'') $E^x[f(X_{s+t}) \mid \mathcal{F}_{s+}] = U_t f(X_s)$

$\triangleq g(X_s) \quad f \in C_b(\mathbb{R}^d)$

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Then we have a strong Markov family.

In fact, one can assume that the optional times are bounded in that. if

$$E^x[f(X_{S+t}) \mid \mathcal{F}_{S+}] = U_t f(X_S)$$

P^x a.s. $\forall$ bdd optional S

then

$$E^x[f(X_{S+t}) \mid \mathcal{F}_{S+}] = U_t f(X_S)$$

P^x a.s. $\forall$ optional S.

B.M. is a Strong Markov Process.

Proof is very technical

Here is a simpler proof, under same nicer assumptions.....

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Thm 1

$(\Omega, \mathcal{F}, P)$, $\mathbb{F}$ given with $\mathbb{F}$ satisfying the usual conditions and τ a bounded (optional) stopping time. Let B be d-dim B.M. (starting at same $x \in \mathbb{R}^d$)

Define

$$\{W_t \triangleq B_{t+\tau} - B_\tau\}_{t\geq 0} \text{ (component-wise } i=1,\dots,d)$$

$$\{\hat{\mathcal{F}}_t \triangleq \mathcal{F}_{\tau+t}\}_{t\geq 0}$$

Then W is a $\hat{\mathbb{F}}$ ~~B.A.~~ d-dim B.M. starting at $\vec{0}$. In particular,

$W_t - W_s \perp\!\!\!\perp \mathcal{F}_\tau \quad \forall \; 0 \leq s < t.$

pf

HW: already showed W is a $\hat{\mathbb{F}}$ martingale.

Indeed: for $i=1,\dots,d$

$$E[W_t^i \mid \hat{\mathcal{F}}_s] = E[B_{t+\tau}^i - B_\tau^i \mid \mathcal{F}_{s+\tau}]$$

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$$= E[B^i_{t+\tau} \mid \mathcal{F}_{s+\tau}] - B^i_\tau$$

$$= B^i_{s+\tau} - B^i_\tau \qquad (\text{OST}: \tau \text{ bdd})$$

$$= W^i_s.$$

Also, for $i, j = 1, \dots, d$

$$E[W^i_t W^j_t \mid \mathcal{F}_s] = E[(B^i_{t+\tau} - B^i_\tau)(B^j_{t+\tau} - B^j_\tau) \mid \mathcal{F}_{s+\tau}]$$

$$= E[B^i_{t+\tau} B^j_{\tau+t} \mid \mathcal{F}_{s+\tau}]$$

$$- B^i_\tau E[B^j_{t+\tau} \mid \mathcal{F}_{s+\tau}]$$

$$- B^j_\tau E[B^i_{t+\tau} \mid \mathcal{F}_{s+\tau}]$$

$$+ B^i_\tau B^j_\tau.$$

$i = j$, since $\{(B^i_t)^2 - t\}_{t \geq 0}$ is a mgle, by OST.

$$= (B^i_{s+\tau})^2 - (s+\tau) + (t+\tau) \cancel{- (B^i_\tau) \ldots}$$

$$\cancel{- B^j_\tau B^i_{s+\tau}} - 2B^i_\tau B^i_{s+\tau} + (B^i_\tau)^2$$

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$$= (B^i_{s+\tau} - B^i_\tau)^2 - s + t$$

so

$$E[(W^i_t)^2 - t \mid \mathcal{F}_s] = (W^i_s)^2 - s.$$

$i \neq j$

$M_t = B^i_t B^j_t$ is a ~~mag~~ mgle, so by OST

$$= B^i_{s+\tau} B^j_{s+\tau} - B^i_\tau B^j_{s+\tau} - B^j_\tau B^i_{s+\tau} + B^i_\tau B^j_\tau$$

$$= (B^i_{s+\tau} - B^i_\tau)(B^j_{s+\tau} - B^j_\tau)$$

$$= W^i_s W^j_s.$$

$\therefore$ ~~$\langle W \rangle$~~ $\langle W^i, W^j \rangle_t = \delta^{ij} t$

$\therefore$ Lévy says W is a d-dim B.M. starting at 0 ∎.

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Thus, for $f : \mathbb{R}^d \mapsto \mathbb{R}$ bdd, mble.

$$E[f(B_{t+\tau}) \mid \mathcal{F}_\tau]$$

$$= E[f(W_t + B_\tau) \mid \mathcal{F}_\tau]$$

now, $W_t \perp\!\!\!\perp \hat{\mathcal{F}}_0 = \mathcal{F}_\tau$.

B_τ is $\mathcal{F}_\tau$ mbl.

so, by the independence Lemma *

$$E[f(W_t + B_\tau) \mid \mathcal{F}_\tau]$$

multiple dimensional version.

$$= g(B_\tau)$$

$$g(y) = E[f(W_t + y)]$$

$\therefore$ B is Strong Markov.