Recommended Problems for the Lectures from 11/23/15 through 12/9/15

(1) (Øksendal exercise 5.11 on page 79). For a fixed $a, b \in \mathbb{R}$ consider the following 1-dimensional equation
$$dY_t = \frac{b - Y_t}{1-t}dt + dB_t; \qquad 0 \le t < 1, Y_0 = a.$$
where B is a standard 1-dimensional Brownian motion. Y is called the *Brownian Bridge* from on $[0,1]$ from a to b.
(a) Verify that
$$Y_t = a(1-t) + bt + (1-t)\int_0^t \frac{1}{1-s}dB_s \qquad 0 \le t < 1.$$
(b) Prove that $\lim_{t\uparrow 1} Y_t = b$ with probability one.
(c) Using the density for the passage times τ_b for B show that $M_t = \max_{s\le t} B_s$ is distributed like $|B_t|$.
(d) Set $M_1^{a,b} = \max_{s\le 1} Y_s$ as the maximum of the Brownian bridge from a to b on $[0,1]$ (here, we have defined $Y_1 = b$ in view of part (b)). Show that $M^{a,b}$ has density
$$f^{a,b}(y) = 2(2y - b - a)e^{-2(y-a)(y-b)}, \qquad y > \max\{a, b\}.$$

(2) (Taken from Prof. Iyer). Let b, σ be uniformly Lipschitz on $\mathbb{R}^d$. Note that this implies b, σ are of linear growth. As such, let X be the corresponding solution starting at x under the measure $\mathbb{P}^x$.
(a) Show that
$$\lim_{t\downarrow 0}\frac{1}{t}E^x\left[X_t - x\right] = b(x)$$
$$\lim_{t\downarrow 0}\frac{1}{t}E^x\left[(X_t^i - x^i)(X_t^j - x^j)\right] = \Sigma^{ij}(x)$$
where $\Sigma = \sigma\sigma^{\mathrm{T}}$.
(b) Show for each $\varepsilon > 0$ that $\lim_{t\downarrow 0}\frac{1}{t}\mathbb{P}^x\left[|X_t - x| > \varepsilon\right] = 0$.
(c) Now, assume that $\sigma(x) = 1_d$ is the identity matrix and that b is bounded (as well as Lipschitz). For any $\delta > 0$ set $\tau_\delta = \inf\{t \ge 0 \mid |X_t - x| \ge \delta\}$. Show that $\lim_{t\downarrow 0}\frac{1}{t}\mathbb{P}^x\left[\tau_\delta > t\right] = 0$.

(3) Let X and Y be solutions to the respective SDE's
$$dX_t = b(X_t)dt + \sigma(X_t)dW_t;$$
$$dY_t = \tilde{b}(Y_t)dt + \sigma(Y_t)d\tilde{W}_t;$$
Here, we assume that $b, \tilde{b}, \sigma$ are locally Lipschitz and such that the X and Y are non-explosive. We also assume that $\sigma = \sqrt{\Sigma}$ in that $\Sigma(x) > 0$ for each $x \in \mathbb{R}^d$ (though not uniformly elliptic) and σ is the unique positive definite square root of Σ. In fact, σ will be locally Lipschitz once

Σ is, thus we are actually assuming Σ is locally Lipschitz and locally elliptic. Thus, strong existence and uniqueness hold for X and Y.

Denote by $\mathbb{P}^x$ and $\mathbb{Q}^y$ the probability on path-space induced by X and Y respectively, where $X_0 = x$ and $Y_0 = y$. On this space the filtration $\mathbb{F}$ is the one generated by the coordinate mapping process.

(a) For each $T \geq 0$ and $x \in \mathbb{R}^d$ show that $\mathbb{P}^x$ is equivalent to $\mathbb{Q}^x$ on $\mathcal{F}_T$. (**Hint:** employ a localization argument where you can use Novikov's condition and then unwind the localization).

(b) Assume $\sigma(x) = 1_d$, $b = 0$ and $\tilde{b}$ is such that $|b(x)| \geq \delta > 0$ for some $\delta > 0$ and all $x \in \mathbb{R}^d$. Show that $\mathbb{P}^x$ and $\mathbb{Q}^x$ are mutually singular on $\mathcal{F}_\infty$.

(4) Problem 3.6.12* on page 209 and 3.6.13* on page 209.

Things to Read and Understand. Please read carefully the following items.

- Section 5.3 D on pages 308-311 where it is shown that weak existence and path-wise uniqueness imply strong existence.
- Theorem 3.6.11 on pages 207-208.