Math 21-880 Final

December 14, 2015

This is a closed book, closed notes exam. No calculators or smart phones are allowed. You have 3 hours to complete the exam. Please mark your answers clearly and put your name on each piece of paper you submit. There are five questions on the exam.

The first two questions concern time changed solutions of stochastic differential equations. More precisely, let W be a standard one-dimensional Brownian motion with $W_0 = 0$ and let $\mathbb{F}^W$ be the augmented filtration generated by W. Let $f : [0,\infty) \mapsto (0,\infty)$ be a (deterministic) strictly increasing smooth function with $f(0) = 0$ and $\lim_{t\uparrow\infty} f(t) = \infty$.

Next, let $b : [0,\infty) \times \mathbb{R}$ be bounded and globally Lipschitz in x (uniformly for $t \geq 0$), and let Y be the corresponding strong solution to the stochastic differential equation (SDE)

$$dY_t = b(t, Y_t)dt + dW_t; \qquad Y_0 = y \in \mathbb{R}.$$

Define the f-time changed process X by

$$X_t = Y_{f(t)}; \qquad t \geq 0; \qquad X_0 = Y_{f(0)} = Y_0 = y.$$

(1) **20 Points.** Assume for some $t_0 > 0$ we have $f(t_0) > t_0$. Show that for any drift and diffusion functions $\tilde{b}(t,x)$, $\tilde{\sigma}(t,x)$, X *cannot* be a strong solution to an SDE with $\tilde{b}, \tilde{\sigma}$ and driving Brownian motion W.

Hint: There is more to do here than you might initially think. Use Girsanov's theorem and properties of Brownian motion to contradict the strong solution nature of X.

(2) **25 Points.** Here, you will show that X can be identified as a weak solution to a certain SDE. Do this in the following steps:

a) **5 Points.** Let $g = f^{-1}$ be the inverse of f. For any continuous function h and $t \geq 0$ show that

$$\int_0^{f(t)} h(v) \left(\frac{d}{dv} \sqrt{\dot{g}(v)} \right) dv = \int_0^t h((f(u)) \left(\frac{d}{du} \frac{1}{\sqrt{\dot{f}(u)}} \right) du.$$

b) **15 Points.** Consider the processes

$$Z_t = W_{f(t)}; \qquad B_t = \int_0^{f(t)} \sqrt{\dot{g}(v)} dW_v; \qquad t \geq 0.$$

Find a filtration $\mathbb{F}$ so that so that B is an $\mathbb{F}$ Brownian motion, and show that

$$\mathbb{P}\left[Z_t = \int_0^t \sqrt{\dot{f}(u)} dB_u; \qquad \forall\, t \geq 0\right] = 1. \tag{1}$$

Hint: Recall that if $\hat{B}$ is a Brownian motion under some filtration $\hat{\mathbb{F}}$ and h is a smooth deterministic function of t then we can define the stochastic integral $\int_0^{\cdot} h(t) d\hat{B}_t$ path-wise via

$$\left(\int_0^{\cdot} h(t) d\hat{B}_t\right)(\omega) \triangleq \left(h(\cdot)\hat{B}_{\cdot} - \int_0^{\cdot} \dot{h}(t)\hat{B}_t dt\right)(\omega)$$

c) **5 Points.** Find $\tilde{b}, \tilde{\sigma}$ so that $(\Omega, \mathcal{F}, \mathbb{P})$, $\mathbb{F}$, (X, B) is a weak solution to the SDE with drift $\tilde{b}$ and diffusion $\tilde{\sigma}$.

The next two problems deal with explosions and Martingales. In particular, when stochastic exponential local Martingales are defined in terms of solutions to SDEs, we seek to weaken the Novikov condition.

3) **15 Points.** Let Z be a strictly positive local martingale with respect to some probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and filtration $\mathbb{F}$ satisfying the usual conditions. Assume that $\{\tau_n\}$ is an increasing sequence of stopping times such that $\tau_n \uparrow \infty$ almost surely and such that the stopped process Z^n defined by $Z^n = Z_{t\wedge\tau_n}$ is a Martingale for each n.

   Prove the following: let $T > 0$ and define the measure $\mathbb{Q}^n$ on $\mathcal{F}_T$ via

$$\left.\frac{d\mathbb{Q}^n}{d\mathbb{P}}\right|_{\mathcal{F}_T} = Z^n_T.$$

   If, for each $T > 0$ we have $\lim_{n\uparrow\infty} \mathbb{Q}^n\left[T \leq \tau_n\right] = 1$ then Z is a true martingale.

4) **20 Points.** The CIR (Cox-Ingersoll-Ross) process is popular in math finance for modeling the interest rate. We say X is a CIR process if it has dynamics

$$dX_t = \kappa(\theta - X_t)dt + \xi\sqrt{X_t} dW_t; \qquad X_0 = x > 0.$$

   Here, W is a standard d-dimensional Brownian motion with respect to some filtration satisfying the usual conditions, and $\kappa, \theta, \xi > 0$ are constants. The state space for the process is $D = (0, \infty)$ and the process is said not to explode if for all $T > 0$ we have that

$$\mathbb{P}\left[X_t \in D, 0 \leq t \leq T \mid X_0 = x\right] = 1.$$

It can be shown (you do not have to do this) that if $\kappa > 0$, $\kappa\theta > \xi^2/2$ then the process does not explode. Now, let $A, B \in \mathbb{R}$. Find parameter restrictions upon A, B so that the process

$$Z_t = \mathcal{E}\left(\int_0^\cdot \left(\frac{A}{\sqrt{X_t}} + B\sqrt{X_t}\right) dW_t\right)_t; \qquad t \geq 0,$$

is a Martingale.

5) **20 Points.** Let W, B be two independent Brownian motions and let $-1 < \rho < 1$. Fix $x, y > 0$ and define the processes

$$X_t^x \triangleq x + W_t; \qquad Y_t^y \triangleq y + \rho W_t + \sqrt{1-\rho^2} B_t; \qquad t \geq 0.$$

Set $\tau^x = \inf\{t \geq 0 \mid X_t^x = 0\}$ and $\sigma^y = \inf\{t \geq 0 \mid Y_t^y = 0\}$ and note that $\tau^x, \sigma^y < \infty$ almost surely for all $x, y > 0$, since the sample paths of Brownian motions are unbounded.

Let $D = (0, \infty)^2$. We say a function $u \in C^2(D)$ if u is twice differentiable with continuous derivatives which are bounded on all compact subsets of D. However, we do not necessarily know the behavior of the derivatives of u near the boundary of D. For example, $u(x, y) = (xy)^{-1} \in C^2(D)$ but clearly u is blowing up near $x = 0$ or $y = 0$.

Identify a partial differential equation (differential expression plus spatial boundary conditions) such that if $u \in C^2(D)$ is a bounded solution of the PDE then u admits the representation

$$u(x, y) = \mathbb{P}\left[\tau^x < \sigma^y\right].$$

Be careful when dealing with the local Martingales here (e.g. make sure to stop the processes before you lose control over the stochastic integrands!).