1

Properties of Brownian Motion and Markov Processes.

We now come back to Brownian Motion and discuss several important features, with generalizations to other processes where appropriate.

1) Wiener Measure on $(C[0,\infty), \mathcal{B}(C[0,\infty)))$

using Kolmogorov-Daniell and Kolmogorov-Čentsov we established existence of B.M. on the probability space $(\mathbb{R}^{[0,\infty)}, \mathcal{B}(\mathbb{R}^{[0,\infty)}))$ where $\mathcal{B}(\mathbb{R}^{[0,\infty)})$ is generated by the cylinder sets.

- we can also think of Brownian Motion through its law.

②

Let $C[0,\infty)$ denote the space of continuous functions on $[0,\infty)$.

Define the metric

$$\rho(\omega_1,\omega_2) = \sum_{n=1}^{\infty} 2^{-n}\left(1 \wedge \max_{t\le n} |\omega_1(t) - \omega_2(t)|\right)$$

facts: 1) $(C[0,\infty), \rho)$ is a complete, separable metric space

2) With

$$\mathcal{C} = \{ C = \{\omega \mid (\omega_{t_1},\dots,\omega_{t_n}) \in A\},\ n\ge 1,\ 0\le t_1\le \dots \le t_n,\ A\in \mathcal{B}(\mathbb{R}^n)\}\}.$$

$$\mathcal{C}_t = \{\text{same, but } 0\le t_1 \le \dots \le t_n \le t\}$$

we have

$$\mathcal{A} = \sigma(\mathcal{C}) = \mathcal{B}(C[0,\infty), \rho)$$
$$\mathcal{A}_t = \sigma(\mathcal{C}_t) = \mathcal{B}_t(C[0,\infty), \rho)$$

↑Borel sets on $[0,t]$.

(3) Then, if B is a B.M. (one-dim) on same $(\Omega, \mathcal{F}, P)$ then the law of B is a measure on $(C[0,\infty), \mathcal{B}(C[0,\infty)))$ which we call Wiener measure P_*, P^0 or W. (in various settings).

Specifically, if we consider the probability space $(\Omega, \mathcal{F}, P) = (C[0,\infty), \mathcal{B}(C[0,\infty)), P_*)$ then with

$$X_t(\omega) = \omega_t \qquad \text{"coordinate mapping"}$$

$$\mathbb{F} = \mathbb{F}^X \qquad \text{"natural filtration"}$$

we have that X is a B.M. under P_*.

2) d-dim. B.M. Revisited.

Recall: $B = (B^1, \dots, B^d)$ is a d-dim B.M. with initial distribution μ if.

1) $B_0 \sim \mu$

2) $\forall\ 0 \leq s < t$, $B_t - B_s$ is a) $\perp\!\!\!\perp$ of $\sigma(B_r; r \leq s)$ and b) normally distributed with mean $\vec{0}$ and covariance $(t-s)\mathbb{1}_d$.

Constructing B

- one method which will help us when we talk about Markov proccessses.

a) start with d $\perp\!\!\!\perp$ copies of Wiener measure:

$$\Omega = C[0,\infty)^d \ ; \ \mathcal{F} = \mathcal{B}(C[0,\infty)^d)$$

$$P^0 = P_* \times P_* \times \ldots \times P_*.$$

Then the coordinate mapping process B is a d-dim B.M. starting at 0.

b) fix $x \in \mathbb{R}^d$. Define P^x via

$$P^x(F) = P^0[\{\omega \mid \omega(\cdot) + x \in F\}]$$

$$F \in \mathcal{B}(C[0,\infty)^d).$$

⑤

Clarify: Coordinate Mapping process is a d-dim B.M. starting at x.

e.g. $F = \{\tilde{\omega} \mid \tilde{\omega}_t - \tilde{\omega}_s \in A\}$.

$\Rightarrow \omega(\cdot) + x \in F \Rightarrow \omega(t) - \omega(s) \in A$.

$P^x(F) = \tilde{P}^x(B_t - B_s \in A)$ → Coordinate Mapping B.

$= P^0(B_t - B_s \in A)$

c) For μ a Borel measure on $\mathbb{R}^d$, set

$$P^\mu(F) \triangleq \int_{\mathbb{R}^d} P^x(F)\,\mu(dx) \qquad F \in \mathcal{B}(C[0,\infty)^d).$$

then B (coordinate mapping) is a d-dim B.M. with initial dist. μ if

$x \mapsto P^x(F)$ is Borel mbl map from $\mathbb{R}^d$ to $[0,1]$ for each $F \in \mathcal{B}(C[0,\infty)^d)$.

But, $x \mapsto P^x(F)$ is Borel mbl.

~~e.g.~~ $\cancel{F = \{\omega \mid (\omega_{t_1}, \dots, \omega_{t_n}) \in A_n\}}$.

e.g. $F = \{\omega \mid \omega_{t_0} \in \Gamma_0, \omega_{t_1} \in \Gamma_1, \dots, \omega_{t_n} \in \Gamma_n\}$.

then

$$P^x(F) = 1_{\Gamma_0}(x) \int_{\Gamma_1} \cdots \int_{\Gamma_n} p(t_1, x, y_1)\, p(t_2 - t_1, y_1, y_2) \cdots p(t_n - t_{n-1}, y_{n-1}, y_n)\, dy_n \cdots dy_1.$$

$$p(t, x, y) = \frac{1}{(2\pi t)^{d/2}} e^{-\frac{1}{2t}|x-y|^2}$$

- clearly Borel mbl.

Result follows from a Dynkin system argument.

- went through this to stress measurability of $x \mapsto P^x(F)$.

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Markov Property of B.M.

Basic Idea: B: 1-dim standard B.M. w.r.t. some $(\Omega, \mathcal{F}, P)$, $\mathbb{F}$.

Suppose we ~~observe~~ are at time s and want to know about B at a later time B_{t+s}, $t \geq 0$.

i.e. we want to compute

$$E[f(B_{t+s}) \mid \mathcal{F}_s]$$

Now, we know that $B_{t+s} = B_s + B_{t+s} - B_s$ and $B_{t+s} - B_s \perp\!\!\!\perp \mathcal{F}_s$, and that B_s is $\mathcal{F}_s$ mbl.

$\Rightarrow$

$$E[f(B_{t+s}) \mid \mathcal{F}_s] = E[f(B_{t+s} - B_s + B_s) \mid \mathcal{F}_s]$$

so, we should be able to do two things

1) replace $\mathcal{F}_s$ with $\sigma(B_s)$

$B_{t+s} - B_s \perp\!\!\!\perp \mathcal{F}_s \quad (\perp\!\!\!\perp \sigma(B_s))$

B_s is knowable w.r.t. smaller σ-alg $\sigma(B_s)$.

(8)

- i.e. the whole past does not matter, just the present value.

2) "Pretend" B_s is constant given $\sigma(B_s)$

so

$$E[f(B_{s+t}) \mid \mathcal{F}_s] = E[f(B_{s+t} - B_s + B_s) \mid \mathcal{F}_s]$$

$$= E[f(B_{s+t} - B_s + B_s) \mid B_s]$$

$$= g(B_s)$$

where

$$g(y) \triangleq E[f(B_{s+t} - B_s + y)]$$

$$= \frac{1}{\sqrt{2\pi t}} \int_{-\infty}^{\infty} f(y+z)\, e^{-\frac{z^2}{2t}}\, dz.$$

$$= \int_{-\infty}^{\infty} f(x) \cdot \underbrace{\frac{1}{\sqrt{2\pi t}}\, e^{-\frac{(x-y)^2}{2t}}}_{p(t,y,x):\ \text{transition density.}}\, dx$$

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This argument is made precise through the Independence Lemma.

Independence Lemma.

X, Y r.v. Y is $\mathcal{A}$ mbl, X is $\perp\!\!\!\perp$ of $\mathcal{A}$.

f: bounded Borel mbl function.

Then

$$E[f(X+Y)\,|\,\mathcal{A}] = E[f(X+Y)\,|\,Y]$$
$$= g(Y)$$

for

$$g(y) \triangleq E[f(X+y)].$$

In particular, $E[f(X+Y)\,|\,Y=y] = g(y)$ *

for PY^{-1} ~~a~~ a.e. y (i.e. $P(Y$ s.t. * doesn't hold$) = 0$).

- proof is easy building up from indicators.

For a general process X, we say it is Markov if the above statements hold.

<u>Def.</u>

$(\Omega, \mathcal{F}, P^\mu)$, $\mathbb{F}$ given. Say a d-dim adapted process X is a Markov process with initial dist. μ if.

1) $P^\mu[X_0 \in \Gamma] = \mu(\Gamma)$ $\quad \Gamma \in \mathcal{B}(\mathbb{R}^d)$.

2) for $0 \leq s < t$, $\Gamma \in \mathcal{B}(\mathbb{R}^d)$

$$P^\mu[X_{t+s} \in \Gamma \mid \mathcal{F}_s]$$
$$= P^\mu[X_{t+s} \in \Gamma \mid X_s] \quad P^\mu \text{ a.s.}$$

-get rid of past....

As for the second statement, we need some additional definitions. since we don't a-priori know how P^μ was built.

(11)

Def : $(\Omega, \mathcal{F})$, $\mathbb{F}$ given. Also, we have $\{P^x\}_{x \in \mathbb{R}^d}$ given. Say a d-dim adapted process X, together with the family $\{P^x\}_{x \in \mathbb{R}^d}$ is a Markov family if.

1) for $F \in \mathcal{F}$, $x \mapsto P^x(F)$ is "universally" measurable

- slight extension of Borel measurability needed when $\mathcal{F}$ is larger than $\mathcal{B}(C[0,\infty)^d)$.

- pg 73.

2) $P^x[X_0 = x] = 1 \quad \forall\ x \in \mathbb{R}^d$

3) $x \in \mathbb{R}^d$, $s \le t$, $\Gamma \in \mathcal{B}(\mathbb{R}^d)$

$$P^x[X_{t+s} \in \Gamma \mid \mathcal{F}_s] = P^x[X_{t+s} \in \Gamma \mid X_s]$$

P^x a.s.

4) $x \in \mathbb{R}^d$, $s \le t$, $\Gamma \in \mathcal{B}(\mathbb{R}^d)$

$$P^x[X_{s+t} \in \Gamma \mid X_s = y]$$

$$= P^y[X_t \in \Gamma]$$

$$P^x X_s^{-1} \quad \text{a.s. } y$$

$$P^x X_s^{-1}(A) = P^x(X_s \in A) \qquad A \in \mathcal{B}(\mathbb{R}^d).$$

So, we have shown that (at least for $d=1$)

1) d-dim B.M. with initial distribution μ is a Markov process.

2) d-dim B.M. with measures $\{P^x\}$ dictating the starting point is a Markov family.

Note: often, $\Omega = C[0,\infty)^d$, $\mathcal{F} = \mathcal{B}(C[0,\infty)^d)$ and $X_t(\omega) = \omega_t$ is the coordinate mapping process.

Here, the markov family really <u>IS</u> the measures $\{P^x\}_{x \in \mathbb{R}^d}$. So, you will often read "Let $\{P^x\}_{x \in \mathbb{R}^d}$ be a Markov process".

- they mean the coordinate process together with $\{P^x\}_{x \in \mathbb{R}^d}$ is a Markov family.

<u>Notes.</u>

1) Markov $\not\Longleftrightarrow$ Martingale.

a) Markov $\not\Rightarrow$ Martingale

B_t^2 is Markov (why?) but <u>not</u> a martingale.

warning: to show B_t^2 is Markov, you must show

$$E[f(B_{t+s}^2) \mid \mathcal{F}_s] = g(B_s^2),$$ <u>not</u> just $g(B_s)$ which we already know.

b) Martingale $\not\Rightarrow$ Markov.

- trickier, but heuristically this holds because to get a Markov process we must verify

$$E[f(X_{t+s}) \mid \mathcal{F}_s] = g(X_s)$$

$\forall$ Borel mbl (bdd) f, not just $f(x) = x$.

Equivalent formulations of the Markov Property

- there are many (see p.p. 75 - 79), we will give one.

- Let $(\Omega, \mathcal{F})$, $\mathbb{F}$, X and $\{P^x\}$ be given.

- assume $x \mapsto P^x(F)$ is Borel mbl (or for universally mbl).
- define, for bdd mbl f

$$U_t f(x) \triangleq E^x[f(X_t)]$$

(15)

Note that $U_t f$ is bdd, Borel (universally) mbl.

<u>Prop.</u>

$X, \{P^x\}$ is Markov iff

1) $x \mapsto P^x(F)$ is Borel (universally) mbl.

2) $P^x[X_0 = x] = 1 \quad \forall x$

3) $x \in \mathbb{R}^d, s \le t, \Gamma \in \mathcal{B}(\mathbb{R}^d)$

$$P^x[X_{t+s} \in \Gamma \mid \mathcal{F}_s] = U_t 1_\Gamma(X_s) \quad P^x \text{ a.s.}$$

<u>pf.</u> (assuming $x \mapsto P^x(F)$ is Borel).

$$P^x[X_{t+s} \in \Gamma \mid X_s = y] = U_t 1_\Gamma(y) \quad P^x X_s^{-1} \text{ a.s.}$$

Since $y \mapsto U_t 1_\Gamma(y)$ is Borel mbl we know

$$P^x[X_{t+s} \in \Gamma \mid X_s] = U_t 1_\Gamma(X_s) \quad P^x \text{ a.s.}$$

so 3) holds.

If 3) holds, $P^x[X_{t+s} \in \Gamma \mid \mathcal{F}_s] = U_t 1_\Gamma(X_s)$ is $\sigma(X_s)$ mbl so $P^x[X_{t+s} \in \Gamma \mid \mathcal{F}_s]$ $= P^x[X_{t+s} \in \Gamma \mid X_s]$ and if $X_s = y$, this is $U_t 1_\Gamma(y)$. ∎