Math 21-880 Midterm

Week of October 12-16 2015

This is a closed book, closed notes, take home exam. No calculators or smart phones are allowed. You must return the exam to my office (Wean Hall 8214) by no later than 5PM on Friday, October 16, 2015. You must complete this exam within a single two-hour time period. In the fields below state when you started the exam and when you finished. You are on your honor to abide by the two hour window. Please mark your answers clearly and put your name on each piece of paper you submit. There are five questions on the exam, each worth twenty points.

- Name:
- Date and Time you started the exam:
- Date and time you finished the exam:
- Signature stating you kept to the two hour window and did not consult your notes or any other course-related material while completing the exam:

Unless otherwise specified, throughout the exam the filtered probability space $(\Omega, \mathbb{F}, \mathcal{F}, \mathbb{P})$ is given on $[0, T]$ or $[0, \infty)$. Additionally, the filtration $\mathbb{F}$ is assumed to satisfy the usual conditions and all stochastic processes $X = \{X_t\}_{t \leq T}$ or $X = \{X_t\}_{t \geq 0}$ are assumed to be measurable and adapted.

(1) **20 Points** (Problem 1.3.29 - you are being asked to do it again because it is important in Math Finance!). Let X be a continuous, non-negative supermartingale. Set $T = \inf\{t \geq 0 : X_t = 0\}$ as the hitting time of 0. Show that almost surely on $\{T < \infty\}$ we have

$$X_{T+t} = 0; \quad 0 \leq t < \infty.$$

(2) **20 Points** Recall that we say X is *bounded in* L^p if $\sup_{t \geq 0} E\left[|X_t|^p\right] < \infty$. Assume X is a right-continuous sub-martingale starting at zero. Define the maximal process Y by $Y_t = \sup_{s \leq t} X_s$ and note that $Y_\infty = \lim_{t \uparrow \infty} Y_t$ exists almost surely.

(a) **10 Points** If X has a last element, show that $Y_\infty < \infty$ almost surely.

(b) **10 Points** If X is bounded in L^p for some $p > 1$ show that $Y_\infty \in L^p$.

(3) **20 Points** Let W be a Brownian motion and define X via $X_t = e^{W_t - t/2}$.

(a) **10 Points** Without using Itô 's formula, show that X is a (strictly positive) martingale.

(b) **10 Points** Now, fix $T > 0$ and note from part (a) that since $X_0 = 1$, we may define a new measure $\mathbb{Q}$ on $\mathcal{F}_T$ via

$$\mathbb{Q}\left[A\right] = E\left[X_T 1_A\right]; \qquad A \in \mathcal{F}_T.$$

Without using Girsanov's theorem or the Lévy characterization of Brownian motion, show that Y defined by $Y_t = W_t - t$ is a $\mathbb{Q}$ Brownian motion on $[0, T]$.

Hint: Use the moment generating function characterization of independence. Also, to ease the calculation burden here, just consider Y at times $Y_t - Y_s$ and $Y_t - Y_s, Y_s$.

(4) **20 Points** Let W be a Brownian motion. Fix $T > 0$ and for each integer $n = 1, 2, \ldots$ define the process X^n via

$$X_t^n = \frac{1}{\sqrt{T - t \wedge (T - 1/n)}}; \qquad t \geq 0.$$

(a) **5 Points** Show that for each n, $X^n \in \mathcal{L}_W^*$ and hence the stochastic integral $M^n = I^W(X^n)$ is in $\mathcal{M}_2^c$.

(b) **10 Points** Show that for $t < T$, the family of random variables $\{M_t^n\}_{n=1,2,\ldots}$ is uniformly integrable, but that the family of random variables $\{M_T^n\}_{n=1,2,\ldots}$ is *NOT* uniformly integrable.

(c) **5 Points** Show that for any $\lambda > 0$ we have

$$\lim_{n\uparrow\infty} \mathbb{P}\left[\sup_{t\leq T} |M_T^n| \geq \lambda\right] = 1.$$

Hint: in the above problem you may use without proof that for any continuous martingale N with $\langle N\rangle_T \leq C$ almost surely the process $e^{N_t - (1/2)\langle N\rangle_t}$ is a martingale on $[0, T]$.

(5) **20 Points** Let $M \in M^{c,\mathrm{loc}}$ be given. Assume that M is non negative and starts at 1. For $k > 0$, denote by τ_k the hitting time to level $k > 0$: i.e. $\tau_k = \inf\{t \geq 0 \mid M_t = k\}$.

(a) **10 Points** Repeat the argument which shows that without loss of generality, for the associated sequence of stopping times $\{T_n\}_{n=1,2,\ldots}$, with $M_t^n = M_{t\wedge T_n}$, we may assume M^n is a uniformly (in $t \geq 0$) bounded martingale, where the bounding constant only depends upon n.

Hint: The hard part here is to show that after adjusting the stopping times, the resultant process is still an $\mathbb{F}$ martingale. To do this, you may invoke (without proof) the homework problem you did relating $E\left[Z \mid \mathcal{F}_T\right]$ and $E\left[Z \mid \mathcal{F}_{S\wedge T}\right]$ for integrable random variables Z and stopping times S, T.

(b) **10 Points** Show that for $k > 1$ we have $\mathbb{P}[\tau_k < \infty] \leq 1/k$. In particular, it holds with positive probability that $\tau_k = \infty$.