Math 21-880 Final

December 8, 2014

This is a closed book, closed notes exam. No calculators or smart phones are allowed. You have 3 hours to complete the exam. Please mark your answers clearly and put your name on each piece of paper you submit. There are five questions on the exam.

(1) **20 Points.** Let W be a standard one-dimensional Brownian motion with $W_0 = 0$. Set $\sigma = \inf\{t \geq 0 \mid W_t = 1\}$ and $\tau = \inf\{t \geq \sigma \mid W_t = -1\}$. Note that by definition $\tau > \sigma$. Compute $\mathbb{P}[\tau < t]$.

(2) **20 Points.** Consider the SDE

$$dX_t = b(X_t)dt + dW_t; \qquad X_0 = x \in \mathbb{R}^d.$$

Here, W is a standard d-dimensional Brownian motion and b is a bounded, Lipschitz function. Show that for any Borel set $A \subseteq \mathbb{R}^d$ with positive (Lebesgue) volume that $\mathbb{P}[X_t^x \in A] > 0$ for all $x \in \mathbb{R}^d$ and $t \geq 0$.

(3) **20 Points.** Let X solve the SDE

$$dX_t = \sigma(X_t)dW_t; \qquad X_0 = x \in \mathbb{R}^d.$$

Here, W is again a d-dimensional Brownian motion and we assume σ is Lipschitz, symmetric and point-wise (in x) positive definite. Now, think of X as the share price of some traded asset: i.e. $X_t(\omega)$ is the price at (t, ω). Let $\pi = \{\pi_t\}_{t \geq 0}$ denote a trading strategy: i.e. $\pi_t(\omega) \in \mathbb{R}^d$ is the number of shares of X we hold at (t, ω). For a given initial wealth $w_0 \in \mathbb{R}$, the wealth processes associated to π is denoted by $\mathcal{W}^\pi$ and satisfies the formula

$$\mathcal{W}_t^\pi = w_0 + \int_0^t \pi_u^{\mathrm{T}} dX_u = w_0 + \int_0^t \pi_u^{\mathrm{T}} \sigma(X_u) dW_u,$$

provided the integrals are well-defined. Note that in particular we require π to be adapted to the (augmented) filtration generated by W.

i) **6 Points.** We say the (X, W) market is complete on $[0, T]$ if for any bounded $\mathcal{F}_T$ measurable random variable H, there is some initial capital w_0 and trading strategy π such that $H = \mathcal{W}_T^\pi$ with probability one. $\mathcal{W}^\pi$ is called the "replicating" process for H. Show that the (X, W) market is complete.

ii) **8 Points.** Now assume that $H = g(X_T)$ for a bounded continuous function g. Identify a partial differential equation such that if $u \in C^{1,2}((0,T)\times\mathbb{R}^d)$ solves the PDE then u "should" take the form $u(t,y) = E\left[g(X_T)\,|X_t = y\right]$. Show that if $u \in C^{1,2}((0,T)\times\mathbb{R}^d)$ is a bounded solution of this PDE then u does admit the representation $u(t,y) = E\left[g(X_T) \mid X_t = y\right] = E\left[g(X^y_{T-t}\right]$ where X^y is the solution of the above SDE starting at y.

iii) **6 Points.** In the setting of b), assume u solves the PDE and admits the stochastic representation. Identify the initial capital w_0 and trading strategy π explicitly for the replicating wealth process of $g(X_T)$.

(4) **20 Points.** Let W be a standard Brownian motion starting at 0. Recall that the local time of W near a on $[0,t]$ is given by

$$L_t(a) = \lim_{\varepsilon\downarrow 0}\frac{1}{2\varepsilon}\mathrm{Leb}\left[s \leq t \mid |W_s - a| \leq \varepsilon\right],$$

in that we showed the limit exists almost surely and satisfies Tanaka's formula.

i) **10 Points.** Sketch the proof of Tanaka's formula using the approximating functions g^ε from lecture (or some other approximating functions if you wish). Provide as much detail as you can, but don't spend the whole exam time filling in every step if you are stuck!

ii) **10 Points.** Recall that we also showed for all Borel measurable non-negative functions h on $\mathbb{R}$ the almost sure identity

$$\int_0^t h(W_s)ds = \int_{\mathbb{R}} h(a)L_t(a)da \tag{1}$$

Now, let $f \in C^2(\mathbb{R})$ be strictly increasing with $f(\pm\infty) = \pm\infty$. Define the process $Y_t = f(W_t)$. Motivated by (1) above, we define the local time $L^Y_t(a)$ as the two parameter random field such that (amongst other properties) for all non-negative Borel measurable functions h on $\mathbb{R}$ we have almost surely that

$$\int_0^t h(Y_s)d\langle Y\rangle_s = \int_{\mathbb{R}} h(a)L^Y_t(a)da; \qquad t \geq 0.$$

Assuming it exists, identify $L^Y_t(a)$ explicitly.

(5) **20 Points.** For a fixed $T > 0$, identify the Laplace transform of $\int_0^T W_t^2dt$ where W is a standard one dimensional Brownian motion starting at 0, i.e. compute

$$E\left[e^{-\lambda\int_0^T W_t^2dt}\right]; \qquad \lambda > 0. \tag{2}$$

Complete this answer through the following steps:

i) **5 Points.** If $\phi \in L^2[0,T]$ is deterministic show that $\int_0^T \phi_u dB_u \sim N(0, \int_0^T \phi_u^2 du)$ where B is a standard one-dimensional Brownian motion.

ii) **5 Points.** Define Y as the solution of the SDE $dY_t = -\gamma Y_t dt + dB_t, Y_0 = 0$ where B is a standard one-dimensional Brownian motion and $\gamma > 0$. Compute the distribution of Y_T and use this to evaluate $E\left[e^{\beta Y_T^2}\right]$ for $\beta \in \mathbb{R}$. Are there any restrictions upon β?

iii) **10 Points.** Use your answer above to compute (2) for any $\lambda > 0$.