Brownian Motion.

Brownian Motion is the canonical continuous time, continuous path square integrable martingale

- plays a fundamental role in Mathematical Finance and in the theory of stochastic integration.

Definition 1.

$(\Omega, \mathcal{F}, P)$; $\mathbb{F} = \{\mathcal{F}_t\}_{t \geq 0}$ given.

We say that an adapted process $B = \{B_t\}_{t \geq 0}$ is a standard, one dimensional Brownian Motion if

1) $B_0 = 0$ a.s.

2) the map $t \mapsto B_t$ is continuous with probability one

3) for all $0 \leq s < t$, $B_t - B_s$ is

②

a) independent of $\mathcal{F}_s$

b) normally distributed with mean 0 and variance $t-s$.

In fact, B can be (and is often) defined independently of the filtration.

Definition 2

$B = \{B_t\}_{t \geq 0}$ is a Brownian Motion if

1) $B_0 = 0$ a.s. ; $t \mapsto B_t$ is continuous a.s.

2) $\forall$ $0 \leq s < t \leq u < v$, $B_v - B_u$ is $\perp\!\!\!\perp$ of $B_t - B_s$.

3) $\forall$ $0 \leq s < t$, $B_t - B_s \sim N(0, t-s)$.

3)

Claim: If B is as in Definition 2 then with $\mathbb{F} = \mathbb{F}B$, B is a B.M. in the sense of Definition 1.

pf steps

$\mathcal{D} \triangleq \{A \in \mathcal{F}_s^B \mid A \perp\!\!\!\perp B_t - B_s\}$ $\quad 0 \le s < t$.

$\mathcal{D}$ is a Dynkin system

1) $\Omega \in \mathcal{D}$ ✓

2) $A, B \in \mathcal{D}$, $A \subseteq B \Rightarrow \forall\ E \subseteq \mathbb{R}$ Borel

$$P(A \cap \{B_t - B_s \in E\}) = P(A) P(B_t - B_s \in E)$$
$$P(B \cap \{B_t - B_s \in E\}) = P(B) \quad ''$$

$$\Rightarrow P(B \setminus A \cap \{B_t - B_s \in E\}) = P(B \setminus A) P(B_t - B_s \in E)$$

$\Rightarrow B \setminus A \in \mathcal{D}$.

3) $\{A_n\} \subseteq \mathcal{D}$, $A_n \nearrow \Rightarrow \bigcup A_n \in \mathcal{D}$ (easy to check.

$$\mathcal{C} = \{ \sigma(B_{t_1}, B_{t_2}-B_{t_1}, \ldots, B_{t_n}-B_{t_{n-1}}) ;$$
$$0 < t_1 < \ldots < t_n \le 1 ; \ n \in \mathbb{N} \}.$$

$\rightarrow$ $\mathcal{C}$ closed under pairwise intersection
- take enlarged mesh.

$\Rightarrow \mathcal{D} \supseteq \sigma(\mathcal{C}) = \mathcal{F}^B_1$ ∎.

We are interested in proving that B.M. exists:

- finding $(\Omega, \mathcal{F}, P)$ and B s.t. B is a B.M. as in Definition 2.

There are numerous ways to do this.

- we will give a heuristic motivation for why B.M. exists as a scaled limit of random walks.

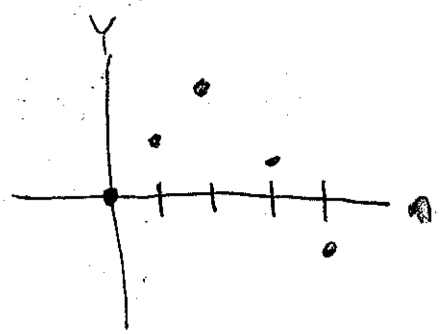
- we will rigorously prove B.M. exists using an abstract result.

5)

Brownian Motion as a Limit of Random Walks

- $\{\xi_n\}_{n=1,2,3,\ldots}$ iid r.v. which are uniformly bounded and have mean 0, variance 1.

- $Y_0 = 0$; $Y_n = \sum_{i=1}^{n} \xi_i \quad n = 1,2,\ldots$



  Y: random walk.

  Y is a martingale

- We create a continuous process B^n by speeding up time and scaling down the jump size.

$$B^n_t = \frac{1}{\sqrt{n}} Y_{\lfloor nt \rfloor} + (nt - \lfloor nt \rfloor)\, \xi_{\lfloor nt \rfloor + 1}/\sqrt{n}$$

correction term to make continuous.

- What happens for large n?

  Ignore $\frac{1}{\sqrt{n}}(nt - \lfloor nt \rfloor)\xi_{\lfloor nt \rfloor + 1}$ b/c it is $\le K/\sqrt{n}$ a.s.

⑥

so, for $s<t$

a) $B_t^{(n)} - B_s^{(n)} \approx \frac{1}{\sqrt{n}} \sum_{i=\lfloor ns \rfloor+1}^{\lfloor nt \rfloor} \xi_i \quad \perp\!\!\!\perp \quad B_r^{(n)} \; r \le s$

b/c $B_r^{(n)}$ $r \le s$ only depends upon (at most) the first $\lfloor ns \rfloor + 1$ ξ_i

• $\perp\!\!\!\perp$ increments.

b) $B_t^{(n)} - B_s^{(n)} \approx \sqrt{\frac{\lfloor nt \rfloor - \lfloor ns \rfloor}{n}} \cdot \frac{1}{\sqrt{\lfloor nt \rfloor - \lfloor ns \rfloor}} \sum_{i=\lfloor ns \rfloor+1}^{\lfloor nt \rfloor} \xi_i$

1) $\sqrt{\frac{\lfloor nt \rfloor - \lfloor ns \rfloor}{n}} \to (t-s)$ as $n \nearrow \infty$

2) CLT: $\frac{1}{\sqrt{\lfloor nt \rfloor - \lfloor ns \rfloor}} \sum_{i=\lfloor ns \rfloor+1}^{\lfloor nt \rfloor} \xi_i \xrightarrow{\mathcal{D}} N(0,1)$ as $n \nearrow \infty$

$\Rightarrow B_t^{(n)} - B_s^{(n)} \xrightarrow{\mathcal{D}} (t-s)N(0,1) \overset{\mathcal{D}}{=} N(0, t-s).$

∴ Brownian Motion...

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This construction can be made precise using notions of convergence for continuous processes: see Ch 2.4.

We will prove B.M. exists using an abstract argument which is shorter (and important).

Abstract Argument Part 1:

Kolmogorov - Daniell Thm

- constructs a probability measure on path space consistent with a given set of finite dimensional distributions.

$\Omega = \mathbb{R}^{[0,\infty)}$: set of all real-valued functions on $[0,\infty)$.

$\mathcal{F} = \mathcal{B}(\mathbb{R}^{[0,\infty)})$

to construct $\mathcal{F}$:

Define an n-dimensional cylinder set by taking

$$C = \{\omega \in \Omega[0,\infty) \mid (\omega_{t_1}, \dots, \omega_{t_n}) \in A\}$$

$A \in \mathcal{B}(\mathbb{R}^n)$ a Borel set.

$t_i \geq 0$, $i = 1, \dots, n$.

Set $\mathcal{C}$ denote the field of all finite dimensional cylinder sets and $\mathcal{B}(\Omega[0,\infty))$ as the smallest σ-field containing $\mathcal{C}$.

Finite Dimensional Distributions.

Set $T = \{\underset{\sim}{t} = (t_1, \dots, t_n) \mid t_i \geq 0, \ n = 1, 2, 3, \dots\}$.

For each $\underset{\sim}{t} \in T$, suppose we have a probability measure ~~on~~ $Q_{\underset{\sim}{t}}$ on $\mathcal{B}(\mathbb{R}^n)$

Then $\{Q_{\underset{\sim}{t}}\}_{\underset{\sim}{t} \in T}$ is called a set of finite dimensional distributions.

(9)

Suppose we have a measure P on $(\mathbb{R}^{[0,\infty)}, \mathcal{B}(\mathbb{R}^{[0,\infty)}))$. Define

$$Q_{\underset{\sim}{t}}(A) = P[(\omega_{t_1}, \ldots, \omega_{t_n}) \in A] \quad A \in \mathcal{B}(\mathbb{R}^n)$$

Then

1) If $\underset{\sim}{s} = (t_{i_1}, \ldots, t_{i_n})$ is a permutation of $\underset{\sim}{t} = (t_1, \ldots, t_n)$ then

$$Q_{\underset{\sim}{s}}(A_{i_1} \times \ldots \times A_{i_n}) = Q_{\underset{\sim}{t}}(A_1 \times \ldots \times A_n)$$

for $A_1, \ldots, A_n \in \mathcal{B}(\mathbb{R})$.

2) If $\underset{\sim}{t} = (t_1, \ldots, t_n)$, $\underset{\sim}{s} = (t_1, \ldots, t_{n-1})$ and $A \in \mathcal{B}(\mathbb{R}^{n-1})$ then

$$Q_{\underset{\sim}{t}}(A \times \mathbb{R}) = Q_{\underset{\sim}{s}}(A)$$

Now, going in reverse, we say that $\{Q_{\underset{\sim}{t}}\}_{\underset{\sim}{t} \in T}$ is <u>consistent</u> if 1), 2) above hold.

⑩ The Kolmogorov-Daniell theorem says that if $\{Q_{\underset{\sim}{t}}\}_{\underset{\sim}{t} \in T}$ is consistent, we can find a P on $(\mathbb{R}^{[0,\infty)}, \mathcal{B}(\mathbb{R}^{[0,\infty)}))$ s.t.

$$* \quad Q_{\underset{\sim}{t}}(A) = P[(\omega_{t_1}, \ldots, \omega_{t_n}) \in A] \quad \forall\, t_1, \ldots, t_n,$$
$$n = 1, 2, \ldots \text{ and } A \in \mathcal{B}(\mathbb{R}^n).$$

- i.e. the theorem allows us to build P.

<u>Thm</u> (Kolmogorov-Daniell).

Let $\{Q_{\underset{\sim}{t}}\}_{\underset{\sim}{t} \in T}$ be a consistent family of finite dimensional distributions. Then there exists a measure P on $(\mathbb{R}^{[0,\infty)}, \mathcal{B}(\mathbb{R}^{[0,\infty)}))$ s.t. $*$ holds.

<u>pf</u> - sketch

1) Define a set function on the field $\mathcal{C}$, denoted Q, by the assignment

$$Q(C) = Q_{\underset{\sim}{t}}(A)$$

if $C = \{(\omega_{t_1}, \ldots, \omega_{t_n}) \in A\}$

ii) Show that Q is well defined

(i.e. if $C = \{(\omega_{t_1}, \ldots, \omega_{t_n}) \in A\}$

$= \{(\omega_{s_1}, \ldots, \omega_{s_m}) \in \tilde{A}\}$

then $Q_{\underset{\sim}{t}}(A) = Q_{\underset{\sim}{s}}(\tilde{A})$)

- follows via consistency.

iii) Show that Q is finitely additive

- also follows via consistency

iv) Show that Q is countably additive

- proof is highly technical - read in pgs 50 - 52.

v) Thus, by the Carathéodory extension theorem, there exists a measure P on $\sigma(\mathcal{C}) = \mathcal{B}(\mathbb{R}^{[0,\infty)})$ which is an extension of Q on $\mathcal{C}$. ∎

Application to Brownian Motion:

Let $0 < s_1 < s_2 < \dots < s_n$ and define

$$F_{(s_1,\dots,s_n)}(x_1,\dots,x_n)$$

$$= \int_{-\infty}^{x_1}\int_{-\infty}^{x_2}\cdots\int_{-\infty}^{x_n} p(s_1;0,y_1)\cdots p(s_n-s_{n-1};y_{n-1},y_n)\, dy_n \cdots dy_1$$

~~joint cdf~~

$$p(t;x,y) = \frac{1}{\sqrt{2\pi t}}\, e^{-\frac{1}{2t}(x-y)^2} \quad t>0,\ x,y\in\mathbb{R}.$$

Clearly, if $(B_{s_1},\dots,B_{s_n})$ has joint distribution $F_{(s_1,\dots,s_n)}$ then $(B_0=0)$

$\{B_{s_j}-B_{s_{j-1}}\}_{j=1,\dots,n}$ are $\perp\!\!\!\perp$ r.v.

$B_{s_j}-B_{s_{j-1}} \sim N(0, s_j-s_{j-1})$ $j=1,\dots,n$.

- easy to check.

Let $\underset{\sim}{t} = \{t_1,\dots,t_n\}$ and, after ordering, let $(B_{t_1},\dots,B_{t_n})$ have the distribution F.

Then, for $A \in \mathcal{B}(\mathbb{R}^n)$ let $Q_{\underline{t}}(A)$ be the probability $(B_{t_1}, \dots, B_{t_n})$ is in A.

- clearly consistent

$\therefore$ There exists a measure on $(\mathbb{R}^{[0,\infty)}, \mathcal{B}(\mathbb{R}^{[0,\infty)}))$ s.t. if $B = \{B_t\}$ is the coordinate mapping process $B_t(\omega) = \omega_t$ for $\omega \in \mathbb{R}^{[0,\infty)}$ then

$$B_0 = 0 \quad \text{a.s.}$$

$$B_t - B_s \sim N(0, t-s) \quad 0 \le s < t$$

and B has $\perp\!\!\!\perp$ increments.

- we would have a Brownian Motion <u>except</u> that we have not shown that B has almost surely continuous paths.

In fact, we have a problem that

$$E = \{\omega \in \mathbb{R}^{[0,\infty)} \mid \omega \text{ has continuous trajectory}\} \notin \mathcal{B}(\mathbb{R}^{[0,\infty)}).$$

so, we cannot even hope that P is concentrated on E b/c E is not measurable!

problem: typical set in $\mathcal{B}(\Omega^{[0,\infty)})$ takes the form:

$$C = \{ (\omega_{t_1}, \omega_{t_2}, \dots) \in A_1 \times A_2 \times \dots \}$$

and we can find many ω s.t.

1) $(\omega_{t_1}, \dots, \omega_{t_2}, \dots) \in A_1 \times A_2 \times \dots$

2) ω is either continuous or discontinuous

- restriction to countable times <u>not</u> enough to ~~~~ identify continuity.
- $\mathcal{B}(\Omega^{[0,\infty)})$ is too small in some sense.

Fortunately, there is a work-around...

Abstract Proof Part 2:

Kolmogorov - Čentsov Thm.

- allows us to find (locally Hölder) continuous modifications of a given process provided paths are γ - Hölder continuous on average.

Thm (Kolmogorov - Čentsov)

$(\Omega, \mathcal{B}, P)$ given. $X = \{X_t\}_{t \leq T}$ a given process. for $T > 0$

If $\exists$ $\alpha, \beta, C > 0$ s.t.

$$E\left[|X_t - X_s|^\alpha\right] \leq C|t-s|^{1+\beta} \quad 0 \leq s < t \leq T.$$

Then $\exists$ a continuous modification $\tilde{X}$ of X which is locally Hölder continuous with exponent γ for any $\gamma \in (0, \beta/\alpha)$

$$P\left[\sup_{\substack{0 < t-s < h(\omega) \\ s,t \in [0,T]}} \frac{|\tilde{X}_t - \tilde{X}_s|}{|t-s|^\gamma} \leq \delta\right] = 1 \quad \text{some } \delta, \; h: \text{r.v.}$$

Application to "Brownian Motion" B from Kolmogorov-Daniell

easy: $E[(B_t - B_s)^4] = 3(t-s)^2$

$\Rightarrow \quad \alpha = 4, \; \beta = 1$

moderate: $E[(B_t - B_s)^{2n}] = C_n (t-s)^n$

$\Rightarrow \quad \alpha = 2n \,, \; \beta = n-1$

$\Rightarrow$ $\exists$ a modification W_T of B on $[0,T]$ with locally γ-Hölder paths for

$\gamma \in \left(0, \frac{n-1}{2n}\right) \quad \forall n \qquad$ i.e. for $\gamma \in (0, 1/2)$

Set $\Omega_T = \{ W^T_t = B_t \quad \forall \; t \in [0,T] \cap \mathbb{Q} \}$.

$$\tilde{\Omega} = \bigcap_{T = 1,2,\ldots} \Omega_T$$

$\Rightarrow W_t^{T_1} = W_t^{T_2} \quad \forall\, t \in \mathbb{Q} \cap [0, T_1 \wedge T_2]$ on $\tilde{\Omega}$

$\Rightarrow W_t^{T_1} = W_t^{T_2} \quad \forall\, t \in [0, T_1 \wedge T_2]$ by continuity.

Define W_t as the common value on $\tilde{\Omega}$
$W_t = 0$ on Ω.

$\therefore$ W is a Brownian Motion. In fact, W is locally γ-Hölder continuous $\forall\, \gamma \in (0, 1/2)$ with probability one.

$\therefore$ Brownian Motion Exists!

Back to Kolmogorov-Čentsov:

Pf (T=1). Note, for $a > 0$

$$P[|X_t - X_s| \geq a] \leq \frac{E[|X_t - X_s|^\alpha]}{a^\alpha} \leq \frac{C|t-s|^{1+\beta}}{a^\alpha}$$

$\Rightarrow X_t \mapsto X_s$ in probability (though not necessarily almost surely).

(18)

Also, with $t = k/2^n$, $s = \frac{k-1}{2^n}$, $a = 2^{-\gamma n}$

for $\gamma \in (0, \beta/2)$

$$P\left(|X_{k/2^n} - X_{\frac{k-1}{2^n}}| \geq 2^{-\gamma n}\right) \leq C 2^{\gamma \alpha n} 2^{-n(1+\beta)}$$
$$= C 2^{-n(1+\beta-\alpha\gamma)}$$

$$\Rightarrow P\left(\max_{1 \leq k \leq 2^n} |X_{k/2^n} - X_{\frac{k-1}{2^n}}| \geq 2^{-\gamma n}\right)$$
$$\leq C \cdot 2^{-n(1+\beta-\alpha\gamma)} \cdot 2^n = C 2^{-n(\beta-\alpha\gamma)}$$

$\therefore$ since $\sum_{n=1}^{\infty} 2^{-n(\beta-\alpha\gamma)} < \infty$ Borel-Cantelli gives a set Ω^* of prob 1 s.t. $\omega \in \Omega^* \Rightarrow$

$$\max_{1 \leq k \leq 2^n} |X_{\frac{k}{2^n}} - X_{\frac{k-1}{2^n}}|(\omega) < 2^{-\gamma n} \quad n \geq n(\omega).$$

$\uparrow$ r.v.

<u>Now.</u>

Set $D_n = \{k/2^n ; k = 0, 1, \ldots, 2^n\}$, $D = \bigcup_n D_n$ as the dyadic rationals.

We claim that for $\omega \in \Omega^*$, $t \mapsto X_t(\omega)$ for $t \in D$ is uniformly continuous. In fact, if $s,t \in D$ with $0 < t-s < h(\omega) \triangleq 2^{-n^*(\omega)}$ then

$$|X_t - X_s|(\omega) \leq \delta |t-s|^{\gamma}$$

$\delta > 0$ $\perp$ of s,t.

pf

a) we know $\max\limits_{1\leq k\leq 2^n} |X_{\frac{k}{2^n}} - X_{\frac{k-1}{2^n}}|(\omega) \leq 2^{-\gamma n}$ $\quad n \geq n^*(\omega)$

b) use a) to show $\forall\, m > n$ that

$$|X_t - X_s|(\omega) \leq 2 \cdot \sum_{j=n+1}^{m} 2^{-\gamma j}$$

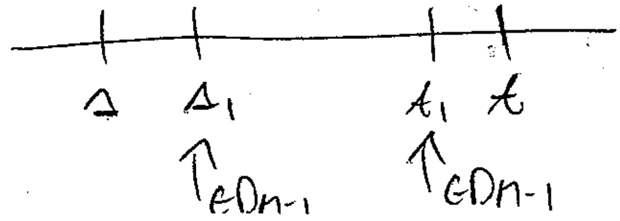
$s,t \in D_m$
$t-s \leq 2^{-n}$

<u>induction</u>

a) $m = n+1 \Rightarrow t = k/2^m$, $s = \frac{k-1}{2^m}$ so result holds by a)

b) assume true for $m = n+1, \dots, M-1$

Take $s,t \in D_n$. Note:



$s_1 = t_1$ or
$s = s_1$ or
$t = t_1$ or

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$\Rightarrow \quad |X_{s_1} - X_s| \le 2^{-\gamma M} \; ; \; |X_t - X_{t_1}| \le 2^{-\gamma M} \quad (a))$

$|X_{t_1} - X_{s_1}| \le 2 \cdot \sum_{j=n+1}^{M-1} 2^{-\gamma j}$ (induction step)

so result holds.

c) take $s, t \in D$ with $0 < t - s < 2^{-n^*(\omega)}$.

$\exists \; n \geq n^*(\omega)$ s.t. $2^{-(n+1)} \le t - s < 2^{-n^*(\omega)}$

$\exists \; M \geq n$ s.t. $s, t \in D_M$.

$\Rightarrow \; |X_t - X_s| \le 2 \sum_{j=n+1}^{M} 2^{-\gamma j} \le 2 \sum_{j=n+1}^{\infty} 2^{-\gamma j}$

$= 2^{-\gamma(n+1)} \times \delta \triangleq \frac{2}{2 - 2^{-\gamma}}$

$\le |t - s|^{\gamma} \, \delta.$

$\therefore$ uniform continuity holds.

Lastly

for $\omega \notin \Omega^*$ set $\tilde{X}_t(\omega) = 0 \quad t \le 1.$

for $\omega \in \Omega^*$ and $t \in D$ set $\tilde{X}_t(\omega) = X_t(\omega)$

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for $\omega \in \Omega^*$ and $t \in [0,1] \cap D^c$ pick $\{s_n\} \subseteq D$ s.t. $s_n \to t$. We know $X_{s_n}(\omega)$ converges to a limit $\underline{\parallel}$ of $\{s_n\}$ by uniform continuity.

Define $\tilde{X}_t = \lim_n X_{s_n}(\omega)$

This $\tilde{X}$ also satisfies

$$|\tilde{X}_t - \tilde{X}_s|(\omega) \leq \hat{\delta}|t-s|^\gamma \quad (\hat{\delta} = \delta) \text{ for } 0 < t-s < 2^{-n^*(\omega)}$$

so local holder continuity follows.

<u>Now</u>

i) $\tilde{X}_t = X_t$ a.s. for $t \in D$

ii) for $t \in [0,1] \cap D^c$ $\tilde{X}_t = \lim_{s_n} X_{s_n}$ $\{s_n\} \in D$ a.s. But, $X_{s_n} \to X_t$ in probability so $\tilde{X}_t = X_t$ a.s.

$\therefore \tilde{X}$ is a modification.

Now that we know B.M. exists we can state some basic properties.

1) B is a martingale

pf
$$E[B_t \mid \mathcal{F}_s] = E[B_t - B_s + B_s \mid \mathcal{F}_s]$$
$$= B_s + E[B_t - B_s]$$
$$= B_s$$

2) B has quadratic variation $\langle B \rangle_t = t$ a.s.

pf
Note: $E[B_t^2] = t$ so B is square integrable and hence in M_c.

Also
$$E[B_t^2 - t \mid \mathcal{F}_s] = E[(B_t - B_s)^2 + 2B_tB_s - B_s^2 - t \mid \mathcal{F}_s]$$
$$= (t-s) + B_s^2 - t$$
$$= B_s^2 - s.$$

Thus, by definition of quadratic variation

$\langle B \rangle_t = t$ a.s.

Alternative way to see this:

Π: uniform partition of $[0,t]$

$$0 = \frac{0t}{n} < t/n < 2t/n < \dots < \frac{nt}{n} = t.$$

$$\sum_{i=1}^{n} \left(B_{\frac{it}{n}} - B_{\frac{(i-1)t}{n}}\right)^2 = t/n \sum_{i=1}^{n} Z_i^2$$

$$Z_i = \sqrt{n/t}\left(B_{\frac{it}{n}} - B_{\frac{(i-1)t}{n}}\right)$$

$\{Z_i\}_{i=1,2,\dots}$ iid $N(0,1)$ r.v.

Strong Law: $t/n \sum_{i=1}^{n} Z_i^2 \to t$ a.s.

$\Rightarrow V_t^{(2)}(\Pi) \Rightarrow t$ a.s.

(at least along uniform partitions)

Transformations of B.M. which are B.M.

1) $W_t = -B_t \; ; \; t \geq 0$ is a B.M.

- clear

2) $W_t = B_{t+t_0} - B_{t_0}$ is a B.M. $\forall \; t_0 \geq 0$

- also clear.

3) $W_t = \frac{1}{\sqrt{a}} B_{at}$ is a B.M. $\forall \; a > 0$

- also clear since $\frac{1}{\sqrt{a}}(B_{at} - B_{as}) \sim N(0, t-s)$

4) $W_t = t B_{1/t}$, $W_0 \triangleq 0$ is a B.M.

- Leave aside continuity at 0 for now.
- Let $0 < s < t \leq u < v$. A tedious calculation shows

$$\begin{pmatrix} s B_{1/s} \\ t B_{1/t} - s B_{1/s} \\ u B_{1/u} - t B_{1/t} \\ v B_{1/v} - u B_{1/u} \end{pmatrix} = \begin{pmatrix} s & s & s & s \\ t-s & t-s & t-s & -s \\ u-t & u-t & -t & 0 \\ v-u & -u & 0 & 0 \end{pmatrix} \begin{pmatrix} B_{1/v} \\ B_{1/u} - B_{1/v} \\ B_{1/t} - B_{1/u} \\ B_{1/s} - B_{1/t} \end{pmatrix}$$

$\Rightarrow \left(s B_{1/s},\ t B_{1/t} - s B_{1/s},\ u B_{1/u} - t B_{1/t},\right.$

$\left. v B_{1/v} - u B_{1/u} \right)$ is jointly normal

with mean vector $\vec{0}$.

Also $\left(\frac{1}{v} < \frac{1}{u} \le \frac{1}{t} < 1/s \right)$

$E\left[(v B_{1/v} - u B_{1/u})(t B_{1/t} - s B_{1/s}) \right]$

$= E\left[(v B_{1/v} - u B_{1/u})(t B_{1/t} - s E[B_{1/s} \mid \mathcal{F}_{1/t}]) \right]$

$= (t-s) E\left[(v B_{1/v} - u B_{1/u}) E[B_{1/t} \mid \mathcal{F}_{1/u}] \right]$

$= (t-s)\ E\left[v B_{1/v} B_{1/u} - u B_{1/u}^2 \right]$

$= (t-s)\ E\left[v B_{1/v} E[B_{1/u} \mid \mathcal{F}_{1/v}] - 1 \right]$

$= (t-s)\ E\left[v B_{1/v}^2 - 1 \right]$

$= (t-s)\ \left[v \cdot 1/v - 1 \right] = 0$

$\therefore\ v B_{1/v} - u B_{1/u} \perp\!\!\!\perp t B_{1/t} - s B_{1/s}.$

Also

$$E\left[(tB_{1/t} - sB_{1/s})^2\right] = t^2 E\left[B_{1/t}^2\right] - 2st\, E\left[B_{1/t}B_{1/s}\right] + s^2 E\left[B_{1/s}^2\right]$$

$$= t^2 \cdot \frac{1}{t} - 2st\, E\left[B_{1/t}^2\right] + s^2/s$$

$$= t - 2st\frac{1}{t} + s = t - s$$

$$\Rightarrow tB_{1/t} - sB_{1/s} \sim N(0, t-s).$$

So, if $\lim\limits_{t \to 0} tB_{1/t} = 0$ we have a B.M.

Claim: $\forall\, \alpha > 1/2$, $\lim\limits_{t \to \infty} \frac{|B_t|}{t^\alpha} = 0$ a.s.

In particular $\lim\limits_{t \to \infty} \frac{|B_t|}{t} = 0$ a.s.

so $\lim\limits_{t \to 0} tB_{1/t} = 0$ a.s.

$\therefore W$ is a B.M.

"Strong Law" for B.M.

(27)

~~pf~~ for $\sigma < \tau$, $\varepsilon > 0$, since B^2 is a sub-mart.

$$P\left(\sup_{\sigma \le t \le \tau} \frac{|B_t|}{t^\alpha} \geq \varepsilon \right)$$

$$\le P\left(\sup_{\sigma \le t \le \tau} |B_t|^2 \geq \varepsilon^2 \sigma^{2\alpha} \right)$$

$$\le \frac{1}{\varepsilon^2 \sigma^{2\alpha}} E[B_\tau^2] = \frac{\tau}{\varepsilon^2 \sigma^{2\alpha}}$$

$\sigma = 2^n$; $\tau = 2^{n+1}$

$$P\left(\sup_{2^n \le t \le 2^{n+1}} |B_t|/t^\alpha \geq \varepsilon \right) \le \frac{1}{\varepsilon^2} 2^{n+1-2\alpha n}$$

$$= \frac{1}{\varepsilon^2} 2^{(2\alpha-1)n+1}$$

$\therefore$ Borel-Cantelli gives with prob 1

$\exists$ $n^*(\omega)$ s.t. $n \geq n^*(\omega)$ implies

$$\sup_{2^n \le t \le 2^{n+1}} \frac{|B_t|(\omega)}{t^\alpha} < \varepsilon \qquad n \geq n^*(\omega)$$

$$\Rightarrow \overline{\lim_{n \to \infty}} \frac{|B_t|(\omega)}{t^\alpha} \le \varepsilon \quad \forall \varepsilon > 0 \text{ a.s.} \blacksquare$$

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What about at 0?

Claim: $\forall\ 0 \le s$:

$$\lim_{t \to s^+} \frac{|B_t - B_s|}{|t-s|^\alpha} = 0 \quad \text{a.s.} \quad \forall\ \alpha < 1/2$$

pf - Exact Same (suffices to consider $s=0$)

$$P\Big(\sup_{2^{-(n+1)} \le t \le 2^{-n}} \frac{|B_t|}{t^\alpha} \geq a \Big)$$

$$\le P\Big(\sup_{2^{-(n+1)} \le t \le 2^{-n}} |B_t|^2 \geq \varepsilon^2 2^{2\alpha(n+1)} \Big)$$

$$\le \frac{2^{-n}}{\varepsilon^2 2^{-2\alpha(n+1)}} = \frac{2^{2\alpha}}{\varepsilon^2} 2^{-(1-2\alpha)n}.$$

Addendum.

Correction to an earlier statement:

We said a right-continuous sub-martingale X has a last element X_∞, or "is closable"

if

1) X_∞ is $\mathcal{F}_\infty = \sigma\left(\bigcup_{t\geq 0} \mathcal{F}_t\right)$ mbl

2) $E[|X_\infty|] < \infty$

3) $E[X_\infty | \mathcal{F}_t] \geq X_t$ a.s. $\forall\, t \geq 0$

We also know that if $\sup_{t\geq 0} E[X_t^+] < \infty$ then $X_t \to \hat{X}_\infty$ a.s. and $\hat{X}_\infty$ is integrable.

However, it may be that $\hat{X}_\infty$ is <u>not</u> a last element for X as in the definition of X_∞.

Now if X has a last element X_∞ then

$$X_t^+ \leq E[X_\infty | \mathcal{F}_t]^+ \leq E[X_\infty^+ | \mathcal{F}_t] \leq E[|X_\infty| | \mathcal{F}_t]$$

so $E[X_t^+] \leq E[|X_\infty|]$

and $\tilde{X}_\infty$ exists. Also, $\forall A \in \mathcal{F}_t$

$$E[X_t 1_A] \leq E[X_\infty 1_A]$$

now: $X_t 1_A \to \tilde{X}_\infty 1_A$

$$|X_t| 1_A \leq 1_A E[|X_\infty| | \mathcal{F}_t]$$

so $\{|X_t| 1_A\}_{t \geq 0}$ is u.i. and

$$E[\tilde{X}_\infty 1_A] \leq E[X_\infty 1_A]$$

$$\Rightarrow X_t \leq E[X_T | \mathcal{F}_t] \to E[\tilde{X}_\infty | \mathcal{F}_t]$$

$\therefore$ $\tilde{X}_\infty$ is a last element.

But, if $\sup_{t \geq 0} E[X_t^+] < \infty$ so X_∞ exists, it may be that X_∞ is <u>not</u> a last element of X.

<u>Ex</u>

$$X_t = e^{B_t - t/2} \qquad t \geq 0$$

- can show X is a martingale. (≥ 0)

$$E[e^{B_t - t/2} \mid \mathcal{F}_s] = e^{B_s - t/2} E[e^{B_t - B_s}] = e^{B_s - s/2}.$$

- since $B_t/t \to 0$ a.s. $X_t \to 0 = X_\infty$ a.s.

$$E[X_t^+] = 1 \qquad \forall\ t \geq 0$$

- But, $e^{B_t - t/2} \not\leq E[0 \mid \mathcal{F}_t] = 0.$