Stopping Time σ-Algebras and Optional Sampling

$(\Omega, \mathcal{F}, \mathbb{F}, P)$ given.

T: $\mathbb{F}$ stopping time.

Q. What do we mean by the information up to T?

- we want $A \in \mathcal{F}$ s.t. $\forall\, t \geq 0$ if we have observed T (i.e. $T \leq t$) then we also know if A occurred.

i.e. we want $A \cap \{T \leq t\} \in \mathcal{F}_t$.

So, define

$$\mathcal{F}_T = \{A \in \mathcal{F} \mid A \cap \{T \leq t\} \in \mathcal{F}_t \ \forall\, t \geq 0\}.$$

facts

1) $\mathcal{F}_T$ is a σ-algebra

- make sure you can show this.

2) T is $\mathcal{F}_T$ mbl

$$\{T \le s\} \cap \{T \le t\} = \{T \le s \wedge t\}$$

$$\in \mathcal{F}_{s \wedge t} \subseteq \mathcal{F}_t \quad \blacksquare.$$

Similarly, if T is optional, define

$$\mathcal{F}_{T+} = \{A \in \mathcal{F} \mid A \cap \{T \le t\} \in \mathcal{F}_{t+} \ \forall t \geq 0\}$$

$\mathcal{F}_{T+}$ is also a σ-algebra.

Basic Properties. (S, T stopping)

1) $A \in \mathcal{F}_S \Rightarrow A \cap \{S \le T\} \in \mathcal{F}_T$

so if $S \le T$ then $\mathcal{F}_S \subseteq \mathcal{F}_T$.

pf: $A \cap \{S \le T\} \cap \{T \le t\}$

$= A \cap \{S \le t\} \cap \{T \le t\}$

$\cap \{S \wedge t \le T \wedge t\} \quad \blacksquare.$

2) $\mathcal{F}_{S \wedge T} = \mathcal{F}_S \cap \mathcal{F}_T$.

3) $\{T \le S\}, \{S \le T\}, \{T < S\}, \{S < T\}, \{S = T\}$

all in $\mathcal{F}_{T \wedge S}$.

4) $E[Z \mid \mathcal{F}_T] = E[Z \mid \mathcal{F}_{S \wedge T}]$ a.s. on $\{T \le S\}$

$\forall$ integrable Z

5) $E[E[Z \mid \mathcal{F}_T] \mid \mathcal{F}_S] = E[Z \mid \mathcal{F}_{T \wedge S}]$

- Tower Property

(4, 5) are very important, so they will be HW exercises.)

6) If T is optional then

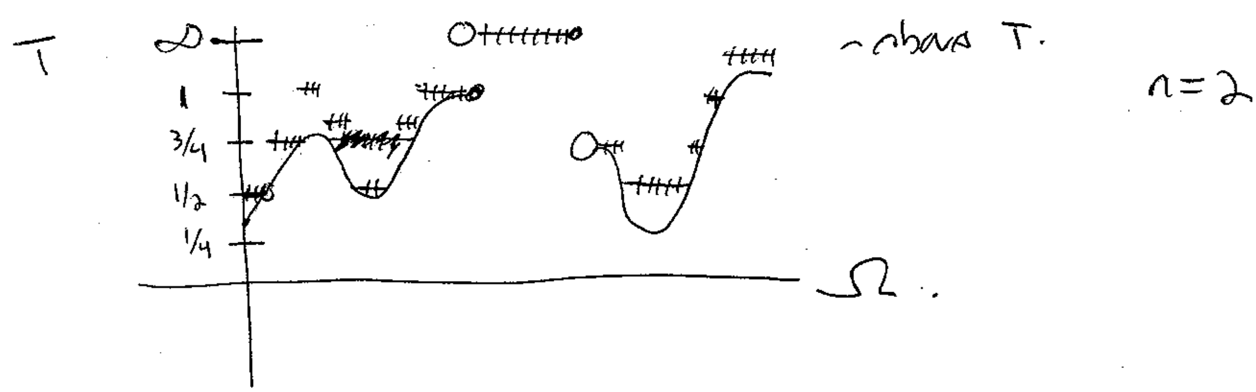
$$\mathcal{F}_{T+} = \{A \in \mathcal{F} \mid A \cap \{T < t\} \in \mathcal{F}_t \ \forall t\}$$

as well and T is $\mathcal{F}_{T+}$ mbl.

④

7) If T is optional, set

$$T_n = \begin{cases} T & \text{on } \{T = \infty\} \\ \frac{k}{2^n} & \text{on } \left\{\frac{k-1}{2^n} \le T < \frac{k}{2^n}\right\} \end{cases} \qquad n, k \geq 1$$



Then

T_n are stopping (easy)

$T_n \to T$ (easy)

$A \in \mathcal{F}_{T+} \Rightarrow A \cap \{T_n = \frac{k}{2^n}\} \in \mathcal{F}_{\frac{k}{2^n}}$

- relatively easy.

Recall, for a process X (mbl) we defined the process stopped at T (random time) as

$$X_T(\omega) = X_{T(\omega)}(\omega) \quad \text{on } \{T < \infty\}.$$

(5)

In fact if X_∞ is defined we can define $X_T = X_\infty$ on $\{T=\infty\}$ as well.

With a little more regularity in T, X we can say more about X_T.

- X prog mbl, T stopping

$\Rightarrow$ 1) X_T is $\mathcal{F}_T$ mbl

2) The "stopped process"

$$X^T_t = X_{T \wedge t} \qquad t \geq 0$$

is prog mbl too

- in particular, these conclusions hold for r-cont, adapted processes
- important, because we want to stop processes when they hit levels and not lose adaptivity properties.

Optional Sampling

Basic Idea: If X is a sub-martingale (resp. mart, super-mart) we know $E[X_t|\mathcal{F}_s] \geq X_s$ $\forall\ 0 \leq s \leq t$.
(resp $=, \leq$)

Q. Can we replace the deterministic $s \leq t$ above with (optional) stopping times and get the same result?

i.e. $E[X_T|\mathcal{F}_S] \geq X_S$ $S \leq T$

(resp. $=, \leq$).

Interpretation: If so, then for a mart. X we have $E[X_T|\mathcal{F}_S] = X_S$

- no way to "beat the game" by judiciously choosing when to quit.

(7)

A. $\boxed{\text{YES}}$, we can replace $s \le t$ with $S \le T$ stopping (optional) but we have to be careful about ∞ since S, T can take this value.

Definition (Last Element)

X: sub-mart. X_∞: $\mathcal{F}_\infty$ measurable, integrable r.v.

$$\mathcal{F}_\infty = \sigma(\mathcal{F}_t ; t \geq 0) \subseteq \mathcal{F}.$$

If $\forall\, t \geq 0$ we have

$$X_t \le E[X_\infty \mid \mathcal{F}_t] \qquad t \geq 0$$

then we say X is a sub-mart with <u>last element</u> X_∞.

- Similar definition for a super-mart.

Warning

The X_∞ of this definition does not necessarily have to be the Limit $\lim_{t\to\infty} X_t$ we encountered in the sub-martingale convergence theorem.

-unfortunate notation.

ex.

$$X_t = e^{W_t - t/2} \qquad t \geq 0$$

W a B.M. (have not actually defined this yet). One can show

1) X is a ≥ 0 mart so

$$\sup_{t \geq 0} E[X_t] = 1 < \infty.$$

2) $\Rightarrow X_t \to \tilde{X}_\infty$ a.s. And,

$\tilde{X}_\infty = 0.$

But, $\tilde{X}_\infty = 0$ cannot be a last element b/c $X_t \geq 0$, $E[\tilde{X}_\infty | \mathcal{F}_t] = 0$.

But, if X has a last element then $E[X_t^+] \leq E[|X_\infty|] < \infty$ so $X_t \to \tilde{X}_\infty$.

- be careful with this.
- this definition is sometimes referred to as "X being closable".

Thm (Optional Sampling)

X: r-cont sub-mart with last element

T, S optional. $S \leq T$.

Then

$E[X_T | \mathcal{F}_{S+}] \geq X_S$ S optional

$E[X_T | \mathcal{F}_S] \geq X_S$ S stopping

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In particular, $E[X_T] \geq E[X_0]$ with equality for martingales.

pf

a) Discrete Time Results.

- Chung pp. 338-342.

- we will prove for S, T bounded.

- extension to unbounded delicate and we don't have time.

pf

Let $A \in \mathcal{F}_S$. Then for $k \leq j$

$$A \cap \{S = j\} \cap \{T > k\}$$

$$\triangleq A_j \cap \{T > k\}$$

$$\in \mathcal{F}_k$$

$$\rightarrow A \cap \{S = j\} \in \mathcal{F}_j \subseteq \mathcal{F}_k$$

$$\{T > k\} = \{T \leq k\}^c \subseteq \mathcal{F}_k.$$

(11)

So $\quad E[X_k 1_{A_j \cap \{T > k\}}] \le E[X_{k+1} 1_{A_j \cap (T > k)}]$

or

$$E[X_k 1_{A_j \cap \{T \geq k\}}] \le E[X_k 1_{A_j \cap \{T = k\}}] + E[X_{k+1} 1_{A_j \cap \{T > k\}}]$$

or

$$E[X_k 1_{A_j \cap \{T \geq k\}}] - E[X_{k+1} 1_{A_j \cap \{T \geq k+1\}}] \le E[X_T 1_{A_j \cap \{T = k\}}]$$

sum for $k = j$ to m where $m \geq T$.

$$E[X_j 1_{A_j \cap \{T \geq j\}}] - E[X_{m+1} 1_{A_j \cap \{T \geq m+1\}}] \le E[X_T 1_{A_j \cap \{j \le T \le m\}}]$$

$\Rightarrow$

$$E[X_j 1_{A_j \cap \{T \geq j\}}] \le E[X_T 1_{A_j \cap \{T \geq j\}}]$$

But $A_j = A \cap \{S = j\} \cap \{T \geq j\}$
$= A \cap \{S = j\}$.

so

$$E[X_j 1_{A \cap \{S=j\}}] \leq E[X_T 1_{A \cap \{S=j\}}]$$

or

$$E[X_S 1_{A \cap \{S=j\}}] \leq E[X_T 1_{A \cap \{S=j\}}]$$

summing for $j = 1, \ldots, m$ gives

$$E[X_S 1_A] \leq E[X_T 1_A] \quad A \in \mathcal{F}_S.$$

Now, assume the full discrete time result holds

- note: optional = stopping (by def) in discrete time.

i.e. $X = \{X_n\}$ sub-mart w/ last element.

S, T optional $S \leq T$

$$\Rightarrow X_S \leq E[X_T | \mathcal{F}_S].$$

- we now lift to continuous time.

2) Approximations

$$S^n = \begin{cases} S & \{S=\infty\} \\ \frac{k}{2^n} & \{\frac{k-1}{2^n} \le S < \frac{k}{2^n}\} \end{cases}$$

$$T^n = \begin{cases} T & \{T=\infty\} \\ \frac{k}{2^n} & \{\frac{k-1}{2^n} \le T < \frac{k}{2^n}\} \end{cases}$$

- discrete optional/stopping by prev. result.

$$\Rightarrow \quad E[X_{T^n} \mid \mathcal{F}_{S^n}] \geq X_{S^n}$$

$$\Rightarrow \quad E[X_{T^n} 1_{A^n}] \geq E[X_{S^n} 1_{A^n}] \quad A^n \in \mathcal{F}_{S^n}$$

fact (HW) $\quad \mathcal{F}_{S+} = \bigcap_{n=1}^{\infty} \mathcal{F}_{S^n}$

$$\Rightarrow A \in \mathcal{F}_{S+} \Rightarrow A \in \mathcal{F}_{S^n} \quad \forall n$$

$$\Rightarrow E[X_{T^n} 1_A] \geq E[X_{S^n} 1_A] \quad A \in \mathcal{F}_{S+}$$

Note: if S is a stopping time, since $S \le S^n$ we know $\mathcal{F}_S \subseteq \mathcal{F}_{S^n}$ $\forall n$

so $\forall A \in \mathcal{F}_S$ we have

$$E[X_{T^n} 1_A] \geq E[X_{S^n} 1_A]$$

c) Uniform Integrability.

Claim: $\{X_{T^n}; \mathcal{F}_{T^n}\}$, $\{X_{S^n}; \mathcal{F}_{S^n}\}$ are backward sub-martingales.

- discrete optional sampling + $T_n, S_n \downarrow$.

And

$E[X_{T^n}] \geq E[X_0]$ (same for S^n)

by discrete optional sampling

$\Rightarrow$ $\{X_{T^n}\}$, $\{X_{S^n}\}$ u.i.

d) finishing touches.

R-continuity: $X_{T^n} \to X_T$, $X_{S^n} \to X_S$

so for $A \in \mathcal{F}_{S+}$ ($A \in \mathcal{F}_S$ S stopping)

$$
\begin{aligned}
E[X_T 1_A] &= \lim_n E[X_{T^n} 1_A] \\
&\geq \lim_n E[X_{S^n} 1_A] \\
&= E[X_S 1_A]. \quad \blacksquare
\end{aligned}
$$

Easy Corollary

If T bdd ($T \leq a$) then we do not require X to have a last element.

Cool Corollary.

X: r-cont, $E[|X_t|] < \infty$ $\forall t \geq 0$

Then X is a sub-mart iff

$E[X_T] \geq E[X_S]$ $\forall$ bdd stopping times $T \geq S$.

| $0 \leq s < t$, $A \in \mathcal{F}_s$. s.t. $S = s 1_A + t 1_{A^c}$, $T = t$, |