①

Two Applications of Itô's Rule.

1) Lévy's Characterization of Brownian Motion

2) Martingale Moment Inequalities.

Lévy's Characterization Result.

Amazingly useful: it says that if $X \in M^{loc}$ and $\langle X \rangle_t = t$ then X is a B.M!

- actually, it even works in higher dimensions
- very useful way to check that a given process is a B.M.

Precise Setup and Statement.

$(\Omega, \mathcal{F}, P)$, $\mathbb{F}$ given (usual conditions)

Let d be an integer ≥ 1, μ a measure on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$.

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Let $X = \{X_t\}$ be a continuous adapted process taking values in $\mathbb{R}^d$. We say X is a d-dimensional B.M. with initial dist. μ if.

1) $X_0 \sim \mu$.

2) For $0 \le s < t$, $X_t - X_s \perp\!\!\!\perp \mathcal{F}_s$ and

$$X_t - X_s \sim N(\vec{0}_d, (t-s) 1_d)$$

$\vec{0}$: d-vector of 0's.

1_d : $d \times d$ identity matrix.

For $\mu = \delta_{\vec{0}}$ we can get a d-dim B.M. by putting d $\perp\!\!\!\perp$ B.M. together.

$$B = (B^1, \ldots, B^d) \quad \{B^i\} \perp\!\!\!\perp \text{ B.M.}$$

③

Now, let B be a d-dim B.M. for $i,j = 1,\dots,d$

$$E\left[B^i_t B^j_t \mid \mathcal{F}_s\right]$$

$$= \begin{cases} E\left[B^i_t \mid \mathcal{F}_s\right] E\left[B^j_t \mid \mathcal{F}_s\right] & i \neq j \\ E\left[(B^i_t)^2 \mid \mathcal{F}_s\right] & i = j \end{cases}$$

$$= \begin{cases} B^i_s B^j_s & i \neq j \\ (t-s) + (B^i_s)^2 & i = j \end{cases}$$

$$\Rightarrow \quad \langle B^i, B^j \rangle_t = \begin{cases} 0 & i \neq j \\ t & i = j \end{cases}$$

– by characterization of $\langle X, Y \rangle$ as unique process of bounded variation s.t. $XY - \langle X, Y \rangle \in M^2_c$ for $X, Y \in M^c_2$.

$\therefore \quad \langle B^i, B^j \rangle_t = t\,\delta_{i=j}$ for B d-dim B.M.

The Lévy Characterization goes in reverse!

Thm

Let $X = (X^1, \dots, X^d)$ be adapted, continuous s.t.

1) $M^i_t \triangleq X^i_t - X^i_0 \in \mathcal{M}^{c, loc}$ $\quad i = 1, \dots, d$

2) $\langle M^i, M^j \rangle_t = t\, \delta_{i=j}$ $\quad i,j = 1, \dots, d$

then X is a d-dim B.M. with initial dist μ = Law of $(X^1_0, \dots, X^d_0)$.

Neat Applications

1) B: 1-dim standard B.M. (start at 0)

$$X_t \triangleq \int_0^t \text{sign}(B_s)\, dB_s \qquad (\text{✓ } Y_s = \text{sign}(B_s) \in \mathcal{L}_B)$$

$$\Rightarrow X \in \mathcal{M}_2^c, \quad \langle X \rangle_t = \int_0^t \text{sign}^2(B_s)\, ds$$

$$= t$$

$\therefore$ X is a B.M.!

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2) More generally, if $M \in \mathcal{M}^{c,loc}$ s.t.

$\langle M \rangle_t = \langle \dot{M} \rangle_t \, dt$ $\quad \langle \dot{M} \rangle_t > 0$ a.s. progm'…

Set

$$X_t = \int_0^t \frac{1}{\sqrt{\langle \dot{M} \rangle_s}} \, dM_s$$

Since

$$\int_0^t \frac{1}{\langle \dot{M} \rangle_s} \langle \dot{M} \rangle_s \, ds = t$$

we know $\frac{1}{\sqrt{\langle \dot{M} \rangle}} \in \mathcal{P}^*_M$ and X is a B.M.

Proof of Lévy

Step 1: it suffices to show for $0 \le s < t$

$$* \quad E\left[e^{i u^T (M_t - M_s)} \,|\, \mathcal{F}_s\right] = e^{-\frac{1}{2}|u|^2 (t-s)} \text{ a.s.}$$

$u \in \mathbb{R}^d$

- note: if $M_t - M_s \perp\!\!\!\perp \mathcal{F}_s$, $M_t - M_s \sim N(\vec{0}, (t-s) I_d)$ then $*$ holds.
- reverse also holds.

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- this follows using regular conditional probabilities : see Lemma 2.6.13

Showing * is easy using Itô

$$f(x) = e^{i\mu^T x} = \cos(\mu^T x) + i\sin(\mu^T x)$$

$$\Rightarrow f_{x_i} = \mu_i(-\sin(\mu^T x) + i\cos(\mu^T x))$$

$$f_{x_i x_j} = -\mu_i\mu_j(\cos(\mu^T x) + i\sin(\mu^T x))$$

$$= -\mu_i\mu_j f(x)$$

So, applying Itô to real/imaginary parts separately:

$$e^{i\mu^T M_t} = e^{i\mu^T M_s} + \sum_{j=1}^{d} \mu_j \int_s^t (-\sin(\mu^T M_r) + i\cos(\mu^T M_r))\, dM^j_r$$

$$- \frac{1}{2}\sum_{j,k=1}^{d} \mu_j\mu_k \int_s^t e^{i\mu^T M_r}\, d\langle M^j, M^k\rangle_r$$

- since $|-\sin(\mu^T M_r) + i\cos(\mu^T M_r)| \leq 2$ the above local martingales are martingales

- since $\langle M^i, M^j\rangle_t = \delta_{ij} t$ we see that.

~~See next stopping~~

$\therefore \left\{ \int_0^\sigma e^{i\mu^T M_r} dM_r^i \right\}_{\sigma \in \mathcal{S}_T}$ is bounded in L^2,

hence, U.I. $\Rightarrow \int_0 e^{i\mu^T M_r} dM_r^i$ is of class DL.

Thus, multiplying the above by $e^{-i\mu^T M_s} 1_A$, $A \in \mathcal{F}_s$ and taking expectations yields

$$E\left[e^{i\mu^T(M_t - M_s)} 1_A\right] = P(A)$$

$$- \frac{1}{2} \sum_{i,j=1}^{d} \mu_i \mu_j \, E\left[e^{-i\mu^T M_s} 1_A \int_s^t e^{i\mu^T M_r} d\langle M^i, M^j \rangle_r \right]$$

$\left(\langle M^i, M^j \rangle_t = t \, \delta_{i=j} \right)$

$$= P(A) - \frac{1}{2} \sum_{i=1}^{d} \mu_i^2 \int_s^t E\left[e^{i\mu^T(M_r - M_s)} 1_A\right] dr$$

Set

$$g(t) = E\left[e^{i\mu^T(M_t - M_s)} 1_A\right] \qquad t \geq s.$$

$$\Rightarrow \quad g(t) = g(s) - \frac{1}{2}|u|^2 \int_s^t g(r)\,dr$$

$$\Rightarrow \quad g(t) = g(s)\, e^{-\frac{1}{2}|u|^2 (t-s)}$$

$$\Rightarrow \quad E\left[e^{i u^T (M_t - M_s)} 1_A\right] = P(A)\, e^{-\frac{1}{2}|u|^2 (t-s)} \quad \blacksquare.$$

Application Bessel Processes.

Let $W = (W^1, \ldots, W^d)$ be a d-dim B.M. starting at x.

Define R via

$$R_t = |W_t| = \sqrt{\sum_{i=1}^{d} (W_t^i)^2} \quad ; R_0 = |x|$$

Note: if $|x| = |y|$ then $y = Qx$ for Q orthogonal. Since one can show

$\hat{W}_t = Q W_t$ is a B.M. starting at y.

(c1)

$$\Rightarrow P(R^{(x)}_\cdot \in A) = P(|W_\cdot| \in A \mid |W_0| = x)$$

$$= P(|\tilde{W}_\cdot| \in A \mid |\tilde{W}_0| = y)$$

$$= P(R^{(y)}_\cdot \in A) \qquad A \in \mathcal{B}(C[0,\infty)).$$

$\Rightarrow$ dist of R depends only upon $r = |x|$.

We say R is a Bessel process of dimension d starting at $r > 0$.

<u>If</u> we could use Itô on $f(x) = \sqrt{\sum_{i=1}^{d} (x^i)^2}$ (i.e. no problems at 0)

$$\partial_i f = \frac{x^i}{|x|} \; ; \; \partial^2_{ij} f = \frac{\delta_{ij}}{|x|} - \frac{x^i x^j}{|x|^3}$$

$\Rightarrow$

$$dR_t = \sum_{i=1}^{d} \frac{W^i_t}{|W_t|} dW^i_t + \frac{1}{2} \sum_{i=1}^{d} \left(\frac{1}{|W_t|} - \frac{(W^i_t)^2}{|W_t|^3} \right) dt.$$

$$= \sum_{i=1}^{d} \frac{W^i_t}{|W_t|} dW^i_t + \frac{d-1}{2R_t} dt.$$

Now, if $M_t = \int_0^t \sum_{i=1}^d \frac{W_u^i}{|W_u|} dW_u^i$ then

$$\langle M \rangle_t = \sum_{i,j=1}^d \int_0^t \frac{W_u^i W_u^j}{|W_u|^2} d\langle W^i, W^j \rangle_u$$

$$= \sum_{i=1}^d \int_0^t \frac{(W_u^i)^2}{|W_u|^2} du$$

$$= t.$$

$$\therefore \quad M_t = B_t \quad \text{a B.M.}$$

<u>Thus</u>

$$R_t = r + \int_0^t \frac{d-1}{2R_u} du + B_t. \qquad *$$

Now, even though ~~f(x) = |x|~~ $f(x) = |x|$ has problems at 0, using an approximation argument ~~on~~ with a function $g^\varepsilon(y)$ which is C^2 and which $\to$ ~~$g(y)$~~ $\sqrt{y}$ as $\varepsilon \downarrow 0$, applied to $Y_t = \sum_{i=1}^d (W_t^i)^2$.

The Bessel process is interesting for many reasons

1) for $d \geq 2$, $r \geq 0$, the Bessel process R_t of dimension d, starting at r satisfies

1) $P(R_t > 0 \ \forall \ t \geq 0) = 1$

- multi-dim B.M. never hits origin

2) With $m = \inf_{t \geq 0} R_t$:

a) if $d = 2$, $m = 0$ a.s. $\forall \ r > 0$

2 dim B.M. comes arbitrarily close to the origin.

b) if $d \geq 3$, $r > 0$ then

$$P(m \leq c) = (c/r)^{d-2} \qquad 0 \leq c \leq r$$

- Beta distribution.

c) If $d \geq 3$, $r \geq 0$ then

$$P(\lim_{t \to \infty} R_t = \infty) = 1$$

- 3 dim B.M. wanders out to infinity

d) if $d \geq 3$, $M_t \triangleq \frac{1}{R_t^{d-2}}$, $t \geq 1$ and $r = 0$ then

1) M is a local martingale

2) $\sup_{t \geq 1} E[M_t^p] < \infty$ $\quad 0 < p < \frac{d}{d-2}$

so M is U.I.

3) M is not a martingale since $M_t \to 0$ a.s. M_t is U.I.

- each result follows by Itô. We will come back to these later.

Martingale Moment Inequalities.

Motivating Example:

$X \in \mathcal{L}_W$ (W B.M.) s.t. $E\left[\int_0^T |X_u|^{2m} du\right] < \infty$

for T > 0, m ≥ 1.

$$\Rightarrow \quad M_t = \int_0^t X_u \, dW_u \in \mathcal{M}^2_c$$

Now, $f(x) = |x|^{2m} \in C^2$ (O.K. b/c $m \geq 1$)

so

$$|M_t|^{2m} = 2m \int_0^t |M_u|^{2m-1} X_u \, dM_u + m(2m-1) \int_0^t |M_u|^{2(m-1)} X_u^2 \, du$$

Now, take $T_n \nearrow \infty$ so that $M^n_t = M_{t \wedge T_n}$ is s.t. $|M^n_t| \leq n$, $\langle M^n \rangle_t \leq n$. and $M = M^n$ on $t \leq T_n$.

$$\Rightarrow \quad E\left[|M^n_t|^{2m}\right] = m(2m-1) \, E\left[\int_0^{t \wedge T_n} |M^n_u|^{2(m-1)} X_u^2 \, du\right]$$

$$\leq m(2m-1) \, E\left[\int_0^t |M^n_u|^{2(m-1)} X_u^2 \, du\right]$$

$$\leq m(2m-1) \int_0^T E\left[|M^n_u|^{2m}\right]^{\frac{m-1}{m}} E\left[|X_u|^{2m}\right]^{1/m} du$$

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$$= T m(2m-1) \frac{1}{T}\int_0^T E[|M_u^n|^{2m}]^{\frac{m-1}{m}} E[|X_u|^{2m}]^{1/m}\, du$$

$$\le T m(2m-1)\left(\frac{1}{T}\int_0^T E[|M_u^n|^{2m}]\, du\right)^{\frac{m-1}{m}}$$

$$\times\left(\frac{1}{T}\int_0^T E[|X_u|^{2m}]\, du\right)^{\frac{1}{m}}$$

$\Rightarrow$

$$E[|M_T^n|^{2m}]^m \le T^{m-1}(m(2m-1))^m\left(\frac{1}{T}\int_0^T E[|M_u^n|^{2m}]\, du\right)^{m-1}$$

$$\int_0^T E[|X_u|^{2m}]\, du$$

$$\le T^{m-1}(m(2m-1))^m E[|M_T^n|^{2m}]^{m-1} \int_0^T E[|X_u|^{2m}]\, du$$

$\Rightarrow$

$$E[|M_T^n|^{2m}] \le T^{m-1}(m(2m-1))^m \int_0^T E[|X_u|^{2m}]\, du$$

$\Rightarrow$ $n \nearrow \infty$ (fatou)

$$E[|M_T|^{2m}] \le K_m \int_0^T E[|X_u|^{2m}]\, du$$

upper bounds on moments of M just

based upon the moments of the integrand X_u.

The Martingale-Moment, or Burkholder-Davis-Gundy (BGD) inequalities are strengthenings of these results.

<u>Thm.</u>

$M \in \mathcal{M}^{c,loc}$, $M_T^* \triangleq \max_{t \le T} |M_t|$. Then $\forall\, m > 0$, $\exists$ constants, depending only on m s.t. for T a stopping time.

$$k_m E[\langle M \rangle_T^m] \le E[(M_T^*)^{2m}] \le K_m E[\langle M \rangle_T^m]$$

at $m = 1/2$, we have

$$E[M_T^*] \le K_{1/2} E[\sqrt{\langle M \rangle_T}].$$

Thus, if $E[\sqrt{\langle M\rangle_\infty}] < \infty$ $\forall\, a>0$ then M is of class DL and M is a Martingale.

We first (essentially) prove this assuming $M, \langle M\rangle$ are bounded.

<u>Thm.</u>

Assume $M, \langle M\rangle$ bounded. Let T be a stopping time. Then

$$E[|M_T|^{2m}] \le C'_m\, E[\langle M\rangle_T^m] \qquad m>0$$

$$B_m E[\langle M\rangle_T^m] \le E[|M_T|^{2m}] \qquad m>1/2$$

$$B_m E[\langle M\rangle_T^m] \le E[(M_T^*)^{2m}] \le C_m\, E[\langle M\rangle_T^m] \quad m>1/2.$$

<u>pf.</u>

Itô: $Y_t = \delta + \varepsilon\langle M\rangle_t + M_t^2 \qquad \delta>0, \varepsilon\ge 0$

$f(x) = x^m.$ $\quad m>0$ (OK b/c $Y_t \ge \delta > 0$)

$$\Rightarrow Y_t^m = Y_0^m + m\int_0^t Y_u^{m-1}\, dY_u + \frac{1}{2}m(m-1)\int_0^t Y_u^{m-2}\, d\langle Y\rangle_u$$

$$= \delta^m + m\int_0^t Y_u^{m-1}\left(\varepsilon\, d\langle M\rangle_u + 2M_u\, dM_u + d\langle M\rangle_u\right)$$

$$+ \frac{1}{2}m(m-1)\int_0^t Y_u^{m-2}\, 4M_u^2\, d\langle M\rangle_u$$

$$= \delta^m + m(1+\varepsilon)\int_0^t Y_u^{m-1}\, d\langle M\rangle_u$$

$$+ 2m(m-1)\int_0^t Y_u^{m-2} M_u^2\, d\langle M\rangle_u$$

$$+ 2m\int_0^t Y_u^{m-1} M_u\, dM_u.$$

Now, M, $\langle M\rangle$ bounded imply (Y bdd, $Y \geq \delta$)

$$E\left[\int_0^t Y_u^{2(m-1)} M_u^2\, d\langle M\rangle_u\right] \leq K_Y^{2m-1} K_M^2 K_{\langle M\rangle}$$

$$\Rightarrow \left\{\int_0^t Y_u^{m-1} M_u\, dM_u\right\}_{t\geq 0} \text{ is u.i.}$$

$\Rightarrow$ Optional sampling: $E\left[\int_0^T Y_u^{m-1} M_u\, dM_u\right] = 0$

$\Rightarrow$ (18)

$$E[Y_T^m] = \delta^m + m(1+\varepsilon) E\left[\int_0^T Y_u^{m-1} d\langle M\rangle_u\right] + 2m(m-1) E\left[\int_0^T Y_u^{m-2} M_u^2 \, d\langle M\rangle_u\right]$$

or

$$E\left[(\varepsilon\langle M\rangle_T + (\delta + M_T^2))^m\right]$$

$$* \quad = \delta^m + m(1+\varepsilon) E\left[\int_0^T (\varepsilon\langle M\rangle_u + (\delta + M_u^2))^{m-1} d\langle M\rangle_u\right] + 2m(m-1) E\left[\int_0^T (\varepsilon\langle M\rangle_u + (\delta + M_u^2))^{m-2} M_u^2 \, d\langle M\rangle_u\right]$$

* Big Equality from which results follow.

~~Note~~

~~$\varepsilon = 0 \Rightarrow$~~

~~$$E[(\delta + M_T^2)^m] = \delta^m + m E\left[\int_0^T (\delta + M_u^2)^{m-1} d\langle M\rangle_u\right] + 2m(m-1) E\left[\int_0^T (\delta + M_u^2)^{m-2} M_u^2 \, d\langle M\rangle_u\right]$$~~

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Too technical to do all cases, so lets focus on $m > 1$

Lower Bound.

OK to pull limit...

↓

In *, $2m(m-1) \geq 0$ so at $\delta = 0$

$$E\left[\left(\varepsilon\langle M\rangle_T + M_T^2\right)^m\right]$$

$$\geq m(1+\varepsilon)\, E\left[\int_0^T \left(\varepsilon\langle M\rangle_u + M_u^2\right)^{m-1} d\langle M\rangle_u\right]$$

$$\geq m(1+\varepsilon)\varepsilon^{m-1} E\left[\int_0^T \langle M\rangle_u^{m-1} d\langle M\rangle_u\right] \quad (m>1)$$

$$= (1+\varepsilon)\varepsilon^{m-1} E\left[\langle M\rangle_T^m\right]$$

Now, $f(x) = x^m$ is convex if $m > 1$

$$\Rightarrow \quad x^m + y^m \geq 2^{1-m}(x+y)^m$$

$$\Rightarrow \quad \varepsilon^m E\left[\langle M\rangle_T^m\right] + E\left[|M|_T^{2m}\right]$$

$$\geq (1+\varepsilon)\varepsilon^{m-1} E\left[\langle M\rangle_T^m\right] \cdot 2^{1-m}$$

$\Rightarrow$ (20)

$$E\left[|M|_T^{2m}\right] \geq \underbrace{\left((1+\varepsilon)\left(\varepsilon/2\right)^{m-1} - \varepsilon^m\right)}_{B_m} E\left[\langle M\rangle_T^m\right]$$

- provided $\varepsilon > 0$ small enough.

<u>Upper Bound.</u>

* at $\varepsilon = \delta = 0$

$$E\left[|M|_T^{2m}\right] = m\, E\left[\int_0^T |M|_u^{2(m-1)}\, d\langle M\rangle_u\right] + 2m(m-1)\, E\left[\int_0^T |M_u|^{2(m-1)}\, d\langle M\rangle_u\right]$$

$$= 2m(m-1/2)\, E\left[\int_0^T |M_u|^{2(m-1)}\, d\langle M\rangle_u\right].$$

Using i) $x^m + y^m \geq 2^{1-m}(x+y)^m$

ii) $E\left[\left(\varepsilon\langle M\rangle_T + (\delta + M_T^2)\right)^m\right]$

$$\geq \delta^m + m(1+\varepsilon)\, E\left[\int_0^T \left(\varepsilon\langle M\rangle_u + (\delta + M_u^2)\right)^{m-1} d\langle\ \rangle\right]$$

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We got

$$2^{m-1} E\left[\varepsilon^m \langle M\rangle_T^m + (\delta + M_T^2)^m\right]$$

$$\geq \delta^m + m(1+\varepsilon)\, E\left[\int_0^T \left(\varepsilon\langle M\rangle_u + (\delta+M_u^2)\right)^{m-1} d\langle M\rangle_u\right]$$

$$\geq \delta^m + m(1+\varepsilon)\, E\left[\int_0^T (\delta+M_u^2)^{m-1} d\langle M\rangle_u\right]$$

at $\delta = 0$

$$2^{m-1} E\left[\varepsilon^m \langle M\rangle_T^m + |M|_T^{2m}\right]$$

$$\geq m(1+\varepsilon)\, E\left[\int_0^T |M_u|^{2(m-1)} d\langle M\rangle_u\right]$$

$$= \frac{m(1+\varepsilon)}{2m(m-1/2)}\, E\left[|M_T|^{2m}\right]$$

$\Rightarrow$

$$E\left[|M_T|^{2m}\right] \leq \underbrace{\frac{2^{m-1}\varepsilon^m}{\frac{(1+\varepsilon)}{2(m-1/2)} - 2^{m-1}}}_{C'_m} E\left[\langle M\rangle_T^m\right]$$

C'_m provided ε large enough

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Now, assumed we have B_m, C'_m s.t.

$$B_m E[\langle M\rangle_T^m] \le E[|M_T|^{2m}] \le C'_m E[\langle M\rangle_T^m]$$

consider the continuous martingale $M_{t\wedge T}$:

$$\Rightarrow B_m E[\langle M\rangle_{T\wedge t}^m] \le E[|M_{T\wedge t}|^{2m}]$$

$$\le E[(M^*_{T\wedge t})^{2m}]$$

$$\le \left(\frac{2m}{2m-1}\right)^{2m} E[|M_{T\wedge t}|^{2m}] \quad (2m > 1)$$

$X_t = |M_{T\wedge t}|$ sub-mart.

$$\le \underbrace{\left(\frac{2m}{m-1}\right)^{2m} C'_m}_{C_m} E[\langle M\rangle_{T\wedge t}^m]$$

$\sup_{\tau\le t} X_t = M^*_t$

$$\Rightarrow B_m E[\langle M\rangle_{T\wedge t}^m] \le E[(M^*_{T\wedge t})^{2m}]$$

$$\le C_m E[\langle M\rangle_{T\wedge t}^m]$$

as $t\nearrow\infty$ we got result by M.C.T.

Now Assume for $M, \langle M \rangle$ bdd we have

$$B_m \, E[\langle M \rangle_T^m] \le E[(M_T^*)^{2m}] \le C_m \, E[\langle M \rangle_T^m]$$

Let $M \in \mathcal{M}^{c,loc}$ $T_n \nearrow \infty$ s.t. $M_t^n = M_{t \wedge T_n}$ is bdd with $\langle M \rangle_{t \wedge T_n}$ bdd.

$\max_{t \le T \wedge T_n} M_t = \max_{t \le T \wedge T_n} M_{t \wedge T_n}$

$$\Rightarrow B_m \, E[\langle M \rangle_{T \wedge T_n}^m] \le E[(M_{T \wedge T_n}^*)^{2m}]$$

$$\le C_m \, E[\langle M \rangle_{T \wedge T_n}^m]$$

Again By MCT we can take $n \nearrow \infty$ and preserve the inequalities.

Note: extension of results to $0 < m \le 1/2$ is technical