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Doob-Meyer Decomposition

- decomposition of a sub-martingale into a martingale + an increasing process.

Q. Why should we expect this?

Discrete Time

$X = \{X_n\}_{n=0,1,2,\dots}$ sub-martingale w.r.t. $(\Omega, \mathcal{F}, P)$

and $\{\mathcal{F}_n\}_{n=0,1,2,\dots}$

We are looking for a martingale M and increasing process A (i.e. $A_n - A_{n-1} \geq 0$)

s.t.

1) A is pre-visible : A_n is $\mathcal{F}_{n-1}$ mbl. $n=1,2,\dots$

with $A_0 = 0$

2) $X_n = M_n + A_n \qquad n=0,1,2,\dots$

Why do we have this?

-assume we already have the decomposition:

$\Rightarrow X_n - X_{n-1} = M_n - M_{n-1} + A_n - A_{n-1} \qquad n=1,2,\dots$

$\Rightarrow E[X_n - X_{n-1} | \mathcal{F}_{n-1}] = A_n - A_{n-1}$ (pre-visible)

2)

$\Rightarrow$ $A_0 = 0$ and

$$A_n = \sum_{k=1}^{n} E[X_k - X_{k-1} \mid \mathcal{F}_{k-1}]$$

$\uparrow$ so, A must take this form.

Note: A is $\nearrow$ since X a submartingale.

$\therefore$ for A as above and M via

$M_n = X_n - A_n \qquad n = 0,1,2,\ldots$

we have our result.

1) check a) $E[|M_n|] < \infty \quad \forall n = 0,1,2,\ldots$

b) $E[M_n - M_{n-1} \mid \mathcal{F}_{n-1}] = 0 \quad \forall n = 1,2,3,\ldots$

Furthermore, we have uniqueness in the class of A which are previsible:

$$X_n = M_n + A_n = M'_n + A'_n$$

$$\Rightarrow M_{n-1} - M'_{n-1} = E[M_n - M'_n \mid \mathcal{F}_{n-1}]$$

$$= E[A'_n - A_n \mid \mathcal{F}_{n-1}]$$

$$= A'_n - A_n = A'_n - A'_{n-1}$$

3)

$n=1 \Rightarrow A_1 = A_1', \eta_1 = \eta_1'$

$n=2 \Rightarrow A_2 = A_2', \eta_2 = \eta_2'$

and so on...

$\therefore P(A_n = A_n', \eta_n = \eta_n' \ \forall n = 0,1,2,\ldots) = 1.$

- indistinguishability.

We wish to extend this to continuous time.

Basic Idea: i) Partition Time

ii) Use discrete time results

iii) Let the partition "fill up" time.

For the third ~~set~~ step to hold we need some uniform integrability

Definition.

Let $X = \{X_t\}_{t\geq 0}$ be a right-continuous process adapted to $\mathbb{F}$.

④

Let $\mathcal{S}$ denote the class of all $\mathbb{F}$ stopping times s.t. $T<\infty$ a.s.

We say X is of class D ("Doob") if $\{X_\tau \mid \tau \in \mathcal{S}\}$ is u.i.

Let $a \geq 0$ and set $\mathcal{S}_a$ as the class of all $\mathbb{F}$ stopping times s.t. $P(T \leq a) = 1$.

We say X is of class DL ("Doob localized") if $\{X_\tau \mid \tau \in \mathcal{S}_a\}$ is u.i. $\forall\ a \geq 0$.

<u>Note</u>

$X \in D \Rightarrow X \in DL$.

$X \geq 0$, X r-cont sub-martingale

$\Rightarrow X \in DL$ $\quad (E[X_\tau 1_{X_\tau \geq \lambda}] \leq E[X_a 1_{X_a \geq \lambda}]$

$\uparrow$ optional sampling.

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we also need a continuous time notion of pre-visible.

- something like $E[A_t | \mathcal{F}_{t-}] = A_t$ b/c A_t is $\mathcal{F}_{t-}$ mbl.

note: in discrete time if A is predictable and M is a bounded martingale, since

$$M_{k-1}(A_k - A_{k-1}) = M_k(A_k - A_{k-1})$$
$$- (M_k - M_{k-1})(A_k - A_{k-1})$$

we see that

$$E[M_{k-1}(A_k - A_{k-1})] = E[M_k(A_k - A_{k-1})]$$

so

$$* \quad E\left[\sum_{k=1}^{n} M_{k-1}(A_k - A_{k-1})\right] = E\left[\sum_{k=1}^{n} M_k(A_k - A_{k-1})\right]$$

$\forall$ bounded martingale.

Note:

$$\sum_{k=1}^{n} M_k(A_k - A_{k-1}) = \sum_{k=1}^{n} M_k A_k - \sum_{k=0}^{n-1} M_{k+1} A_k.$$

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$$= M_n A_n + \sum_{k=1}^{n-1} M_k A_k - \sum_{k=1}^{n-1} M_{k+1} A_k$$

$$= M_n A_n - \sum_{k=1}^{n-1} A_k (M_{k+1} - M_k)$$

so, since A is predictible, M a martingale.

$$E\left[\sum_{k=1}^{n} M_k (A_k - A_{k-1})\right] = E[M_n A_n]$$

Thus $*$ is equivalent to

$$** \quad E[M_n A_n] = E\left[\sum_{k=1}^{n} M_k (A_k - A_{k-1})\right]$$

$$= E\left[\sum_{k=1}^{n} M_{k-1} (A_k - A_{k-1})\right]$$

$\forall$ bounded M martingale.

We say that A is <u>natural</u> if $**$ holds $\forall$ bounded martingale M.

so predictible $\Rightarrow$ natural and in fact, predictible $\Leftrightarrow$ natural in discrete time (Prop 1.4.3)

The continuous time analog of $**$ is, for increasing processes,

$$E[M_t A_t] = E\left[\int_{(0,t]} M_s \, dA_s\right] = E\left[\int_{(0,t]} M_{s-} \, dA_s\right] \quad ☆$$

$\forall$ bounded right-continuous martingale M

Notes

- Technically we say A is increasing if
  0) A is adapted
  1) $A_0 = 0$
  2) $t \mapsto A_t(\omega)$ is non-decreasing and right continuous w/ prob 1.

- For A increasing

$$\int_{(0,t]} X_s \, dA_s \, (\omega)$$

is the Stiltjes integral defined ω by ω.

and, if X is prog. mbl then so is the integral, provided it exists

- If $\mathbb{F}$ satisfies the usual conditions then A natural, increasing implies A_t is $\mathcal{F}_{t-}$ mbl. $\forall\ t \geq 0$.

Theorem (Doob-Meyer)

$\mathbb{F}$: satisfies the usual conditions

X: r-cont, sub-mart of class DL

Then

$$X_t = M_t + A_t \qquad t \geq 0$$

where

1) M is a right-continuous mart.

2) A is natural, increasing.

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The decomposition is unique up to indistinguishability among the class where A is natural.

Sketch of Proof.

1) Uniqueness: By A if A is natural, increasing then $E[\tilde{M}_t A_t] = E[\int_{(0,t]} \tilde{M}_{s-} dA_s]$ $\forall$ bdd, r-cont mart.

So, if $X_t = M_t + A_t = M'_t + A'_t$ then

$M_t - M'_t = A'_t - A_t$ is a mart. of bounded variation and $\forall$ $\tilde{M}$ as above

$$E[\tilde{M}_t(A'_t - A_t)] = E[\int_{(0,t]} \tilde{M}_{s-} d(A'-A)_s]$$

$$= \lim_n E[\sum_{k=1}^{n} \tilde{M}_{t_{k-1}}((A'-A)_{t_k} - (A'-A)_{t_{k-1}})]$$

$$= \lim_n E[\sum_{k=1}^{n} \tilde{M}_{t_{k-1}}((M-M')_{t_k} - (M-M')_{t_{k-1}})]$$

$$= 0 \qquad (M, M' \text{ mart})$$

– dominated convergence, $\tilde{M}$ bdd.

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Let $\tilde{M}$ be a r-cont modification of

$$E[1_{A'_{t_0}-A_{t_0}\geq a} \mid \mathcal{F}_t] \quad \text{or}$$

$$E[1_{A'_{t_0}-A_{t_0}\leq -a} \mid \mathcal{F}_t] \qquad t \leq t_0.$$

gives

$$0 = E[\tilde{M}_{t_0}(A'_{t_0}-A_{t_0})] \qquad t=t_0$$

$$= E[(A'_{t_0}-A_{t_0})\, 1_{A'_{t_0}-A_{t_0} \substack{\geq a \\ \leq -a}}]$$

$\Rightarrow \quad P(A_t = A'_t) = 1 \quad \forall \; t \geq 0$

$\Rightarrow$ indistinguishability b/y r-continuity.

11) existence.

1) establish existence on $[0,a]$ $\forall$ $a>0$
- extence on $[0,\infty)$ follows by uniqueness on $[0,a]$ (process restricted to a well defined).

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b) Set $Y_t = X_t - E[X_0 | \mathcal{F}_t]$ $t \le 0$

• Y non-positive sub-mart w/ $Y_0 = 0$.

c) Fix n, consider the dyadic partition

$$\Pi_n = \left\{ \frac{j}{2^n} \cdot 0 \;\middle|\; j = 0, 1, \ldots, 2^n \right\}$$

Discrete result:

$\exists$ A^n predictible, M^n martingale so that

$$Y_{t_j^n} = A^n_{t_j^n} + M^n_{t_j^n} \qquad t_j^n = j/2^n \cdot 0$$

$$= A^n_{t_j^n} + E[M^n_0 | \mathcal{F}_{t_j^n}]$$

$$= A^n_{t_j^n} - E[A^n_0 | \mathcal{F}_{t_j^n}] \quad (Y_0 = 0)$$

d) (key step) Show that

$$\{A^n_0\}_{n=1,2,\ldots} \quad \text{is} \quad \text{u.i.}$$

- uses DL property / very technical.

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e) $\{A_0^n\}_{n=1,2,\ldots}$ U.I. $\Rightarrow \exists$ a sub-sequence (still labeled n) and an integrable r.v. A_0 s.t.

1) $A_0^n \xrightarrow{w-L^1} A_0$ i.e.

$$E[\xi A_0^n] \to E[\xi A_0] \quad \forall \text{ bdd } \mathcal{F}_0 \text{ r.v. } \xi.$$

in particular

$$E[\xi A_0^n] \to E[\xi A_0] \quad \forall \text{ bdd } \mathcal{F}_t \text{ r.v. } \xi, \ t \le 0.$$

f) Define $A = \{A_t\}_{t \le 0}$ is a r-cont modification of

$$A_t = Y_t + E[A_0 | \mathcal{F}_t] \qquad t \le 0$$

has r-cont mod

$$= X_t + E[(A_0 - X_0) | \mathcal{F}_t]$$

$$= X_t - m_t \;;\; m_t = E[X_\infty - A_\infty \mid \mathcal{F}_t]$$

(r-cont. modification)

g) show that A is natural, increasing.

Corollary

If X is of class D then M is u.i. and A is integrable in that $E[A_\infty] < \infty$.

Also

$X_t = M_t + A_t \to X_\infty$ a.s. and in L^1

– follows from f) above.

Prop

If X is continuous, ≥ 0, sub-mart then A is continuous

– note: X is DL. Proof of this on pg 28/29. Too technical to be presented here.