①

Construction of the Stochastic Integral

Goal: Assign meaning to and analyze the object

$$I_t = \int_0^t X_u \, dM_u \qquad t \geq 0$$

X: given (adapted, ???) process

$M \in \mathcal{M}_2^c$

Problem: with probability one, the paths of M have unbounded variation (first) so I cannot be defined pathwise.

Workaround: exploit the Itô isometry, density of simple processes to define I as an L^2 limit.

Plan: 1) construct I for simple processes.

2) Extend I to "general" processes when $M = W$, a B.M.

3) Extend I to "general" processes for general $M \in \mathcal{M}_2^c$, $M \in \mathcal{M}^{c,loc}$.

Preliminaries:

$(\Omega, \mathcal{F}, P)$, $\mathbb{F}$ given. $\mathbb{F}$: satisfies usual conditions (important here).

$M \in \mathcal{M}_2^c$

A) Integral for Simple Processes.

We say $X = \{X_t\}_{t \geq 0}$ is <u>simple</u> if X admits the form

$$X_t(\omega) = 1_0(t)\xi_0(\omega) + \sum_{i=0}^{\infty} \xi_i(\omega) 1_{(t_i, t_{i+1}]}(t)$$

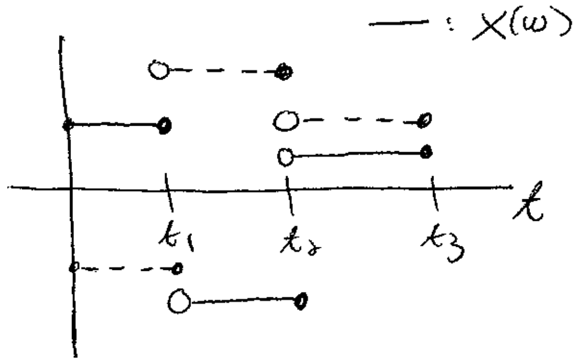
with

i) $t_i \nearrow \infty$ ii) ξ_i is $\mathcal{F}_{t_i}$ mbl

iii) $\sup_{i,\omega} |\xi_i(\omega)| \leq C < \infty$.

3)

Simple processes are random step functions



note: 1) X is left continuous, bounded

2) X is progressively measurable

For X simple, $M \in \mathcal{M}_2^c$ we define the integral pathwise by

left endpoint

$$I_t(\omega) = I(X)_t(\omega) \triangleq \sum_{i=0}^{\infty} \xi_i(\omega)\left(M_{t \wedge t_{i+1}} - M_{t \wedge t_i}\right)(\omega)$$

- essentially the discrete time martingale transform.

- note: $0 \le t < t_1$: $I_t = \xi_0 M_t$

$t_1 \le t < t_2$: $I_t = \xi_0 M_{t_1} + \xi_1(M_t - M_{t_1})$

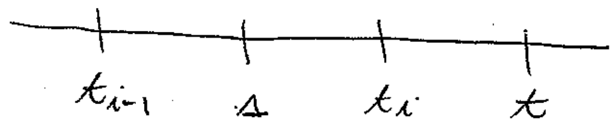
$\vdots$

Properties of I for X simple.

1) $I_0 = 0$ a.s. (clear)

2) I is a martingale.

pf.



(other cases similar)

$$I_t = \sum_{j=0}^{i-1} \xi_j (M_{t_{j+1}} - M_{t_j}) + \xi_i (M_t - M_{t_i})$$

$$I_s = \sum_{j=0}^{i-2} \xi_j (M_{t_{j+1}} - M_{t_j}) + \xi_{i-1} (M_s - M_{t_{i-1}})$$

$$\Rightarrow \quad I_t - I_s = \xi_i (M_t - M_{t_i}) + \xi_{i-1}(M_{t_i} - M_{t_{i-1}}) - \xi_{i-1}(M_s - M_{t_{i-1}})$$

$$= \xi_i (M_t - M_{t_i}) + \xi_{i-1}(M_{t_i} - M_s)$$

$$E[\xi_i (M_t - M_{t_i}) \mid \mathcal{F}_s] = E[\xi_i \, E[M_t - M_{t_i} \mid \mathcal{F}_{t_i}] \mid \mathcal{F}_s] = 0$$

$$E[\xi_{i-1}(M_{t_i} - M_s) \mid \mathcal{F}_s] = \xi_{i-1} E[M_{t_i} - M_s \mid \mathcal{F}_s] = 0$$

$\blacksquare$

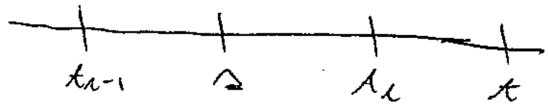
3) I is continuous (clear)

4) $I \in \mathcal{M}_2^c$ with $E[I_t^2] = E[\int_0^t X_u^2 \, d\langle M\rangle_u]$

and $\langle I \rangle_t = \int_0^t X_u^2 \, d\langle M\rangle_u$

- Itô isometry.

pf.



(other cases similar)

$$E[(I_t - I_s)^2 \mid \mathcal{F}_s]$$

$$= E\Big[\xi_i^2 (M_t - M_{t_i})^2 + 2\xi_i\xi_{i-1}\cancelto{0}{(M_t - M_{t_i})}(M_{t_i} - M_s) + \xi_{i-1}^2 (M_{t_i} - M_s)^2 \,\Big|\, \mathcal{F}_s\Big]$$

$$= E\Big[\xi_i^2(\langle M\rangle_t - \langle M\rangle_{t_i}) + \xi_{i-1}^2(\langle M\rangle_{t_i} - \langle M\rangle_s) \,\Big|\, \mathcal{F}_s\Big]$$

$$+ E\Big[\xi_i^2\big(\cancelto{0}{(M_t - M_{t_i})^2 - (\langle M\rangle_t - \langle M\rangle_{t_i})}\big) + \xi_{i-1}^2\big(\cancelto{0}{(M_{t_i} - M_s)^2 - (\langle M\rangle_{t_i} - \langle M\rangle_s)}\big) \,\Big|\, \mathcal{F}_s\Big]$$

⑥

$$= E\left[\int_{\Delta}^{t} X_u^2 \, d\langle M\rangle_u \,\middle|\, \mathcal{F}_\Delta \right].$$

$$- \quad X_u = \xi_{i-1} \qquad u \in (\Delta, t_i]$$

$$= \xi_i \qquad u \in (t_i, t].$$

$\therefore$

1) $$E[I_t^2] = E[E[I_t^2 \mid \mathcal{F}_0]]$$

$$= E\left[E\left[\int_0^t X_u^2 \, d\langle M\rangle_u \,\middle|\, \mathcal{F}_0 \right]\right]$$

$$= E\left[\int_0^t X_u^2 \, d\langle M\rangle_u \right]$$

$\Rightarrow \quad E[I_t^2] < \infty \quad \forall t \quad$ since $|X| \le C$.

$\Rightarrow \quad I \in \mathcal{M}_2^c$

2) $\left(I_t^2 - \int_0^t X_u^2 \, d\langle M\rangle_u \right)_{t \ge 0}$ is a martingale

$$\Rightarrow \langle I\rangle_t = \int_0^t X_u^2 \, d\langle M\rangle_u. \quad \blacksquare$$

(subtle point: X prog. mbl ~~⇒~~ simple $\Rightarrow \int_0^t X_u^2 d\langle M\rangle_u$ adapted)

5) I is linear in that $I(\alpha X + BY)$

$= \alpha I(X) + B I(Y)$

- clear after enlarging partition to contain both $\{t_n\}$, $\{s_n\}$.

Summarizing

X simple, $M \in \mathcal{M}_2^c$ implies $I(X) \in \mathcal{M}_2^c$ with $\langle I \rangle_t = \int_0^t X_u^2 \, d\langle M \rangle_u$. The map $X \to I(X)$ is linear.

Extending I to General Integrands.

Basic Idea: find X^n simple s.t. $X^n \to X$ in some sense. Then, show that I^n converges to some process $\hat{I}$. Define $I(X) = \hat{I}$.

Approximation.

Lemma.

Let $X = \{X_t\}_{t \geq 0}$ be bounded (in (t, ω)) adapted, measurable. There is a sequence of simple processes s.t.

$$\sup_{T>0} \lim_{n \to \infty} E\left[\int_0^T |X_t^n - X_t|^2 \, dt\right] = 0$$

pf.

suffices to find $\{X^{(n,T)}\}_n$ simple s.t.

$$\lim_{n \to \infty} E\left[\int_0^T |X_t^{n,T} - X_t|^2 \, dt\right] = 0 \qquad \forall T > 0$$

$\Rightarrow \forall m, \exists n_m$ s.t.

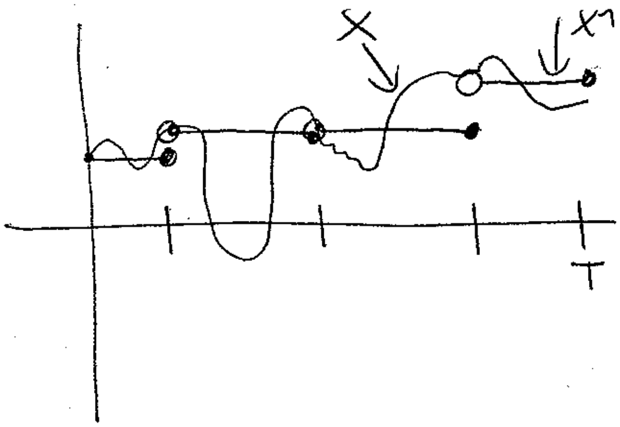
$$E\left[\int_0^m |X^{m, n_m, m} - X_t|^2 \, dt\right] \leq 1/n$$

so $\{X^{m, n_m}\}$ works.

(9)

fix $T>0$. a) If X is continuous set

$$X^n_t(\omega) = X_0(\omega)\mathbb{1}_{\{0\}}(t) + \sum_{k=0}^{2^n-1} X_{\frac{kT}{2^n}}(\omega)\mathbb{1}_{\left(\frac{kT}{2^n}, \frac{(k+1)T}{2^n}\right]}(t).$$



X^n simple b/c X bounded, adapted, mbl.

$$E\left[\int_0^T |X^n_t - X_t|^2\,dt\right] \to 0$$ b/c bounded conv thm.

b) If X is progressively measurable then

1) for fixed m, approximate X by $\tilde{X}^m$ via.

$$F_t(\omega) = \int_0^{t\wedge T} X_s(\omega)\,ds \quad ;$$

$$\tilde{X}_t^{(m)} = m\left(F_t(\omega) - F_{(t-1/m)\vee 0}(\omega)\right)$$

(10)

technical: need X prog mbl to have F adapted, hence X adapted

$\tilde{X}^m$ is continuous, bounded so $\exists$ $\tilde{X}^{n,m}$

s.t. $\lim_n E\left[\int_0^T |\tilde{X}^{n,m}_t - \tilde{X}^m_t|^2 \, dt\right] = 0.$

But, for fixed ω the set

$A_\omega = \{t \le T \mid \lim_m \tilde{X}^m_t(\omega) \ne X_t(\omega)\}$ or Lim d.n.e.

is $B[0,T]$ mbl (X prog. mbl) and has Lebesgue measure 0 (FTOC)

$\Rightarrow \tilde{X}^m_t \mapsto X_t$ a.s. $P \times Leb[0,T]$.

$\Rightarrow \lim_{m\to\infty} E\left[\int_0^T |\tilde{X}^m_t - X_t|^2 \, dt\right] = 0$ by

bounded convergence

$\Rightarrow \exists$ X^n simple s.t. $\lim_n E\left[\int_0^T |X^n_t - X_t|^2 \, dt\right] = 0$

- double subsequences.

(12)

c) X bounded, mbl, adapted

- $F_t = \int_0^{t\wedge T} X_s\,ds$ might not be adapted

- X has a prog mbl modification. Y

a) $G_t = \int_0^{t\wedge T} Y_s\,ds$ is adapted.

b) $P(G_t = F_t) = 1 \quad \forall t.$

c) F_t is adapted since $\mathcal{F}_t$ contains all P null sets.

- repeat argument from b).

Extension of Integral for $M=W$

Notation:

$$\mu_W(A) \triangleq E\left[\int_0^\infty 1_A(t,\omega)\,dt\right] \quad A \in \mathcal{B}[0,\infty)\otimes\mathcal{F}.$$

$$[X]_T^2 = E\left[\int_0^T X_t^2\,dt\right]$$

(L^2 norm on $\Omega\times[0,T]$ for μ_W.

$$[X] = \sum_{n=1}^\infty \frac{1\wedge[X]_n}{2^n}$$

(13)

$\mathcal{L}_W = \{X \text{ mbl, adapted s.t. } [X]_T < \infty \ \forall T \geq 0\}$

$[\ ]$: metric on $\mathcal{L}$.

$\mathcal{L}_0 = \{X \in \mathcal{L} \mid X \text{ simple}\} = \{X \text{ simple}\}$.

<u>Lemma</u>

$\mathcal{L}_0$ is dense in $\mathcal{L}_W$.

pf/

If $X \in \mathcal{L}_W$ is bounded, it follows by previous lemma. Else, fix n and define

$$X_t^n(\omega) = X_t(\omega) \mathbb{1}_{|X_t(\omega)| \leq n}$$

By Dominated Convergence.

$$[X^n - X]_T^2 = E\left[\int_0^T X_t^2 \mathbb{1}_{|X_t| > n} \, dt\right] \xrightarrow[n \to \infty]{} 0$$

Thus, result follows by approximating X^n and then X.

(14)

Note: if $t \mapsto \langle M \rangle_t$ is a.c. w/ prob 1 the above result also works for

$$\mathcal{L}_M = \{ X \text{ mbl adapted s.t. } [X]^{(M)} < \infty \}$$

$$[X](M)^2_T = E\left[\int_0^T X_t^2 \, d\langle M \rangle_t \right]$$

$$[X](M) = \sum_n \frac{1 \wedge [X](M)_n}{2^n}$$

Idea: X bounded, adapted, mbl $\Rightarrow X^n \to X$ in that $\sup_{T>0} E\left[\int_0^T |X^n_t - X_t|^2 dt \right] = 0$

$\Rightarrow \exists X^{n_k}$ s.t. $P \otimes Leb\left([\{t,\omega\} \mid X^{n_k}_t(\omega) = X_t(\omega)]^c \right) \to 0$

$\Rightarrow [X - X^n](M)_T = E\left[\int_0^T |X^n_t - X_t|^2 \langle \dot{M} \rangle_t \, dt \right]$

$\to 0$ by bounded convergence.

15

Integral for $X \in \mathcal{L}_W$:

- recall the metric $\|\cdot\|$ for $\mathcal{M}_2^c$ given by $\|X\| = \sum_{n=1}^{\infty} \frac{1 \wedge E[X_n^2]}{2^n}$

- Itô Isometry: X, Y simple implies for ~~$I \in \mathcal{M}_2^c$~~ $I(X), I(Y) \in \mathcal{M}_2^c$ that

$$E\left[(I(X)-I(Y))_t^2\right]$$

$$= E\left[I(X-Y)_t^2\right] \quad \text{(linearity)}$$

$$= E\left[\int_0^t (X-Y)_u^2 \, du\right] \quad \text{(Isometry for } \langle M \rangle_t = t\text{)}$$

$$= [X-Y]_t$$

$$\Rightarrow \|I(X)-I(Y)\| = [X-Y]$$

So, here is what we do:

Let $X \in \mathcal{L}_W$. Take $X^n \in \mathcal{L}_0$ s.t. $[X^n - X] \to 0$. Thus

$$\|I(X^n) - I(X^m)\| = [X^n - X^m] \to 0 \quad n, m \to \infty$$

$\Rightarrow \{I(X^n)\}$ is Cauchy in $(\mathcal{M}_2^c, \|\cdot\|)$

$\Rightarrow \exists$ a process $I \in \mathcal{M}_2^c$ ← unique up to indistinguishability s.t.

$$\|I(X^n) - I(X)\| \to 0$$

We define the stochastic integral to be $I(X)$ and write

$$I(X)_t = \int_0^t X_u \, dW_u \qquad t \geq 0.$$

1) $I \in \mathcal{M}_2^c \Rightarrow I_0 = 0$ a.s., I is a martingale.

(17)

11) Fix $s < t$. Note $\{I_t(X^n)\}_n$, $\{I_s(X^n)\}_n$ converge in mean-square to $I_t(X)$, $I_s(X)$.

$$\Rightarrow E[1_A(I(X)_t - I(X)_s)^2]$$

$$= \lim_n E[1_A(I(X^n)_t - I(X^n)_s)^2]$$

$$= \lim_n E[1_A \int_s^t (X_u^n)^2 du] \quad \text{(Isometry)}$$

$$= E[1_A \int_s^t X_u^2 du]$$

$$\Rightarrow E[(I(X)_t - I(X)_s)^2 | \mathcal{F}_s] = E[\int_s^t X_u^2 du | \mathcal{F}_s] *$$

now

for $X \in \mathcal{L}$, one can show $\int_0^\cdot X_u^2 du$ is adapted

$$\Rightarrow \langle I \rangle_t = \int_0^t X_u^2 du$$

Also $*$ gives

$$E[I(X)_t^2] = E[\int_0^t X_u^2 du] \Rightarrow \|I(X)\| = [X].$$

Lastly, linearity and that $I(X)$ is well defined are clear.

Integral for General $M \in \mathcal{M}_2^c$.

- If $t \mapsto \langle M \rangle_t$ is a.c. then since $\mathcal{L}_0$ is dense in $\mathcal{L}_M$ w.r.t. $[\cdot](M)$ the construction of I is the exact same

1) Take $X^n \in \mathcal{L}_0$ s.t. $[X^n - X](M) \to 0$

2) $\| I(X^n) - I(X^m) \| = [X^n - X^m](M) \to 0$

$$\therefore \quad I(X^n) \mapsto I(X) \in \mathcal{M}_2^c$$

$$I(X)_t \stackrel{\Delta}{=} \int_0^t X_u \, dM_u.$$

$$\langle I \rangle_t = \int_0^t X_u^2 \, d\langle M \rangle_u$$ since $\int_0^t X_u^2 \, d\langle M \rangle_u$ is adapted

$$\| I(X) \| = [X](M).$$

- If $t \mapsto \langle M \rangle_t$ is <u>not</u> a.c. we slightly modify our argument.

1) Set $\mathcal{L}_M^* = \{X$ prog. mbl s.t. $[X]_T(M) < \infty$ $\forall_T > 0\}$

$$[X](M)_T^2 = E\left[\int_0^T X_t^2 \, d\langle M \rangle_t\right].$$

2) Show $\mathcal{L}_0$ is dense in $\mathcal{L}_M^*$ w.r.t. $[\cdot](M)$

- technical proof on pgs 135 -> 137.

3) For $X \in \mathcal{L}_M^*$ construct $I(X)$ exactly as before

$$\Rightarrow \quad I(X) \in \mathcal{M}_2^c$$

$$\langle I \rangle_t = \int_0^t X_u^2 \, dM_u$$

Analysis of the Integral.

Going forward we assume $X \in \mathcal{L}_M^*$ for $M \in \mathcal{M}_2^c$. Results also hold for $X \in \mathcal{L}_M$ if $t \to \langle M \rangle_t$ is a.c. a.s.

<u>Prop</u>

Let $X, Y \in \mathcal{L}_M^*$, $S \le T$ stopping. Then

1) $$E\left[(I_{t\wedge T}(X) - I_{t\wedge S}(X))(I_{t\wedge T}(Y) - I_{t\wedge S}(Y)) \mid \mathcal{F}_S\right]$$

$$= E\left[\int_{t\wedge S}^{t\wedge T} X_u Y_u \, d\langle M \rangle_u \mid \mathcal{F}_S\right] \quad t > 0.$$

$$\Rightarrow \quad S = s \le t$$

$$E\left[(I_t(X) - I_s(X))(I_t(Y) - I_s(Y)) \mid \mathcal{F}_s\right]$$

$$= E\left[\int_s^t X_u Y_u \, d\langle M \rangle_u \mid \mathcal{F}_s\right]$$

$$\Rightarrow \langle I(X), I(Y) \rangle_t = \int_0^t X_u Y_u \, d\langle M \rangle_u.$$

(21)

pf Since $I(X)$ is a continuous martingale we have $E[I_{t\wedge T}(X) \mid \mathcal{F}_s] = I_{t\wedge s}(X)$

pf (Prob 1.3.24 (ii)) Opt. Sampl. gives

$$E[I_{t\wedge T}(X) \mid \mathcal{F}_{s\wedge t}] = I_{s\wedge t}(X).$$

on $\{s \le t\}$, prob 1.2.17 (i) at $T=S$, $S=t$ gives

$$E[I_{t\wedge T}(X) \mid \mathcal{F}_{s\wedge t}] = E[I_{t\wedge T}(X) \mid \mathcal{F}_s]$$

on $\{s>t\} \in \mathcal{F}_{s\wedge t}$ for $A \in \mathcal{F}_{s\wedge t}$ we have

$$E[1_A I_{t\wedge T}(X) 1_{s>t}] = E[1_A I_{t\wedge s}(X) 1_{s>t}]$$

so on $\{s>t\}$,

$$E[I_{t\wedge T}(X) \mid \mathcal{F}_{s\wedge t}] = I_{t\wedge s}(X) \blacksquare.$$

Applying this same result to the Martingale $I_t(X)^2 - \langle I \rangle_t$ ~~gives~~ for $X \in \mathcal{L}^*_n$ gives.

(22)

$$E[(I_{t\wedge T}(\tilde{X}) - I_{t\wedge S}(\tilde{X}))^2 \mid \mathcal{F}_S]$$

$$= E[I_{t\wedge T}(\tilde{X})^2 - I_{t\wedge S}(\tilde{X})^2 \mid \mathcal{F}_S]$$

$$= E\left[\int_{t\wedge S}^{t\wedge T} \tilde{X}_u^2 \, d\langle M\rangle_u \mid \mathcal{F}_S\right]$$

Now, use this for $\tilde{X} = \frac{1}{4}(X+Y)$ and $\tilde{X} = \frac{1}{4}(X-Y)$ and subtract. ∎

ii) $I_{t\wedge T}(X) = I_t(\tilde{X})$ $\quad \tilde{X}_t = X_t 1_{t\le T}$

stopped integral = integral for killed integrand

pf

$$E[(I_{t\wedge T}(X-\tilde{X}) - I_{\Delta\wedge T}(X-\tilde{X}))^2 \mid \mathcal{F}_\Delta]$$

$$= E\left[\int_{\Delta\wedge T}^{t\wedge T} (X-\tilde{X})_u^2 \, d\langle M\rangle_u \mid \mathcal{F}_\Delta\right] = 0$$

- note this follows from i) by first using i) at $S = \Delta\wedge T \le T$ to get result for $\mathcal{F}_{\Delta\wedge T}$ then using problem 1.2.17 (i) to switch to $\mathcal{F}_\Delta$ on $\{\Delta \le T\}$

23

then noting that on $\{\Delta > T\}$ both sides of the above are 0.

$\Rightarrow$ $I_{\cdot \wedge T}(X - \tilde{X})$ has 0 quad var

$\Rightarrow$ $I_{\cdot \wedge T}(X - \tilde{X}) = 0$ (indistinguishability).

Similarly, for $N_t = I_t(\tilde{X}) - I_{t \wedge T}(\tilde{X}) \in \mathcal{M}_2^c$

$$E[N_t^2] = E\left[\int_{t \wedge T}^{t} \tilde{X}_u^2 \, d\langle M \rangle_u\right] \quad \text{(opt. sampling)}$$

$$= 0$$

so

$$I_{t \wedge T}(X) - I_t(\tilde{X}) = I_{t \wedge T}(X - \tilde{X}) - (I_t(\tilde{X}) - I_{t \wedge T}(\tilde{X})) = 0$$

(indistinguishable) ∎.

Thm.

$M, N \in \mathcal{M}_2^c$. $X \in \mathcal{L}_M^*$, $Y \in \mathcal{L}_N^*$.

Then

$$\langle I^M(X), I^N(Y) \rangle_t = \int_0^t X_u Y_u \, d\langle M, N \rangle_u$$

(24)

Very important/ Practical Result.

pf $X, Y \in \mathcal{L}_0$ (simple), this follows via a direct calculation.

e.g. $X_t = \xi_0 \mathbb{1}_{t \le t_0}$; $Y_t = \theta_0 \mathbb{1}_{t \le t_1}$

$$\Rightarrow I^M(X)_t = \xi_0 M_{t \wedge t_0}$$

$$I^N(Y)_t = \theta_0 N_{t \wedge t_1}$$

$$\Rightarrow \langle I^M(X), I^N(Y) \rangle_t = \xi_0 \theta_0 \int_0^{t \wedge t_0 \wedge t_1} d\langle M, N \rangle_u$$

$$= \int_0^t \xi_0 \mathbb{1}_{u \le t_0} \theta_0 \mathbb{1}_{u \le t_1} \, d\langle M, N \rangle_u$$

$$= \int_0^t X_u Y_u \, d\langle M, N \rangle_u.$$

to extend to $X \in \mathcal{L}_M^*$, $Y \in \mathcal{L}_N^*$ we proceed as follows:

25

1) (Kunita + Watanabe) with Prob 1

$$\int_0^t |X_u||Y_u|\, d\check{\xi}_u \le \left(\int_0^t |X_u|^2 d\langle M\rangle_u\right)^{1/2}$$

$$\times\left(\int_0^t |Y_u|^2 d\langle N\rangle_u\right)^{1/2}$$

$\check{\xi}$: total variation (ω-by-ω) of $\langle M, N\rangle$.

pf

$\check{\xi}(\omega) << \varphi(\omega) \triangleq \frac{1}{4}(\langle M\rangle(\omega) + \langle N\rangle(\omega))$

$\langle M\rangle(\omega) << \varphi(\omega)$

$\langle N\rangle(\omega) << \varphi(\omega)$

$\Rightarrow$ (think ω fixed in a set of prob 1)

$$\langle M\rangle_t(\omega) = \int_0^t f_1(s,\omega)\, d\varphi_s(\omega)$$

$$\langle N\rangle_t(\omega) = \int_0^t f_2(s,\omega)\, d\varphi_s(\omega)$$

$$\langle M, N\rangle_t(\omega) = \int_0^t f_3(s,\omega)\, d\varphi_s(\omega)$$

- R/N Derivatives of random measure on $(0,\infty)$.

(26)

for $\alpha, \beta \in \mathbb{Q}$ $\exists$ $\Omega^{\alpha,\beta}$ w/ prob 1

s.t.

$$0 \le \langle \alpha M + \beta N \rangle_t - \langle \alpha M + \beta N \rangle_s (\omega)$$

$$= \int_s^t \left(\alpha^2 f_1(\tau,\omega) + 2\alpha\beta f_3(\tau,\omega) + \beta^2 f_2(\tau,\omega) \right) d\varphi_\tau(\omega)$$

$\Rightarrow$ $\forall$ $\tau \notin T_{\alpha,\beta}(\omega) \subseteq [0,\infty)$ with

$$\int_{T_{\alpha,\beta}(\omega)} d\varphi_\tau(\omega) = 0$$ we have

$$\alpha^2 f_1(\tau,\omega) + 2\alpha\beta f_3(\tau,\omega) + \beta^2 f_2(\tau,\omega) \geq 0$$

Set $\tilde{\Omega} = \bigcap_{\alpha,\beta \in \mathbb{Q}} \Omega^{\alpha,\beta}$; $\tilde{T}(\omega) = \bigcup_{\alpha,\beta \in \mathbb{Q}} T_{\alpha,\beta}(\omega)$

for $\omega \in \tilde{\Omega}$, $\tau \notin \tilde{T}(\omega)$ we have
$\forall$ $\alpha, \beta \in \mathbb{Q}$ that

$$* \quad \alpha^2 f_1(\tau,\omega) + 2\alpha\beta f_3(\tau,\omega) + \beta^2 f_2(\tau,\omega) \geq 0.$$

(27)

Thus $*$ holds $\forall\ \alpha, \beta \in \mathbb{R}$.

Now, if $f_3(\tau, \omega) \geq 0$ take

$\alpha = a|X_\tau(\omega)|, \quad \beta = |Y_\tau(\omega)|$

$\Rightarrow$

$$0 \leq a^2 |X_\tau(\omega)|^2 f_1(\tau,\omega) + 2a|X_\tau(\omega)||Y_\tau(\omega)||f_3(\tau,\omega)|$$

$**$ $$+ |Y_\tau(\omega)|^2 f_2(\tau,\omega).$$

If $f_3(\tau,\omega) \leq 0$ take $\alpha = a|X_\tau(\omega)|$

$\beta = -|Y_\tau(\omega)|$ so $**$ holds as

well.

Thus, integrating on $\int_0^t$ against $d\varphi_\tau(\omega)$

and plugging back in.

$$0 \leq a^2 \int_0^t |X_u|^2 d\langle M\rangle_u + 2a \int_0^t |X_u|\cdot|Y_u|\, d\check{S}_u$$

$$+ \int_0^t |Y_u|^2 d\langle N\rangle_u \qquad (\text{a.s. } t \geq 0)$$

result follows by minimizing over a.

(28)

2) For $M, N \in \mathcal{M}_2^c$, $X \in \mathcal{L}_M^*$, $Y \in \mathcal{L}_N^*$:

$$\langle I^M(X), I^N(Y) \rangle_t = \int_0^t X_u Y_u \, d\langle M, N \rangle_u.$$

<u>pf</u>

$X \in \mathcal{L}_M^*$ implies $\exists$ $\{X^n\} \subseteq \mathcal{L}_0$ s.t.

$$\sup_T \lim_n E\left[\int_0^T |X_u^n - X_u|^2 d\langle M \rangle_u\right] = 0$$

$\Rightarrow$ $\forall$ T $\exists$ sub-seq. (still labeled n) s.t.

$$\lim_n \int_0^T |X_u^n - X_u|^2 d\langle M \rangle_u = 0 \quad \text{a.s.}$$

$\Rightarrow$ $t \leq T$ gives

$$|\langle I^M(X^n), N \rangle_t - \langle I^M(X), N \rangle_t|$$

$$\leq \langle I^M(X^n - X) \rangle_t \langle N \rangle_t \quad \text{(HW 1.5.7)}$$

$$\leq \int_0^T |X_u^n - X_u|^2 d\langle M \rangle_u \cdot \langle N \rangle_T$$

$$\to 0 \quad \text{a.s.}$$

29

$$\Rightarrow \langle I^M(X), N\rangle_t = \lim_{n\to\infty} \langle I(X^n), N\rangle_t$$

$$= \lim_{n\to\infty} \int_0^t X_s^n \, d\langle M, N\rangle_u.$$

By Kunita-Watanabe

$$\left| \int_0^t X_s^n \, d\langle M,N\rangle_u - X_u \, d\langle M,N\rangle_u \right|$$

$$\leq \int_0^t |X_u^n - X_u| \, d\xi_u \qquad \xi : \text{total var of } \langle M,N\rangle$$

$$\leq \sqrt{\int_0^t |X_u^n - X_u|^2 d\langle M\rangle_u} \sqrt{N_t}$$

$$\to 0$$

$$\therefore \langle I^M(X), N\rangle_t = \int_0^t X_u \, d\langle M,N\rangle_u$$

Take $N = I^N(Y)$ $Y \in \mathcal{L}_N^*$, switch roles of $I^M(X), N$ above.

$$\langle I^M(X), I^N(Y)\rangle_t = \int_0^t X_u \, d\langle M, I^N(Y)\rangle_u$$

$$= \int_0^t X_u Y_u \, d\langle M, N\rangle_u \quad \blacksquare.$$

Integration wrt $M \in \mathcal{M}^{c,loc}$

- begin w/ a useful characterization of $I^M(X)$:

L. $M \in \mathcal{M}_2^c$, $X \in \mathcal{L}_M^*$. Then $I^M(X)$ is the unique $\Phi \in \mathcal{M}_2^c$ s.t.

$$☆ \quad \langle \Phi, N\rangle_t = \int_0^t X_u \, d\langle M, N\rangle_u \quad \forall N \in \mathcal{M}_2^c.$$

pf. $\int_0^t X_u \, dM_u$ satisfies ☆.

If $\Phi \in \mathcal{M}_2^c$ also satisfies ☆:

$$\langle \Phi - I^M(X), N\rangle_t = \int_0^t X_u \, d\langle M, N\rangle_u - \int_0^t X_u \, d\langle M, N\rangle_u$$

$$= 0 \qquad \text{a.s. } t \geq 0 \quad \forall N.$$

Take $N = \Phi - I^M(X) \in \mathcal{M}_2^c$

$$\Rightarrow \langle \Phi - I^M(X)\rangle_t = 0 \qquad \text{a.s.} \quad t \geq 0$$

$$\Rightarrow \Phi = I^M(X) \qquad \text{(indistinguishability).}$$

Corollary

$M \in \mathcal{M}_2^c$, $X \in \mathcal{L}_M^*$, $N = I^M(X)$, $Y \in \mathcal{L}_N^*$

$\Rightarrow$

1) $XY \in \mathcal{L}_M^*$

2) $I^N(Y) = I^M(XY)$

pf

$$E\left[\int_0^T Y_t^2 \, d\langle N\rangle_t\right] = E\left[\int_0^T Y_t^2 X_t^2 \, d\langle M\rangle_t\right]$$

so $Y \in \mathcal{L}_N^* \Rightarrow XY \in \mathcal{L}_M^*$.

32

Let $\tilde{N} \in \mathcal{M}_2^c$:

$$\langle I^M(XY), \tilde{N} \rangle = \int_0^t X_u Y_u \, d\langle M, \tilde{N} \rangle_u$$

$$= \int_0^t Y_u \, d\langle N, \tilde{N} \rangle_u = \langle I^N(X), \tilde{N} \rangle_t. \blacksquare$$

- this allows us to use the shorthand

$$dI^M(X)_t = X_t \, dM_t$$

$$dI^N(Y)_t = Y_t \, dN_t = Y_t X_t \, dM_t.$$

etc.

Localization Result.

$M, \tilde{M} \in \mathcal{M}_2^c$; $X \in \mathcal{L}_M^*$, $\tilde{X} \in \mathcal{L}_{\tilde{M}}^*$.

Suppose $\exists$ s.t. T s.t.

i) $X_{t \wedge T} = \tilde{X}_{t \wedge T}$

ii) $M_{t \wedge T} = \tilde{M}_{t \wedge T}$

Then $I^M_{t \wedge T}(X) = I^{\tilde{M}}_{t \wedge T}(\tilde{X}) \quad t \geq 0 \quad$ a.s.

(33)

pf

Let $N \in \mathcal{M}_2^c$. Then

$$\langle M - \tilde{M}, N \rangle_{t \wedge T} = 0 \quad \text{a.s.} \quad t \geq 0$$

(use $\langle M - \tilde{M}, N \rangle = \underset{\|\Pi\| \to 0}{p\text{-}\lim} \sum (M_{t_i} - \tilde{M}_{t_i} - (M_{t_{i-1}} - \tilde{M}_{t_{i-1}}))(N_{t_i} - N_{t_{i-1}})$

to see this.)

$$\Rightarrow \langle I^M(X) - I^{\tilde{M}}(\tilde{X}), N \rangle_{t \wedge T}$$

$$= \int_0^t X_u \, d\langle M, N \rangle_u - \int_0^t \tilde{X}_u \, d\langle \tilde{M}, N \rangle_u$$

$$= 0$$

Take $N = I^M(X) - I^{\tilde{M}}(\tilde{X})$.

Using, we extend I^M to $M \in \mathcal{M}^{c,loc}$

Define: $\mathcal{P}_M = \{ X$ mbl, adapted s.t.

$$P\left[\int_0^T X_u^2 \, d\langle M \rangle_u < \infty\right] = 1$$

$\forall \; T \geq 0 \}$.

34

$$P_M^* = \{X \in P_M \mid X \text{ prog mbl}\}.$$

<u>Note</u>

1) $\mathcal{L}_M \subseteq P_M$; $\mathcal{L}_M^* \subseteq P_M^*$, $M \in \mathcal{M}_2^c$.

2) we will define $I^M(X)$ for $X \in P_M^*$.
If $t \mapsto \langle M \rangle_t$ is a.c. we can define $I^M(X)$ for $X \in P_M$.

<u>Main Idea</u>

$M \in \mathcal{M}^{c,loc} \Rightarrow \exists \{S_n\}$, S_n stopping, $S_n \nearrow \infty$ a.s. s.t. $M_{t \wedge S_n} \in \mathcal{M}_2^c$.

If $X \in P_M^*$, set

$$R_n = n \wedge \inf_{t \geq 0} \left\{ \int_0^t X_u^2 \, d\langle M \rangle_u \geq n \right\}$$

$\Rightarrow R_n$ stopping, $R_n \nearrow \infty$ a.s.

(35)

$T_n = S_n \wedge R_n$

$M^n_t = M_{t\wedge T_n}$; $X^n_t = X_t 1_{t\le T_n}$

$\Rightarrow M^n \in \mathcal{M}^c_2$, $X^n \in \mathcal{L}^*_{M^n}$ since

$$\int_0^{t\wedge T_n} (X^n_u)^2 \, d\langle M^n\rangle_u = \int_0^{t\wedge T_n} X_s^2 \, d\langle M\rangle_s \le n.$$

$\Rightarrow I^{M^n}(X^n) \in \mathcal{M}^c_2$ well defined.

Since $M^n = M^m$, $X^n = X^m$ on $[0, T_n \wedge T_m]$

we know

$$I^{M^n}_t(X^n) = I^{M^m}_t(X^m) \quad t \le T_n \wedge T_m$$

so, we define

$$I^M(X)_t \triangleq I^{M^m}(X^m)_t \quad t \le T_m.$$

$\Rightarrow I^M$ continuous,

I^M a local martingale with

$\{T^n\}$ stopping.

In fact, we have:

L/ $M \in \mathcal{M}^{c,loc}$, $X \in \mathcal{P}^*_M$. Then

1) $I^M_0(X) = 0$

2) $I^M(\alpha X + \beta Y) = \alpha I^M(X) + \beta I^M(Y)$

3) $\langle I^M(X)\rangle_t = \int_0^t X_u^2 \, d\langle M\rangle_u$

4) $I^M_{t\wedge T}(X) = I^M_t(\tilde{X})$ $\tilde{X}_t = X_t 1_{t \le T}$

5) $\langle I^M(X), I^N(Y)\rangle_t$

$$= \int_0^t X_u Y_u \, d\langle M, N\rangle_u$$

$N \in \mathcal{M}^{c,loc}$, $Y \in \mathcal{P}^*_N$.

- rule of thumb: sample path properties of $I^M(X)$, $M \in \mathcal{M}^c_2$, $X \in \mathcal{L}^*_M$ carry over to $M \in \mathcal{M}^{c,loc}$, $X \in \mathcal{P}^*_M$. Expectation properties do not carry over.