## Math 21-880 Final: SOLUTIONS

## December 14, 2015

This is a closed book, closed notes exam. No calculators or smart phones are allowed. You have 3 hours to complete the exam. Please mark your answers clearly and put your name on each piece of paper you submit. There are five questions on the exam.

The first two questions concern time changed solutions of stochastic differential equations. More precisely, let W be a standard one-dimensional Brownian motion with  $W_0 = 0$  and let  $\mathbb{F}^W$  be the augmented filtration generated by W. Let  $f : [0, \infty) \mapsto (0, \infty)$  be a (deterministic) strictly increasing smooth function with f(0) = 0 and  $\lim_{t \to \infty} f(t) = \infty$ .

Next, let  $b : [0, \infty) \times \mathbb{R}$  be bounded and globally Lipschitz in x (uniformly for  $t \ge 0$ ), and let Y be the corresponding strong solution to the stochastic differential equation (SDE)

$$dY_t = b(t, Y_t)dt + dW_t; \qquad Y_0 = y \in \mathbb{R}.$$

Define the f-time changed process X by

$$X_t = Y_{f(t)};$$
  $t \ge 0;$   $X_0 = Y_{f(0)} = Y_0 = y$ 

(1) **20 Points.** Assume for some  $t_0 > 0$  we have  $f(t_0) > t_0$ . Show that for any drift and diffusion functions  $\tilde{b}(t, x)$ ,  $\tilde{\sigma}(t, x)$ , X cannot be a strong solution to an SDE with  $\tilde{b}, \tilde{\sigma}$  and driving Brownian motion W.

**Hint:** There is more to do here than you might initially think. Use Girsanov's theorem and properties of Brownian motion to contradict the strong solution nature of X.

- (2) **25 Points.** Here, you will show that X can be identified as a weak solution to a certain SDE. Do this in the following steps:
  - a) **5 Points.** Let  $g = f^{-1}$  be the inverse of f. For any continuous function h and  $t \ge 0$  show that

$$\int_{0}^{f(t)} h(v) \left(\frac{d}{dv}\sqrt{\dot{g}(v)}\right) dv = \int_{0}^{t} h((f(u)) \left(\frac{d}{du}\frac{1}{\sqrt{\dot{f}(u)}}\right) du$$

b) **15 Points.** Consider the processes

$$Z_t = W_{f(t)}; \qquad B_t = \int_0^{f(t)} \sqrt{\dot{g}(v)} dW_v; \qquad t \ge 0.$$

Find a filtration  $\mathbb F$  so that so that B is an  $\mathbb F$  Brownian motion, and show that

$$\mathbb{P}\left[Z_t = \int_0^t \sqrt{\dot{f}(u)} dB_u; \quad \forall t \ge 0\right] = 1.$$
(1)

**Hint:** Recall that if  $\hat{B}$  is a Brownian motion under some filtration  $\hat{\mathbb{F}}$  and h is a smooth deterministic function of t then we can define the stochastic integral  $\int_{0}^{\cdot} h(t) d\hat{B}_{t}$  path-wise via

$$\left(\int_{0}^{\cdot} h(t)d\hat{B}_{t}\right)(\omega) \triangleq \left(h(\cdot)\hat{B}_{\cdot} - \int_{0}^{\cdot} \dot{h}(t)\hat{B}_{t}dt\right)(\omega)$$

c) **5 Points.** Find  $\tilde{b}, \tilde{\sigma}$  so that  $(\Omega, \mathcal{F}, \mathbb{P}), \mathbb{F}, (X, B)$  is a weak solution to the SDE with drift  $\tilde{b}$  and diffusion  $\tilde{\sigma}$ .

The next two problems deal with explosions and Martingales. In particular, when stochastic exponential local Martingales are defined in terms of solutions to SDEs, we seek to weaken the Novikov condition.

3) 15 Points. Let Z be a strictly positive local martingale with respect to some probability space  $(\Omega, \mathcal{F}, \mathbb{P})$  and filtration  $\mathbb{F}$  satisfying the usual conditions. Assume that  $\{\tau_n\}$  is an increasing sequence of stopping times such that  $\tau_n \uparrow \infty$  almost surely and such that the stopped process  $Z^n$ defined by  $Z^n = Z_{t \land \tau_n}$  is a Martingale for each n.

Prove the following: let T > 0 and define the measure  $\mathbb{Q}^n$  on  $\mathcal{F}_T$  via

$$\left. \frac{d\mathbb{Q}^n}{d\mathbb{P}} \right|_{\mathcal{F}_T} = Z_T^n.$$

If, for each T > 0 we have  $\lim_{n \uparrow \infty} \mathbb{Q}^n [T \leq \tau_n] = 1$  then Z is a true martingale.

4) **20 Points.** The CIR (Cox-Ingersoll-Ross) process is popular in math finance for modeling the interest rate. We say X is a CIR process if it has dynamics

$$dX_t = \kappa(\theta - X_t)dt + \xi \sqrt{X_t}dW_t; \qquad X_0 = x > 0.$$

Here, W is a standard d-dimensional Brownian motion with respect to some filtration satisfying the usual conditions, and  $\kappa, \theta, \xi > 0$  are constants. The state space for the process is  $D = (0, \infty)$  and the process is said not to explode if for all T > 0 we have that

$$\mathbb{P}\left[X_t \in D, 0 \le t \le T \mid X_0 = x\right] = 1.$$

It can be shown (you do not have to do this) that if  $\kappa > 0$ ,  $\kappa \theta > \xi^2/2$  then the process does not explode. Now, let  $A, B \in \mathbb{R}$ . Find parameter restrictions upon A, B so that the process

$$Z_t = \mathcal{E}\left(\int_0^{\cdot} \left(\frac{A}{\sqrt{X_t}} + B\sqrt{X_t}\right) dW_t\right)_t; \qquad t \ge 0,$$

is a Martingale.

5) **20 Points.** Let W, B be two independent Brownian motions and let  $-1 < \rho < 1$ . Fix x, y > 0 and define the processes

$$X_t^x \triangleq x + W_t; \qquad Y_t^y \triangleq y + \rho W_t + \sqrt{1 - \rho^2} B_t; \qquad t \ge 0.$$

Set  $\tau^x = \inf \{t \ge 0 \mid X_t^x = 0\}$  and  $\sigma^y = \inf \{t \ge 0 \mid Y_t^y = 0\}$  and note that  $\tau^x, \sigma^y < \infty$  almost surely for all x, y > 0, since the sample paths of Brownian motions are unbounded.

Let  $D = (0, \infty)^2$ . We say a function  $u \in C^2(D)$  if u is twice differentiable with continuous derivatives which are bounded on all compact subsets of D. However, we do not necessarily know the behavior of the derivatives of u near the boundary of D. For example,  $u(x, y) = (xy)^{-1} \in C^2(D)$  but clearly u is blowing up near x = 0 or y = 0.

Identify a partial differential equation (differential expression plus spatial boundary conditions) such that if  $u \in C^2(D)$  is a bounded solution of the PDE then u admits the representation

$$u(x,y) = \mathbb{P}\left[\tau^x < \sigma^y\right].$$

Be careful when dealing with the local Martingales here (e.g. make sure to stop the processes before you lose control over the stochastic integrands!).

## Solutions

(1) Recall that if X is a strong solution (for any  $\tilde{b}, \tilde{\sigma}$ ) with driving Brownian motion W and deterministic starting point y then X must be adapted to  $\mathbb{F}^W$  and hence for the  $t_0$  so that  $f(t_0) > t_0$  we have that  $X_{t_0} = Y_{f(t_0)}$  is  $\mathbb{F}^W_{t_0}$  measurable. Now, since  $f(t_0) > t_0$  this should not be possible. However, we have to prove this rigorously. To do this, note that since b is bounded, for some  $T > t_0$  we may define a measure  $\mathbb{Q}$  on  $\mathcal{F}^W_T$  by

$$\frac{d\mathbb{Q}}{d\mathbb{P}}\bigg|_{\mathcal{F}_T^W} = \mathcal{E}\left(-\int_0^\cdot b(u, Y_u) dW_u\right)_T$$

and that  $W_t^{\mathbb{Q}} = W_t + \int_0^t b(u, Y_u) du = Y_t, t \leq T$  is a  $\mathbb{Q}$  Brownian motion. Thus, we have i)  $Y_{f(t_0)} - Y_{t_0}$  is  $\mathcal{F}_{t_0}^W$  measurable (if X is a strong solution) and ii)  $Y_{f(t_0)} - Y_{t_0}$  is  $\mathbb{Q}$  independent of  $\mathcal{F}_{t_0}^W$  which in turn implies (note  $E^{\mathbb{Q}}[Y_{f(t_0)} - Y_{t_0}] = 0$ ):

$$\mathbb{Q}\left[Y_{f(t_0)} = Y_{t_0}\right] = 1,$$

and hence this equality holds with  $\mathbb{P}$  probability one as well. Coming back to the SDE for Y this implies with  $\mathbb{P}$  probability one

$$W_{f(t_0)} - W_{t_0} = -\int_{t_0}^{f(t_0)} b(u, Y_u) du \le K(f(t_0) - t_0),$$

where K is any bounding constant for b. But, this is a contradiction since  $W_{f(t_0)} - W_{t_0}$  is normally distributed under  $\mathbb{P}$ . Thus, X cannot be adapted to  $\mathbb{F}^W$  and hence there can be no strong solution.

(2) a) Since g(f(u)) = u we have that

$$\dot{g}(f(u)) = \frac{1}{\dot{f}(u)}; \qquad \ddot{g}(u) = -\frac{\ddot{f}(u)}{\dot{f}(u)^3}.$$

Thus, making the substitution u = g(v) or v = f(u) we have

$$\begin{split} \int_0^{f(t)} h(v) \left(\frac{d}{dv}\sqrt{\dot{g}(v)}\right) dv &= \int_0^{f(t)} h(v) \frac{\ddot{g}(v)}{2\sqrt{\dot{g}(v)}} dv \\ &= -\int_0^t h(f(u)) \frac{\ddot{f}(u)}{2\dot{f}(u)^{5/2}} \dot{f}(u) du \\ &= \int_0^t h(f(u)) \left(\frac{d}{du} \frac{1}{\sqrt{\dot{f}(u)}}\right) du \end{split}$$

b) For the filtration we take  $\mathcal{F}_t = \mathcal{F}_{f(t)}^W$  and note that Z, B are  $\mathbb{F}$  adapted continuous processes starting at 0. We first show B is an  $\mathbb{F}$  Brownian motion, but this is easy since for any  $0 \leq a < b$ ,  $\int_a^b \sqrt{\dot{g}(v)} dW_v$  is independent of  $F_a^W$  and normally distributed with mean 0 and variance

$$\int_{a}^{b} \dot{g}(v)dt = g(b) - g(a).$$

So, with a = f(s), b = f(t), s < t we have  $B_t - B_s$  is independent of  $\mathcal{F}_{f(s)}^W = \mathcal{F}_s$  and normally distributed with mean 0 and variance g(f(t)) - g(f(s)) = t - s. Thus, B is a F Brownian motion.

Now, regarding Z, note that Z is a  $\mathbb F$  martingale: indeed, for s < t we have

$$E\left[Z_t - Z_t \mid \mathcal{F}_s\right] = E\left[W_{f(t)} - W_{f(s)} \mid \mathcal{F}_{f(s)}^W\right] = 0.$$

As for (1), we have to work around the fact that we don't really know the dynamics for B in terms of W. To this end we claim that

$$\sqrt{\dot{f}(t)}B_t = Z_t - \sqrt{\dot{f}(t)} \int_0^t Z_u \left(\frac{d}{du}\frac{1}{\sqrt{\dot{f}(u)}}\right) du.$$
 (2)

Admitting this equality and using the hint we have

$$M_t \triangleq Z_t - \int_0^t \sqrt{\dot{f}(u)} dB_u = Z_t - \left(\sqrt{\dot{f}(t)}B_t - \int_0^t B_u\left(\frac{d}{du}\sqrt{\dot{f}(u)}\right)du\right)$$
$$= \sqrt{\dot{f}(t)} \int_0^t Z_u\left(\frac{d}{du}\frac{1}{\sqrt{\dot{f}(u)}}\right)du + \int_0^t B_u\left(\frac{d}{du}\sqrt{\dot{f}(u)}\right)du$$

The right hand side above is a finite variation process and hence  $\langle M \rangle_t = 0$  for the continuous martingale M. Thus, M is indistinguishable from 0 and the result follows. To prove (2) note that by integration by parts:

$$\sqrt{\dot{g}(\tau)}W_{\tau} = \int_0^{\tau} \sqrt{\dot{g}(v)} dW_v + \int_0^{\tau} W_v \frac{d}{dv} \left(\sqrt{\dot{g}(v)}\right) dv$$

so that

$$B_t = \sqrt{\dot{g}(f(t))} W_{f(t)} - \int_0^{f(t)} W_v \left(\frac{d}{dv} \sqrt{\dot{g}(v)}\right) dv$$
$$= \frac{Z_t}{\sqrt{\dot{f}(t)}} - \int_0^t Z_u \left(\frac{d}{du} \frac{1}{\sqrt{\dot{f}(u)}}\right) du$$

where the last equality follows by part *a*). Thus, the result is proved. c) To find  $\tilde{b}, \tilde{\sigma}$  we simply note:

$$X_{t} = Y_{f(t)} = y + \int_{0}^{f(t)} b(u, Y_{u}) du + W_{f(t)} \qquad (v = g(u) \text{ or } u = f(v))$$
$$= y + \int_{0}^{t} b(f(v), X_{v}) \dot{f}(v) dv + \int_{0}^{t} \sqrt{\dot{f}(v)} dB_{v}$$

so that  $(\Omega, \mathcal{F}, \mathbb{P}), \mathbb{F}, (B, X)$  is a weak solution with

$$\tilde{b}(t,x) = b(f(t),x)\dot{f}(t);$$
  $\tilde{\sigma}(t,x) = \sqrt{\dot{f}(t)}.$ 

(3) Since Z is a non-negative local martingale it is a super-martingale and hence to prove that Z is a Martingale, it suffices to show that  $E[Z_T] = 1$  for all  $T \ge 0$ . To this end we have

$$1 \ge E [Z_T] = E [Z_T (1_{T > \tau_n} + 1_{T \le \tau_n})]$$
  
$$\ge E [Z_{T \land \tau_n} 1_{T \le \tau_n}]$$
  
$$= \mathbb{Q}^n [T \le \tau_n]$$

Thus, if  $\lim_{n\uparrow\infty} \mathbb{Q}^n [T \leq \tau_n] = 1$  for all  $T \geq 0$  then  $E[Z_T] = 1$  and the result follows.

(4) Set  $\tau_n = \inf \{t \ge 0 \mid X_t \notin (1/n, n)\}$  and since X does not explode we have that  $\tau_n \to \infty$  almost surely. Next, define  $\Psi_t^n = A/\sqrt{X_{t \wedge \tau_n}} + B\sqrt{X_{t \wedge \tau_n}}$  and  $\Psi_t = A/\sqrt{X_t} + B\sqrt{X_t}$ . Note that

$$|\Psi_t^n| \le \sqrt{n}|A| + \sqrt{n}|B|$$

for all  $t \geq 0$ . Furthermore if we set  $Z_t^n = Z_{t \wedge \tau_n}$  then

$$Z_t^n = e^{\int_0^{t \wedge \tau_n} \Psi_s dW_s - \frac{1}{2} \int_0^{t \wedge \tau_n} \Psi_s^2 ds} = e^{\int_0^t \Psi_s^n \mathbf{1}_{s \le \tau_n} dW_s - \frac{1}{2} \int_0^t (\Psi_s^n \mathbf{1}_{s \le \tau_n})^2 ds}; \qquad t \ge 0.$$

Thus, by the Novikov condition we have that  $Z^n$  is a Martingale, and that for each T > 0 if we define  $\mathbb{Q}^n$  on  $\mathcal{F}_T$  via  $d\mathbb{Q}^n/d\mathbb{P}|_{\mathcal{F}_T} = Z_T^n$  then from Girsanov's theorem we have

$$W_t^n = W_t - \int_0^t \left( \Psi_s^n \mathbf{1}_{s \le \tau_n} \right) ds = W_t - \int_0^{t \land \tau_n} \Psi_s ds$$

is a  $\mathbb{Q}^n$  Brownian motion. Thus, on  $\{t \leq \tau_n\}$ , X has  $\mathbb{Q}^n$  dynamics

$$dX_t = \kappa(\theta - X_t)dt + \xi\sqrt{X_t} \left(dW_t^n + \Psi_t dt\right)$$
$$= (\kappa - \xi B) \left(\frac{\kappa\theta + \xi A}{\kappa - \xi B} - X_t\right) dt + \xi\sqrt{X_t} dW_t^n$$

Since for any constants  $\tilde{\kappa}, \tilde{\theta}$  the functions  $\tilde{\kappa}(\tilde{\theta} - x)$  and  $\xi\sqrt{x}$  are locally Lipschitz it follows that the law of X under  $\mathbb{Q}^n$  on  $t \leq \tau_n$  coincides with the law of  $\hat{X}$  up to the first hitting time of  $(1/n, n)^C$  under any measure  $\hat{\mathbb{P}}$ , supporting a Brownian motion  $\hat{W}$ , of the process

$$d\hat{X}_t = (\kappa - \xi B) \left(\frac{\kappa \theta + \xi A}{\kappa - \xi B} - X_t\right) dt + \xi \sqrt{X_t} dW_t$$

As such we have  $\mathbb{Q}^n [T \leq \tau_n] = \hat{\mathbb{P}} [T \leq \hat{\tau}_n]$  where  $\hat{\tau}_n = \inf \{ t \geq 0 \mid \hat{X}_t \notin (1/n, n) \}$ . Thus, if A and B are such that

$$\begin{aligned} \kappa - \xi B > 0 \Longrightarrow B < \frac{\kappa}{\xi} \\ \kappa \theta + \xi A > \frac{1}{2} \xi^2 \Longrightarrow A > -\frac{1}{\xi} \left( \kappa \theta - \frac{1}{2} \xi^2 \right) \end{aligned}$$

we have that the process under  $\hat{\mathbb{P}}$  does not explode: i.e.  $\lim_{n\uparrow\infty}\hat{\mathbb{P}}\left[T\leq\hat{\tau}_n\right]=1$  and hence  $\lim_{n\uparrow\infty}\mathbb{Q}^n\left[T\leq\tau_n\right]=1$ , giving that Z is a Martingale.

(5) The PDE is

$$u_{xx}(x,y) + 2\rho u_{xy}(x,y) + u_{yy}(x,y) = 0; \qquad (x,y) \in D$$
$$u(0,y) = 1; \qquad y > 0$$
$$u(x,0) = 0; \qquad x > 0$$

Indeed, assume that  $u \in C^2(D)$  is a bounded solution of the above PDE. For each *n* define the stopping times

$$\begin{split} \tau_n^x &= \inf \left\{ t \ge 0 \mid X_t^x = 1/n \right\}, \\ \sigma_n^y &= \inf \left\{ t \ge 0 \mid Y_t^y = 1/n \right\}, \\ R_n^{x,y} &= \inf \left\{ t \ge 0 \mid \max \left\{ X_t^x, Y_t^y \right\} = n \right\}, \\ S_n^{x,y} &= \tau_n^x \wedge \sigma_n^y \wedge R_n^{x,y} \end{split}$$

Note that almost surely

$$\lim_{n \uparrow \infty} S_n^{x,y} = \tau^x \mathbf{1}_{\tau^x < \tau^y} + \sigma^y \mathbf{1}_{\tau^x \ge \tau^y}.$$

Now, we have for n so large that 1/n < x, y < n

$$\begin{split} u\left(X_{t \wedge S_n^{x,y}}, Y_{t \wedge S_n^{x,y}}\right) &= u(x,y) + \int_0^{t \wedge S_n^{x,y}} \left(u_x(X_s, Y_s) + \rho u_y(X_s, Y_s)\right) dW_s \\ &+ \sqrt{1 - \rho^2} \int_0^{t \wedge S_n^{x,y}} u_y(X_s, Y_s) dB_s \end{split}$$

Since  $u \in C^2(D)$  its first derivatives are bounded when  $(x, y) \in (1/n, n)^2$ , as is the case here since we have stopped things appropriately. Thus, taking expectations gives

$$u(x,y) = E\left[u\left(X_{t\wedge S_n^{x,y}}, Y_{t\wedge S_n^{x,y}}\right)\right].$$

We have assumed u was bounded. Thus, taking  $t\uparrow\infty$  and using the bounded convergence theorem gives

$$u(x,y) = E\left[u\left(X_{S_n^{x,y}}, Y_{S_n^{x,y}}\right)\right].$$

Taking  $n \uparrow \infty$  and using the bounded convergence theorem again yields

$$u(x,y) = E\left[u(0, Y_{\tau^x}) \mathbf{1}_{\tau^x < \sigma^y} + u(X_{\sigma^y}, 0) \mathbf{1}_{\tau^x > \sigma^y}\right]$$

Using the boundary conditions (along with  $\tau^x, \sigma^y$  being almost surely finite the processes being continuous) gives

$$u(x,y) = E\left[1_{\tau^x < \sigma^y}\right] = \mathbb{P}\left[\tau^x < \sigma^y\right]$$

which is what we wanted to show.