1

Stochastic Processes
Sigma-fields
Stopping Times

Setup

$(\Omega, \mathcal{F}, P)$: probability space.

Ω : sample space

$\mathcal{F}$: collection of events (sigma-algebra)

P : probability function $P: \mathcal{F} \mapsto [0,1]$
(countably additive)

Stochastic Process

Collection of random variables indexed by time

$X: \mathbb{T} \mapsto S$

$\mathbb{T}$: index set ($\mathbb{T} = \{0,1,2,\dots\}$ "discrete"
or $\mathbb{T} = [0,\infty)$ "continuous time")

S: state space.

S: Polish space (complete, separable metric space w/ Borel σ-algebra $\mathcal{B}(S)$)

$S = \mathbb{R}, \mathbb{R}^d, C[0,T]$, etc.

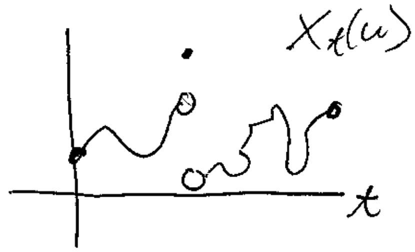
for the time being, assume $S = \mathbb{R}$.

X: s.t. X_t is a r.v. $\forall\, t \in \mathbb{T}$.

$X_t : \Omega \mapsto \mathbb{R}$ s.t. X_t is $\mathcal{F}/\mathcal{B}(\mathbb{R})$ mbl.

$\{ X_t \in A \} \in \mathcal{F} \quad \forall\ A \in \mathcal{B}(\mathbb{R})$.

For fixed $\omega \in \Omega$ the map $t \mapsto X_t(\omega)$ from $\mathbb{T}$ to $\mathbb{R}$ is the sample path of $X(\omega)$



③ For continuous time processes, we say that X is (left, right) continuous if the sample paths are (left, right) continuous with probability one.

Let X, Y be two cont. time stochastic processes

Q. What do we mean by $X = Y$?

0) $X_t(\omega) = Y_t(\omega) \quad \forall\ t \geq 0, \omega \in \Omega$

- too strong, X, Y truly are the same.

1) Indistinguishable

$$P(X_t = Y_t \quad \forall\ t \geq 0) = 1$$

2) Modification

$$P(X_t = Y_t) = 1 \qquad \forall \quad t \geq 0$$

3) Same Finite-Dimensional Distributions (fidis)

$$P((X_{t_1}, \dots, X_{t_n}) \in A) = P((Y_{t_1}, \dots, Y_{t_n}) \in A) = 1$$

④

for $t_1, \dots, t_n \geq 0$, $A \in \mathcal{B}(\mathbb{R}^n)$, $n = 1, 2, 3, \dots$

<u>Clearly</u>

1) $\Rightarrow$ 2) $\Rightarrow$ 3)

But, converses not true:

① 2) $\not\Rightarrow$ 1) $\quad X_t \equiv 0$, $Y_t = \mathbb{1}_{t=Z}$

for $Z \sim \text{Exp}(1)$

$$P(Y_t = 0) = 1 - P(Z = t) = 0.$$

② 3) $\not\Rightarrow$ 2) $\quad X_t = \begin{cases} Z & t = 1 \\ 0 & t \neq 1 \end{cases}$, $Y_t = \begin{cases} -Z & t = 1 \\ 0 & t \neq 1 \end{cases}$

for $Z \sim N(0,1)$.

Also, note that X, Y can be defined on different prob. spaces and still have the same fidis.

Good News: with same path regularity,

2) $\Rightarrow$ 1)

pf. (X, Y continuous path.) follows from

$$\{X_t = Y_t \ \forall \ t \geq 0\} = \bigcap_{t \geq 0 \cap \mathbb{Q}} \{X_t = Y_t\}.$$

- left, right cont. OK too.

Filtrations.

We think of a stochastic process as evolving through time.

We also heuristically think of "information" evolving through time as well.

- how to make this formal?

A collection of σ-algebras $\{\mathcal{F}_t\}_{t \in \mathbb{T}}$ is a filtration if.

i) $\mathcal{F}_t \subseteq \mathcal{F}$ is a σ-alg. $\forall\, t \in \mathbb{T}$

ii) $\mathcal{F}_t \subseteq \mathcal{F}_s$ if $t \le s$.

$\mathbb{T} = \{0, 1, 2, \ldots\}$ "discrete filtration"

$\mathbb{T} = [0, \infty)$ "continuous filtration"

- work primarily w/ continuous filtrations.

<u>Example.</u>

X : cont. time stochastic process

for each $t \ge 0$, set $\mathcal{F}^X_t = \sigma(X_s;\, s \le t)$

then $\{\mathcal{F}^X_t\}_{t \ge 0}$ is the filtration generated by X.

$A \in \mathcal{F}^X_t \Rightarrow$ we know if A has occurred by t.

Write $\mathbb{F} = \{\mathcal{F}_t\}_{t \geq 0}$ for the filtration ($\mathbb{F}^X$ for natural filtration)

As with processes, there is a notion of (left, right) continuity for filtrations

1) $\mathcal{F}_{t-} = \sigma\left(\bigcup_{s<t} \mathcal{F}_s\right) \quad \mathcal{F}_{0-} = \mathcal{F}_0$

- collection of events immediately prior to t.

2) $\mathcal{F}_{t+} = \bigcap_{\varepsilon>0} \mathcal{F}_{t+\varepsilon}$

- collection of events immediately after t.

Say $\mathbb{F}$ is

1) right continuous if $\mathcal{F}_t = \mathcal{F}_{t+}$
2) left continuous if $\mathcal{F}_t = \mathcal{F}_{t-}$

3) continuous if $\mathcal{F}_t = \mathcal{F}_{t+} = \mathcal{F}_{t-}$

ex.

$$X_t = \begin{cases} 0 & t < 1 \\ Z & t \geq 1 \end{cases} \qquad Z \sim N(0,1)$$

$$\Rightarrow \mathcal{F}^X_t = \mathcal{F}^X_{t+}, \quad \mathcal{F}^X_1 \neq \mathcal{F}^X_{1-}.$$

Measurability and Adaptivity.

Let $(\Omega, \mathbb{F}, \mathcal{F}, P)$ be a given filtered probability space

Let $X = \{X_t\}_{t \geq 0}$ be a stochastic process

$X_t : \Omega \mapsto \mathbb{R}$ is $\mathcal{F}/\mathcal{B}(\mathbb{R})$ mbl.

We now develop notions of measurability for the process X taking into account the filtration.

(9)

First, we say that X is mbl if

$$X(t,\omega) : [0,\infty) \times \Omega \mapsto \mathbb{R}$$

is

$$\mathcal{B}([0,\infty) \times \Omega) / \mathcal{B}(\mathbb{R}) \quad \text{mbl.}$$

- joint map is mbl, not just each X_t.

- Fubini: $t \mapsto X_t(\omega)$ is mbl

  $t \mapsto E[X_t]$ is mbl provided

  $E[|X_t|] < \infty \quad \forall t.$

Next, we connect X with the filtration $\mathbb{F}$.

We say X is <u>adapted</u> to $\mathbb{F}$ if X_t is $\mathcal{F}_t$ mbl $\forall\, t$.

$$\{X_t \in A\} \in \mathcal{F}_t \quad \forall \quad A \in \mathcal{B}(\mathbb{R})$$

information in $\mathcal{F}_t$ sufficient to know X_t.

(10)

We say X is progressively mbl if $\forall\, t$

$$X(s,\omega) : [0,t] \times \Omega \mapsto \mathbb{R}$$

is $\mathcal{B}([0,t]) \times \mathcal{F}_t / \mathcal{B}(\mathbb{R})$ mbl.

$$\{(s,\omega) \mid s \le t,\ X_s(\omega) \in A\} \in \mathcal{B}([0,t)) \times \mathcal{F}_t$$

$\forall\ A \subseteq \mathcal{B}(\mathbb{R})$.

Now, if X is adapted to $\mathcal{F}$ then

$\forall\ s \le t$: $\{X_s \in A\} \in \mathcal{F}_s \subseteq \mathcal{F}_t$

but progressive measurability allows to consider all the $s \le t$.

Similarly to the relationship between indistinguisability and modification in the presence of path-regularity adaptivity implies progressive mblity.

(11)

L. If X is (left, right) continuous and adapted, it is progressively mbl.

P. (right-continuity).

Set $X^n_s(\omega) = X_{\frac{(k+1)t}{2^n}}(\omega) \quad s \in \left(\frac{kt}{2^n}, \frac{(k+1)t}{2^n}\right]$

$k = 0, \dots, 2^n - 1, \quad n = 1, 2, 3, \dots$

$X^n_0(\omega) = X_0(\omega)$.

Since

$1_{s \in \left(\frac{kt}{2^n}, \frac{(k+1)t}{2^n}\right]} \in \mathcal{B}[[0,t]]$

$1_{X_{\frac{(k+1)t}{2^n}}(\omega) \in A} \in \mathcal{F}_{\frac{(k+1)t}{2^n}} \subseteq \mathcal{F}_t$

X^n is progressively mbl.

But $X^n_s(\omega) \to X_s(\omega) \quad \forall\ \omega$.

$\therefore X$ is prog. mbl.

Message: paths of X have to be really wild for adaptivity to not imply prog. mblity.

Stopping Times.

We would like to evaluate the process, not only at fixed times, but also at "random times" corresponding to events of interest.

e.g. where and when is X the first time X falls out of some set.

How can we do this?

(13)

A random time T is a map $T : \Omega \mapsto [0, \infty]$ which is $\mathcal{F}$ mbl

- just a non-negative r.v. which may be infinite

If T a random time and X a continuous time stochastic process (jointly measurable in t, ω) define

$$X_T(\omega) \triangleq X_{T(\omega)}(\omega) \quad \text{on } \{T < \infty\}$$

$\Rightarrow X_T$ a r.v. since compositions of rv are r.v.

Now, consider T and a (continuous time) filtration $\mathbb{F}$. We would like to know, for each t, if T has occurred by t, using <u>only</u> our accumulated information.

"Stopping Times" are precisely those random times for which we can do this.

We say a random time T is a <u>stopping time</u> for $\mathbb{F}$ if

$$\{T \leq t\} \in \mathcal{F}_t \qquad t \geq 0$$

We say T is an <u>optional time</u> for $\mathbb{F}$ if

$$\{T < t\} \in \mathcal{F}_t \qquad t \geq 0$$

<u>Proposition.</u>

a) Stopping $\Rightarrow$ Optional

b) Stopping $\Leftrightarrow$ Optional if $\mathbb{F}$ is r-continuous.

15

pf

a) $\{T < t\} = \bigcup_n \{T \le t - 1/n\}$

b) $\{T \le t\} = \bigcap_{n > m} \{T < t + 1/n\} \quad \forall m >$

$\left(\Rightarrow \{T \le t\} \in \mathcal{F}_{t + 1/m} \quad \forall\, m > 0. \right)$

Basic Properties of Optional / Stopping Times.

a) T optional, $\theta > 0$ constant

$\Rightarrow T + \theta$ stopping

– easy

b) T, S stopping

$\Rightarrow T \wedge S, \ T \vee S, \ T + S$ stopping

<u>note</u>

$$\{T+S>t\} = \{T=0, S>t\}$$
$$\cup \{S=0, T>t\} \cup \{T \geq t, S>0\}$$
$$\cup \{0<T<t, S+T>t\}. \quad \text{(check!)}$$

$$= \{T=0, S>t\} \cup \{S=0, T>t\}$$
$$\cup \{T \geq t, S>0\} \cup \left(\bigcup_{\substack{r \in \mathbb{Q} \\ 0<r<t}} \{r<T<t, S>t-r\} \right)$$

(check!) ∎

c) Let $\{T_n\}_{n=1,2,\ldots}$ be optional. Then

$\sup_n T_n$; $\inf_n T_n$; $\overline{\lim}_n T_n$, $\underline{\lim}\, T_n$

are all optional

pf

i) $\{\sup_n T_n \leq t\} = \bigcap_n \{T_n \leq t\}$

ii) $\forall\, m>0$

$$\{\overline{\lim}_n T_n \leq t\} = \bigcap_{m \geq M} \bigcup_n \{\sup_{r \geq n} T_r \leq t + 1/m\}$$

- similar identities for $\inf_n T_n$, $\underline{\lim}_n T_n$.

Q. If the T_n are stopping, which of the above are as well?

Important Stopping Times.

X: mbl stochastic process taking values in $\mathbb{R}^d$

$\mathbb{F} = \mathbb{F}^X$

Γ: Borel subset of $\mathbb{R}^d$

Set

$$\tau_\Gamma = \inf[t \geq 0 \mid X_t \in \Gamma]$$

- "hitting time" of X to Γ
- we wish for this to be stopping b/c if we know X then morally we should ~~not~~ know if X hit Γ.

- but, ~~wh~~ we need some path regularity and set regularity.

Proposition.

a) Γ open, X r-continuous $\Rightarrow \tau_\Gamma$ optional

b) Γ closed, X continuous $\Rightarrow \tau_\Gamma$ stopping

pf

a) $\{\tau_\Gamma < t\} \stackrel{?}{=} \bigcup_{\substack{s<t \\ s\in\mathbb{Q}}} \{X_s \in \Gamma\}$

- show this.

b) Set $\Gamma^n = \{x \mid \inf_{y\in\Gamma} |x-y| < 1/n\}$

use a) + continuity.