transition probability function $P(i, \cdot)$, also has support in N^0. Thus

$$\{P(i, j); (i, j) \in N^0 \times N^0\}$$

is an infinite matrix called the "transition matrix". Show that $P^{(n)}$ as a matrix is just the nth power of $P^{(1)}$. Express the probability $\mathscr{P}\{X_{t_k} = i_k, 1 \le k \le n\}$ in terms of the elements of these matrices. [This is the case of *homogeneous Markov chains.*]

12. A process $\{X_n, n \in N^0\}$ is said to possess the "rth-order Markov property", where $r \ge 1$, iff (6) is replaced by

$$\mathscr{P}\{X_{n+1} \in B \mid X_0, \dots, X_n\} = \mathscr{P}\{X_{n+1} \in B \mid X_n, \dots, X_{n-r+1}\}$$

for $n \ge r - 1$. Show that if $r < s$, then the rth-order Markov property implies the sth. The ordinary Markov property is the case $r = 1$.

13. Let Y_n be the random vector $(X_n, X_{n+1}, \dots, X_{n+r-1})$. Then the vector process $\{Y_n, n \in N^0\}$ has the ordinary Markov property (trivially generalized to vectors) if and only if $\{X_n, n \in N^0\}$ has the rth-order Markov property.

14. Let $\{X_n, n \in N^0\}$ be an independent process. Let

$$S_n^{(1)} = \sum_{j=0}^{n} X_j, \quad S_n^{(r+1)} = \sum_{j=0}^{n} S_j^{(r)}$$

for $r \ge 1$. Then $\{S_n^{(r)}, n \in N^0\}$ has the rth-order Markov property. For $r = 2$, give an example to show that it need not be a Markov process.

15. If $\{S_n, n \in N\}$ is a random walk such that $\mathscr{P}\{S_1 \ne 0\}>0$, then for any finite interval $[a, b]$ there exists an $\epsilon < 1$ such that

$$\mathscr{P}\{S_j \in [a, b], 1 \le j \le n\} \le \epsilon^n.$$

This is just Exercise 6 of Sec. 5.5 again.]

16. The same conclusion is true if the random walk above is replaced by a homogeneous Markov process for which, e.g., there exist $\delta>0$ and $\eta>0$ such that $P(x, \mathscr{R}^1 - (x - \delta, x + \delta)) \ge \eta$ for every x.

9.3 Basic properties of smartingales

The sequence of sums of independent r.v.'s has motivated the generalization to a Markov process in the preceding section; in another direction it will now motivate a *martingale*. Changing our previous notation to conform with later usage, let $\{x_n, n \in N\}$ denote independent r.v.'s with mean zero and write $X_n = \sum_{j=1}^n x_j$ for the partial sum. Then we have

$$\begin{aligned}\mathscr{E}(X_{n+1} \mid x_1, \dots, x_n) &= \mathscr{E}(X_n + x_{n+1} \mid x_1, \dots, x_n) \\ &= X_n + \mathscr{E}(x_{n+1} \mid x_1, \dots, x_n) = X_n + \mathscr{E}(x_{n+1}) = X_n.\end{aligned}$$

Note that the conditioning with respect to $x_1, \dots, x_n$ may be replaced by conditioning with respect to $X_1, \dots, X_n$ (why?). Historically, the equation above led to the consideration of dependent r.v.'s $\{x_n\}$ satisfying the condition

$$(1) \qquad \mathscr{E}(x_{n+1} \mid x_1, \dots, x_n) = 0.$$

It is astonishing that this simple property should delineate such a useful class of stochastic processes which will now be introduced. In what follows, where the index set for n is not specified, it is understood to be either N or some initial segment N_m of N.

DEFINITION OF MARTINGALE. The sequence of r.v.'s and B.F.'s $\{X_n, \mathscr{F}_n\}$ is called a *martingale* iff we have for each n:

(a) $\mathscr{F}_n \subset \mathscr{F}_{n+1}$ and $X_n \in \mathscr{F}_n$;
(b) $\mathscr{E}(|X_n|) < \infty$;
(c) $X_n = \mathscr{E}(X_{n+1} \mid \mathscr{F}_n)$, a.e.

It is called a *supermartingale* iff the "=" in (c) above is replaced by "≥", and a *submartingale* iff it is replaced by "≤". For abbreviation we shall use the term *smartingale* to cover all three varieties. In case $\mathscr{F}_n = \mathscr{F}_{[1,n]}$ as defined in Sec. 9.2, we shall omit $\mathscr{F}_n$ and write simply $\{X_n\}$; more frequently however we shall consider $\{\mathscr{F}_n\}$ as given in advance and omitted from the notation.

Condition (a) is nowadays referred to as: $\{X_n\}$ is *adapted* to $\{\mathscr{F}_n\}$. Condition (b) says that all the r.v.'s are integrable; we shall have to impose stronger conditions to obtain most of our results. A particularly important one is the uniform integrability of the sequence $\{X_n\}$, which is discussed in Sec. 4.5. A weaker condition is given by

$$(2) \qquad \sup_n \mathscr{E}(|X_n|) < \infty;$$

when this is satisfied we shall say that $\{X_n\}$ is L^1-bounded. Condition (c) leads at once to the more general relation:

$$(3) \qquad n < m \Rightarrow X_n = \mathscr{E}(X_m \mid \mathscr{F}_n).$$

This follows from Theorem 9.1.5 by induction since

$$\mathscr{E}(X_m \mid \mathscr{F}_n) = \mathscr{E}(\mathscr{E}(X_m \mid \mathscr{F}_{m-1}) \mid \mathscr{F}_n) = \mathscr{E}(X_{m-1} \mid \mathscr{F}_n).$$

An equivalent form of (3) is as follows: for each $\Lambda \in \mathscr{F}_n$ and $n \leq m$, we have

(4)
$$\int_\Lambda X_n \, d\mathscr{P} = \int_\Lambda X_m \, d\mathscr{P}.$$

It is often safer to use the explicit formula (4) rather than (3), because conditional expectations can be slippery things to handle. We shall refer to (3) or (4) as the *defining relation* of a martingale; similarly for the "super" and "sub" varieties.

Let us observe that in the form (3) or (4), the definition of a smartingale is meaningful if the index set N is replaced by any linearly ordered set, with "$<$" as the strict order. For instance, it may be an interval or the set of rational numbers in the interval. But even if we confine ourselves to a *discrete parameter* (as we shall do) there are other index sets to be considered below.

It is scarcely worth mentioning that $\{X_n\}$ is a supermartingale if and only if $\{-X_n\}$ is a submartingale, and that a martingale is both. However the extension of results from a martingale to a smartingale is not always trivial, nor is it done for the sheer pleasure of generalization. For it is clear that martingales are harder to come by than the other varieties. As between the super and sub cases, though we can pass from one to the other by simply changing signs, our force of habit may influence the choice. The next proposition is a case in point.

Theorem 9.3.1. Let $\{X_n, \mathscr{F}_n\}$ be a submartingale and let φ be an increasing convex function defined on $\mathscr{R}^1$. If $\varphi(X_n)$ is integrable for every n, then $\{\varphi(X_n), \mathscr{F}_n\}$ is also a submartingale.

PROOF. Since φ is increasing, and

$$X_n \leq \mathscr{E}\{X_{n+1} \mid \mathscr{F}_n\}$$

we have

(5)
$$\varphi(X_n) \leq \varphi(\mathscr{E}\{X_{n+1} \mid \mathscr{F}_n\}).$$

By Jensen's inequality (Sec. 9.1), the right member above does not exceed $\mathscr{E}\{\varphi(X_{n+1}) \mid \mathscr{F}_n\}$; this proves the theorem. As forewarned in 9.1, we have left out some "a.e." above and shall continue to do so.

Corollary 1. If $\{X_n, \mathscr{F}_n\}$ is a submartingale, then so is $\{X_n^+, \mathscr{F}_n\}$. Thus $\mathscr{E}(X_n^+)$ as well as $\mathscr{E}(X_n)$ is increasing with n.

Corollary 2. If $\{X_n, \mathscr{F}_n\}$ is a martingale, then $\{|X_n|, \mathscr{F}_n\}$ is a submartingale; and $\{|X_n|^p, \mathscr{F}_n\}$, $1 < p < \infty$, is a submartingale provided that every $X_n \in L^p$; similarly for $\{|X_n| \log^+ |X_n|, \mathscr{F}_n\}$ where $\log^+ x = (\log x) \vee 0$ for $x \geq 0$.

PROOF. For a martingale we have equality in (5) for *any* convex φ, hence we may take $\varphi(x) = |x|$, $|x|^p$ or $|x| \log^+ |x|$ in the proof above.

Thus for a martingale $\{X_n\}$, all three transmutations: $\{X_n^+\}$, $\{X_n^-\}$ and $\{|X_n|\}$ are submartingales. For a submartingale $\{X_n\}$, nothing is said about the last two.

Corollary 3. If $\{X_n, \mathscr{F}_n\}$ is a supermartingale, then so is $\{X_n \wedge A, \mathscr{F}_n\}$ where A is any constant.

PROOF. We leave it to the reader to deduce this from the theorem, but here is a quick direct proof:

$$X_n \wedge A \geq \mathscr{E}(X_{n+1} \mid \mathscr{F}_n) \wedge \mathscr{E}(A \mid \mathscr{F}_n) \geq \mathscr{E}(X_{n+1} \wedge A \mid \mathscr{F}_n).$$

It is possible to represent any smartingale as a martingale plus or minus something special. Let us call a sequence of r.v.'s $\{Z_n, n \in N\}$ an *increasing process* iff it satisfies the conditions:

(**i**) $Z_1 = 0$; $Z_n \leq Z_{n+1}$ for $n \geq 1$;
(**ii**) $\mathscr{E}(Z_n) < \infty$ for each n.

It follows that $Z_\infty = \lim_{n\to\infty} \uparrow Z_n$ exists but may take the value $+\infty$; Z_∞ is integrable if and only if $\{Z_n\}$ is L^1-bounded as defined above, which means here $\lim_{n\to\infty} \uparrow \mathscr{E}(Z_n) < \infty$. This is also equivalent to the uniform integrability of $\{Z_n\}$ because of (i). We can now state the result as follows.

Theorem 9.3.2. Any submartingale $\{X_n, \mathscr{F}_n\}$ can be written as

(6) $$X_n = Y_n + Z_n,$$

where $\{Y_n, \mathscr{F}_n\}$ is a martingale, and $\{Z_n\}$ is an increasing process.

PROOF. From $\{X_n\}$ we define its *difference sequence* as follows:

(7) $$x_1 = X_1, \quad x_n = X_n - X_{n-1}, \quad n \geq 2,$$

so that $X_n = \sum_{j=1}^n x_j$, $n \geq 1$ (cf. the notation in the first paragraph of this section). The defining relation for a submartingale then becomes

$$\mathscr{E}\{X_n \mid \mathscr{F}_{n-1}\} \geq 0,$$

with equality for a martingale. Furthermore, we put

$$y_1 = x_1, \qquad y_n = x_n - \mathscr{E}\{x_n \mid \mathscr{F}_{n-1}\}, \qquad Y_n = \sum_{j=1}^n y_j;$$

$$z_1 = 0, \qquad z_n = \mathscr{E}\{x_n \mid \mathscr{F}_{n-1}\}, \qquad Z_n = \sum_{j=1}^n z_j.$$

Then clearly $x_n = y_n + z_n$ and (6) follows by addition. To show that $\{Y_n, \mathscr{F}_n\}$ is a martingale, we may verify that $\mathscr{E}\{y_n \mid \mathscr{F}_{n-1}\} = 0$ as indicated a moment ago, and this is trivial by Theorem 9.1.5. Since each $z_n \geq 0$, it is equally obvious that $\{Z_n\}$ is an increasing process. The theorem is proved.

Observe that $Z_n \in \mathscr{F}_{n-1}$ for each n, by definition. This has important consequences; see Exercise 9 below. The decomposition (6) will be called *Doob's decomposition*. For a supermartingale we need only change the "+" there into "−", since $\{-Y_n, \mathscr{F}_n\}$ is a martingale. The following complement is useful.

Corollary. If $\{X_n\}$ is L^1-bounded [or uniformly integrable], then both $\{Y_n\}$ and $\{Z_n\}$ are L^1-bounded [or uniformly integrable].

PROOF. We have from (6):

$$\mathscr{E}(Z_n) \leq \mathscr{E}(|X_n|) - \mathscr{E}(Y_1)$$

since $\mathscr{E}(Y_n) = \mathscr{E}(Y_1)$. Since $Z_n \geq 0$ this shows that if $\{X_n\}$ is L^1-bounded, then so is $\{Z_n\}$; and $\{Y_n\}$ is too because

$$\mathscr{E}(|Y_n|) \leq \mathscr{E}(|X_n|) + \mathscr{E}(Z_n).$$

Next if $\{X_n\}$ is uniformly integrable, then it is L^1-bounded by Theorem 4.5.3, hence $\{Z_n\}$ is L^1-bounded and therefore uniformly integrable as remarked before. The uniform integrability of $\{Y_n\}$ then follows from the last-written inequality.

We come now to the fundamental notion of *optional sampling* of a smartingale. This consists in substituting certain random variables for the original index n regarded as the time parameter of the process. Although this kind of thing has been done in Chapter 8, we will reintroduce it here in a slightly different way for the convenience of the reader. To begin with we adjoin a last index ∞ to the set N and call it $N_\infty = \{1, 2, \dots, \infty\}$. This is an example of a linearly ordered set mentioned above. Next, adjoin $\mathscr{F}_\infty = \bigvee_{n=1}^{\infty} \mathscr{F}_n$ to $\{\mathscr{F}_n\}$.

A r.v. α taking values in N_∞ is called *optional* (relative to $\{\mathscr{F}_n, n \in N_\infty\}$) iff for every $n \in N_\infty$ we have

$$\{\alpha \leq n\} \in \mathscr{F}_n. \tag{8}$$

Since $\mathscr{F}_n$ increases with n, the condition in (8) is unchanged if $\{\alpha \leq n\}$ is replaced by $\{\alpha = n\}$. Next, for an optional α, the pre-α field $\mathscr{F}_\alpha$ is defined to be the class of all subsets Λ of $\mathscr{F}_\infty$ satisfying the following condition: for each $n \in N_\infty$ we have

$$\Lambda \cap \{\alpha \leq n\} \in \mathscr{F}_n, \tag{9}$$

where again $\{\alpha \le n\}$ may be replaced by $\{\alpha = n\}$. Writing then

(10) $$\Lambda_n = \Lambda \cap \{\alpha = n\},$$

we have $\Lambda_n \in \mathscr{F}_n$ and

$$\Lambda = \bigcup_n \Lambda_n = \bigcup_n [\{\alpha = n\} \cap \Lambda_n]$$

where the index n ranges over N_∞. This is (3) of Sec. 8.2. The reader should now do Exercises 1–4 in Sec. 8.2 to get acquainted with the simplest properties of optionality. Here are some of them which will be needed soon: $\mathscr{F}_\alpha$ is a B.F. and $\alpha \in \mathscr{F}_\alpha$; if α is optional then so is $\alpha \wedge n$ for each $n \in N$; if $\alpha \le \beta$ where β is also optional then $\mathscr{F}_\alpha \subset \mathscr{F}_\beta$; in particular $\mathscr{F}_{\alpha \wedge n} \subset \mathscr{F}_\alpha \cap \mathscr{F}_n$ and in fact this inclusion is an equation.

Next we assume X_∞ has been defined and $X_\infty \in \mathscr{F}_\infty$. We then define X_α as follows:

(11) $$X_\alpha(\omega) = X_{\alpha(\omega)}(\omega);$$

in other words,

$$X_\alpha(\omega) = X_n(\omega) \quad \text{on} \quad \{\alpha = n\}, \quad n \in N_\infty.$$

This definition makes sense for any α taking values in N_∞, but for an optional α we can assert moreover that

(12) $$X_\alpha \in \mathscr{F}_\alpha.$$

This is an exercise the reader should not miss; observe that it is a natural but nontrivial extension of the assumption $X_n \in \mathscr{F}_n$ for every n. Indeed, all the general propositions concerning optional sampling aim at the same thing, namely to make optional times behave like constant times, or again to enable us to substitute optional r.v.'s for constants. For this purpose conditions must sometimes be imposed either on α or on the smartingale $\{X_n\}$. Let us however begin with a perfect case which turns out to be also very important.

We introduce a class of martingales as follows. For any integrable r.v. Y we put

(13) $$X_n = \mathscr{E}(Y \mid \mathscr{F}_n), \quad n \in N_\infty.$$

By Theorem 9.1.5, if $n \le m$:

(14) $$X_n = \mathscr{E}\{\mathscr{E}(Y \mid \mathscr{F}_m) \mid \mathscr{F}_n\} = \mathscr{E}\{X_m \mid \mathscr{F}_n\}$$

which shows $\{X_n, \mathscr{F}_n\}$ is a martingale, not only on N but also on N_∞. The following properties extend both (13) and (14) to optional times.

Theorem 9.3.3. For any optional α, we have

$$X_\alpha = \mathscr{E}(Y \mid \mathscr{F}_\alpha). \tag{15}$$

If $\alpha \le \beta$ where β is also optional, then $\{X_\alpha, \mathscr{F}_\alpha; X_\beta, \mathscr{F}_\beta\}$ forms a two-term martingale.

PROOF. Let us first show that X_α is integrable. It follows from (13) and Jensen's inequality that

$$|X_n| \le \mathscr{E}(|Y| \mid \mathscr{F}_n).$$

Since $\{\alpha = n\} \in \mathscr{F}_n$, we may apply this to get

$$\int_\Omega |X_\alpha|\, d\mathscr{P} = \sum_n \int_{\{\alpha=n\}} |X_n|\, d\mathscr{P} \le \sum_n \int_{\{\alpha=n\}} |Y|\, d\mathscr{P} = \int_\Omega |Y|\, d\mathscr{P} < \infty.$$

Next if $\Lambda \in \mathscr{F}_\alpha$, we have, using the notation in (10):

$$\int_\Lambda X_\alpha\, d\mathscr{P} = \sum_n \int_{\Lambda_n} X_n\, d\mathscr{P} = \sum_n \int_{\Lambda_n} Y\, d\mathscr{P} = \int_\Lambda Y\, d\mathscr{P},$$

where the second equation holds by (13) because $\Lambda_n \in \mathscr{F}_n$. This establishes (15). Now if $\alpha \le \beta$, then $\mathscr{F}_\alpha \subset \mathscr{F}_\beta$ and consequently by Theorem 9.1.5.

$$X_\alpha = \mathscr{E}\{\mathscr{E}(Y \mid \mathscr{F}_\beta) \mid \mathscr{F}_\alpha\} = \mathscr{E}\{X_\beta \mid \mathscr{F}_\alpha\},$$

which proves the second assertion of the theorem.

As an immediate corollary, if $\{\alpha_n\}$ is a sequence of optional r.v.'s such that

$$\alpha_1 \le \alpha_2 \le \cdots \le \alpha_n \le \cdots, \tag{16}$$

then $\{X_{\alpha_n}, \mathscr{F}_{\alpha_n}\}$ is a martingale. This new martingale is obtained by sampling the original one at the optional times $\{\alpha_j\}$. We now proceed to extend the second part of Theorem 9.3.3 to a supermartingale. There are two important cases which will be discussed separately.

Theorem 9.3.4. Let α and β be two bounded optional r.v.'s such that $\alpha \le \beta$. Then for any [super]martingale $\{X_n\}$, $\{X_\alpha, \mathscr{F}_\alpha; X_\beta, \mathscr{F}_\beta\}$ forms a [super]martingale.

PROOF. Let $\Lambda \in \mathscr{F}_\alpha$; using (10) again we have for each $k \ge j$:

$$\Lambda_j \cap \{\beta > k\} \in \mathscr{F}_k$$

because $\Lambda_j \in \mathscr{F}_j \subset \mathscr{F}_k$, whereas $\{\beta > k\} = \{\beta \leq k\}^c \in \mathscr{F}_k$. It follows from the defining relation of a supermartingale that

$$\int_{\Lambda_j \cap \{\beta > k\}} X_k \, d\mathscr{P} \geq \int_{\Lambda_j \cap \{\beta > k\}} X_{k+1} \, d\mathscr{P}$$

and consequently

$$\int_{\Lambda_j \cap \{\beta \geq k\}} X_k \, d\mathscr{P} \geq \int_{\Lambda_j \cap \{\beta = k\}} X_k \, d\mathscr{P} + \int_{\Lambda_j \cap \{\beta > k\}} X_{k+1} \, d\mathscr{P}$$

Rewriting this as

$$\int_{\Lambda_j \cap \{\beta \geq k\}} X_k \, d\mathscr{P} - \int_{\Lambda_j \cap \{\beta \geq k+1\}} X_{k+1} \, d\mathscr{P} \geq \int_{\Lambda_j \cap \{\beta = k\}} X_\beta \, d\mathscr{P};$$

summing over k from j to m, where m is an upper bound for β; and then replacing X_j by X_α on Λ_j, we obtain

$$(17) \quad \int_{\Lambda_j \cap \{\beta \geq j\}} X_\alpha \, d\mathscr{P} - \int_{\Lambda_j \cap \{\beta \geq m+1\}} X_{m+1} \, d\mathscr{P} \geq \int_{\Lambda_j \cap \{j \leq \beta \leq m\}} X_\beta \, d\mathscr{P}.$$

$$\int_{\Lambda_j} X_\alpha \, d\mathscr{P} \geq \int_{\Lambda_j} X_\beta \, d\mathscr{P}.$$

Another summation over j from 1 to m yields the desired result. In the case of a martingale the inequalities above become equations.

A particular case of a bounded optional r.v. is $\alpha_n = \alpha \Lambda n$ where α is an arbitrary optional r.v. and n is a positive integer. Applying the preceding theorem to the sequence $\{\alpha_n\}$ as under Theorem 9.3.3, we have the following corollary.

Corollary. If $\{X_n, \mathscr{F}_n\}$ is a [super]martingale and α is an arbitrary optional r.v., then $\{X_{\alpha \Lambda n} \mathscr{F}_{\alpha \Lambda n}\}$ is a [super]martingale.

In the next basic theorem we shall assume that the [super]martingale is given on the index set N_∞. This is necessary when the optional r.v. can take the value $+\infty$, as required in many applications; see the typical example in (5) of Sec. 8.2. It turns out that if $\{X_n\}$ is originally given only for $n \in N$, we may take $X_\infty = \lim_{n \to \infty} X_n$ to extend it to N_∞ under certain conditions, see Theorems 9.4.5 and 9.4.6 and Exercise 6 of Sec. 9.4. A trivial case occurs when $\{X_n, \mathscr{F}_n; n \in N\}$ is a positive supermartingale; we may then take $X_\infty = 0$.

Theorem 9.3.5. Let α and β be two arbitrary optional r.v.'s such that $\alpha \leq \beta$. Then the conclusion of Theorem 9.3.4 holds true for any supermartingale $\{X_n, \mathscr{F}_n; n \in N_\infty\}$.

Remark. For a martingale $\{X_n, \mathcal{F}_n; n \in N_\infty\}$ this theorem is contained in Theorem 9.3.3 since we may take the Y in (13) to be X_∞ here.

PROOF. (a) Suppose first that the supermartingale is positive with $X_\infty = 0$ a.e. The inequality (17) is true for every $m \in N$, but now the second integral there is positive so that we have

$$\int_{\Lambda_j} X_\alpha \, d\mathcal{P} \geq \int_{\Lambda_j \cap \{\beta \leq m\}} X_\beta \, d\mathcal{P}.$$

Since the integrands are positive, the integrals exist and we may let $m \to \infty$ and then sum over $j \in N$. The result is

$$\int_{\Lambda \cap \{\alpha < \infty\}} X_\alpha \, d\mathcal{P} \geq \int_{\Lambda \cap \{\beta < \infty\}} X_\beta \, d\mathcal{P}$$

which falls short of the goal. But we can add the inequality

$$\int_{\Lambda \cap \{\alpha = \infty\}} X_\alpha \, d\mathcal{P} = \int_{\Lambda \cap \{\alpha = \infty\}} X_\infty \, d\mathcal{P} = \int_{\Lambda \cap \{\beta = \infty\}} X_\infty \, d\mathcal{P} = \int_{\Lambda \cap \{\beta = \infty\}} X_\beta \, d\mathcal{P}$$

which is trivial because $X_\infty = 0$ a.e. This yields the desired

$$\int_\Lambda X_\alpha \, d\mathcal{P} \geq \int_\Lambda X_\beta \, d\mathcal{P}. \tag{18}$$

Let us show that X_α and X_β are in fact integrable. Since $X_n \geq X_\infty$ we have $X_\alpha \leq \underline{\lim}_{n\to\infty} X_{\alpha\wedge n}$ so that by Fatou's lemma,

$$\mathcal{E}(X_\alpha) \leq \underline{\lim}_{n\to\infty} \mathcal{E}(X_{\alpha\wedge n}). \tag{19}$$

Since 1 and $\alpha\wedge n$ are two bounded optional r.v.'s satisfying $1 \leq \alpha\wedge n$; the right-hand side of (19) does not exceed $\mathcal{E}(X_1)$ by Theorem 9.3.4. This shows X_α is integrable since it is positive.

(b) In the general case we put

$$X'_n = \mathcal{E}\{X_\infty \mid \mathcal{F}_n\}, \quad X''_n = X_n - X'_n.$$

Then $\{X'_n, \mathcal{F}_n; n \in N_\infty\}$ is a martingale of the kind introduced in (13), and $X_n \geq X'_n$ by the defining property of supermartingale applied to X_n and X_∞. Hence the difference $\{X''_n, \mathcal{F}_n; n \in N\}$ is a positive supermartingale with $X''_\infty = 0$ a.e. By Theorem 9.3.3, $\{X'_\alpha, \mathcal{F}_\alpha; X'_\beta, \mathcal{F}_\beta\}$ forms a martingale; by case (a), $\{X''_\alpha, \mathcal{F}_\alpha; X''_\beta, \mathcal{F}_\beta\}$ forms a supermartingale. Hence the conclusion of the theorem follows simply by addition.

The two preceding theorems are the basic cases of *Doob's optional sampling theorem*. They do not cover all cases of optional sampling (see e.g.

Exercise 11 of Sec. 8.2 and Exercise 11 below), but are adequate for many applications, some of which will be given later.

Martingale theory has its intuitive background in gambling. If X_n is interpreted as the gambler's capital at time n, then the defining property postulates that his expected capital after one more game, played with the knowledge of the entire past and present, is exactly equal to his current capital. In other words, his expected gain is zero, and in this sense the game is said to be "fair". Similarly a smartingale is a game consistently biased in one direction. Now the gambler may opt to play the game only at certain preferred times, chosen with the benefit of past experience and present observation, but without clairvoyance into the future. [The inclusion of the present status in his knowledge seems to violate raw intuition, but examine the example below and Exercise 13.] He hopes of course to gain advantage by devising such a "system" but Doob's theorem forestalls him, at least mathematically. We have already mentioned such an interpretation in Sec. 8.2 (see in particular Exercise 11 of Sec. 8.2; note that $\alpha + 1$ rather than α is the optional time there.) The present generalization consists in replacing a stationary independent process by a smartingale. The classical problem of "gambler's ruin" illustrates very well the ideas involved, as follows.

Let $\{S_n, n \in N^0\}$ be a random walk in the notation of Chapter 8, and let S_1 have the Bernoullian distribution $\frac{1}{2}\delta_1 + \frac{1}{2}\delta_{-1}$. It follows from Theorem 8.3.4, or the more elementary Exercise 15 of Sec. 9.2, that the walk will almost certainly leave the interval $[-a, b]$, where a and b are strictly positive integers; and since it can move only one unit a time, it must reach either $-a$ or b. This means that if we set

$$(20) \qquad \alpha = \min\{n \geq 1: S_n = -a\}, \quad \beta = \min\{n \geq 1: S_n = b\},$$

then $\gamma = \alpha \Lambda \beta$ is a finite optional r.v. It follows from the Corollary to Theorem 9.3.4 that $\{S_{\gamma\Lambda n}\}$ is a martingale. Now

$$(21) \qquad \lim_{n\to\infty} S_{\gamma\Lambda n} = S_\gamma \quad \text{a.e.}$$

and clearly S_γ takes only the values $-a$ and b. The question is: with what probabilities? In the gambling interpretation: if two gamblers play a fair coin-tossing game and possess, respectively, a and b units of the constant stake as initial capitals, what is the probability of ruin for each?

The answer is immediate ("without any computation"!) if we show first that the two r.v.'s $\{S_1, S_\gamma\}$ form a martingale, for then

$$(22) \qquad \mathscr{E}(S_\gamma) = \mathscr{E}(S_1) = 0,$$

which is to say that

$$-a\mathscr{P}\{S_\gamma = -a\} + b\mathscr{P}\{S_\gamma = b\} = 0,$$

so that the probability of ruin is inversely proportional to the initial capital of the gambler, a most sensible solution.

To show that the pair $\{S_1, S_\gamma\}$ forms a martingale we use Theorem 9.3.5 since $\{S_{\gamma\wedge n}, n\in N_\infty\}$ is a bounded martingale. The more elementary Theorem 9.3.4 is inapplicable, since γ is not bounded. However, there is a simpler way out in this case: (21) and the boundedness just mentioned imply that

$$\mathscr{E}(S_\gamma) = \lim_{n\to\infty} \mathscr{E}(S_{\gamma\wedge n}),$$

and since $\mathscr{E}(S_{\gamma\wedge 1}) = \mathscr{E}(S_1)$, (22) follows directly.

The ruin problem belonged to the ancient history of probability theory, and can be solved by elementary methods based on difference equations (see, e.g., Uspensky, *Introduction to mathematical probability*, McGraw-Hill, New York, 1937). The approach sketched above, however, has all the main ingredients of an elaborate modern theory. The little equation (22) is the prototype of a "harmonic equation", and the problem itself is a "boundary-value problem". The steps used in the solution—to wit: the introduction of a martingale, its optional stopping, its convergence to a limit, and the extension of the martingale property to include the limit with the consequent convergence of expectations—are all part of a standard procedure now ensconced in the general theory of Markov processes and the allied potential theory.

EXERCISES

1. The defining relation for a martingale may be generalized as follows. For each optional r.v. $\alpha \le n$, we have $\mathscr{E}\{X_n \mid \mathscr{F}_\alpha\} = X_\alpha$. Similarly for a smartingale.

***2.** If X is an integrable r.v., then the collection of (equivalence classes of) r.v.'s $\mathscr{E}(X \mid \mathscr{G})$ with $\mathscr{G}$ ranging over all Borel subfields of $\mathscr{F}$, is uniformly integrable.

3. Suppose $\{X_n^{(k)}, \mathscr{F}_n\}$, $k = 1, 2$, are two [super]martingales, α is a finite optional r.v., and $X_\alpha^{(1)} = [\ge] X_\alpha^{(2)}$. Define $X_n = X_n^{(1)} 1_{\{n\le\alpha\}} + X_n^{(2)} 1_{\{n>\alpha\}}$; show that $\{X_n, \mathscr{F}_n\}$ is a [super]martingale. [HINT: Verify the defining relation in (4) for $m = n + 1$.]

4. Suppose each X_n is integrable and

$$\mathscr{E}\{X_{n+1} \mid X_1, \ldots, X_n\} = n^{-1}(X_1 + \cdots + X_n)$$

then $\{(n^{-1})(X_1 + \cdots + X_n), n \in N\}$ is a martingale.

5. Every sequence of integrable r.v.'s is the sum of a supermartingale and a submartingale.

6. If $\{X_n, \mathscr{F}_n\}$ and $\{X'_n, \mathscr{F}_n\}$ are martingales, then so is $\{X_n + X'_n, \mathscr{F}_n\}$. But it may happen that $\{X_n\}$ and $\{X'_n\}$ are martingales while $\{X_n + X'_n\}$ is not.

[HINT: Let x_1 and x_1' be independent Bernoullian r.v.'s; and $x_2 = x_2' = +1$ or -1 according as $x_1 + x_1' = 0$ or $\neq 0$; notation as in (7).]

7. Find an example of a positive martingale which is not uniformly integrable. [HINT: You win 2^n if it's heads n times in a row, and you lose everything as soon as it's tails.]

8. Find an example of a martingale $\{X_n\}$ such that $X_n \to -\infty$ a.e. This implies that even in a "fair" game one player may be bound to lose an arbitrarily large amount if he plays long enough (and no limit is set to the liability of the other player). [HINT: Try sums of independent but not identically distributed r.v.'s with mean 0.]

***9.** Prove that if $\{Y_n, \mathscr{F}_n\}$ is a martingale such that $Y_n \in \mathscr{F}_{n-1}$, then for every n, $Y_n = Y_1$ a.e. Deduce from this result that Doob's decomposition (6) is unique (up to equivalent r.v.'s) under the condition that $Z_n \in \mathscr{F}_{n-1}$ for every $n \geq 2$. If this condition is not imposed, find two different decompositions.

10. If $\{X_n\}$ is a uniformly integrable submartingale, then for any optional r.v. α we have

(i) $\{X_{\alpha \wedge n}\}$ is a uniformly integrable submartingale;

(ii) $\mathscr{E}(X_1) \leq \mathscr{E}(X_\alpha) \leq \sup_n \mathscr{E}(X_n)$.

[HINT: $|X_{\alpha \wedge n}| \leq |X_\alpha| + |X_n|$.]

***11.** Let $\{X_n, \mathscr{F}_n; n \in N\}$ be a [super]martingale satisfying the following condition: there exists a constant M such that for every $n \geq 1$:

$$\mathscr{E}\{|X_n - X_{n-1}| \,|\, \mathscr{F}_{n-1}\} \leq M \text{ a.e.}$$

where $X_0 = 0$ and $\mathscr{F}_0$ is trivial. Then for any two optional r.v.'s α and β such that $\alpha \leq \beta$ and $\mathscr{E}(\beta) < \infty$, $\{X_\alpha, \mathscr{F}_\alpha; X_\beta, \mathscr{F}_\beta\}$ is a [super]martingale. This is another case of optional sampling given by Doob, which includes Wald's equation (Theorem 5.5.3) as a special case. [HINT: Dominate the integrand in the second integral in (17) by Y_β where $X_0 = 0$ and $Y_m = \sum_{n=1}^m |X_n - X_{n-1}|$. We have

$$\mathscr{E}(Y_\beta) = \sum_{n=1}^{\infty} \int_{\{\beta \geq n\}} |X_n - X_{n-1}| \, d\mathscr{P} \leq M \mathscr{E}(\beta).]$$

12. Apply Exercise 11 to the gambler's ruin problem discussed in the text and conclude that for the α in (20) we must have $\mathscr{E}(\alpha) = +\infty$. Verify this by elementary computation.

***13.** In the gambler's ruin problem take $b = 1$ in (20). Compute $\mathscr{E}(S_{\beta \wedge n})$ for a fixed n and show that $\{S_0, S_{\beta \wedge n}\}$ forms a martingale. Observe that $\{S_0, S_\beta\}$ does not form a martingale and explain in gambling terms the effect of stopping

β at n. This example shows why in optional sampling the option may be taken even with the knowledge of the present moment under certain conditions. In the case here the present (namely $\beta \wedge n$) may leave one no choice!

14. In the gambler's ruin problem, suppose that S_1 has the distribution

$$p\delta_1 + (1-p)\delta_{-1}, \quad p \neq \tfrac{1}{2};$$

and let $d = 2p - 1$. Show that $\mathscr{E}(S_\gamma) = d\mathscr{E}(\gamma)$. Compute the probabilities of ruin by using difference equations to deduce $\mathscr{E}(\gamma)$, and vice versa.

15. Prove that for any L^1-bounded smartingale $\{X_n, \mathscr{F}_n, n \in N_\infty\}$, and any optional α, we have $\mathscr{E}(|X_\alpha|) < \infty$. [HINT: Prove the result first for a martingale, then use Doob's decomposition.]

***16.** Let $\{X_n, \mathscr{F}_n\}$ be a martingale: $x_1 = X_1$, $x_n = X_n - X_{n-1}$ for $n \geq 2$; let $v_n \in \mathscr{F}_{n-1}$ for $n \geq 1$ where $\mathscr{F}_0 = \mathscr{F}_1$; now put

$$T_n = \sum_{j=1}^{n} v_j x_j.$$

Show that $\{T_n, \mathscr{F}_n\}$ is a martingale provided that T_n is integrable for every n. The martingale may be replaced by a smartingale if $v_n \geq 0$ for every n. As a particular case take $v_n = 1_{\{n \leq \alpha\}}$ where α is an optional r.v. relative to $\{\mathscr{F}_n\}$. What then is T_n? Hence deduce the Corollary to Theorem 9.3.4.

17. As in the preceding exercise, deduce a new proof of Theorem 9.3.4 by taking $v_n = 1_{\{\alpha < n \leq \beta\}}$.

9.4 Inequalities and convergence

We begin with two inequalities, the first of which is a generalization of Kolmogorov's inequality (Theorem 5.3.1).

Theorem 9.4.1. If $\{X_j, \mathscr{F}_j, j \in N_n\}$ is a submartingale, then for each real λ we have

(1)
$$\lambda \mathscr{P}\{\max_{1 \leq j \leq n} X_j \geq \lambda\} \leq \int_{\{\max_{1 \leq j \leq n} X_j \geq \lambda\}} X_n \, d\mathscr{P} \leq \mathscr{E}(X_n^+);$$

(2)
$$\lambda \mathscr{P}\{\min_{1 \leq j \leq n} X_j \leq -\lambda\} \leq \mathscr{E}(X_n - X_1) - \int_{\{\min_{1 \leq j \leq n} X_j \leq -\lambda\}} X_n \, d\mathscr{P}$$
$$\leq \mathscr{E}(X_n^+) - \mathscr{E}(X_1).$$

PROOF. Let α be the first j such that $X_j \geq \lambda$ if there is such a j in N_n, otherwise let $\alpha = n$ (optional stopping at n). It is clear that α is optional;

since it takes only a finite number of values, Theorem 9.3.4 shows that the pair $\{X_\alpha, X_n\}$ forms a submartingale. If we write

$$\mathrm{M} = \{\max_{1\le j\le n} X_j \ge \lambda\},$$

then $\mathrm{M} \in \mathscr{F}_\alpha$ (why?) and $X_\alpha \ge \lambda$ on M, hence the first inequality follows from

$$\lambda\mathscr{P}(\mathrm{M}) \le \int_{\mathrm{M}} X_\alpha \, d\mathscr{P} \le \int_{\mathrm{M}} X_n \, d\mathscr{P};$$

the second is just a cruder consequence.

Similarly let β be the first j such that $X_j \le -\lambda$ if there is such a j in N_n, otherwise let $\beta = n$. Put also

$$\mathrm{M}_k = \{\min_{1\le j\le k} X_j \le -\lambda\}.$$

Then $\{X_1, X_\beta\}$ is a submartingale by Theorem 9.3.4, and so

$$\begin{aligned}\mathscr{E}(X_1) \le \mathscr{E}(X_\beta) &= \int_{\{\beta\le n-1\}} X_\beta \, d\mathscr{P} + \int_{\mathrm{M}_{n-1}^c \mathrm{M}_n} X_n \, d\mathscr{P} + \int_{\mathrm{M}_n^c} X_n \, d\mathscr{P} \\ &\le -\lambda\mathscr{P}(\mathrm{M}_n) + \mathscr{E}(X_n) - \int_{\mathrm{M}_n} X_n \, d\mathscr{P},\end{aligned}$$

which reduces to (2).

Corollary 1. If $\{X_n\}$ is a martingale, then for each $\lambda>0$:

$$(3) \qquad \mathscr{P}\{\max_{1\le j\le n} |X_j| \ge \lambda\} \le \frac{1}{\lambda}\int_{\{\max_{1\le j\le n} |X_j|\ge\lambda\}} |X_n| \, d\mathscr{P} \le \frac{1}{\lambda}\mathscr{E}(|X_n|).$$

If in addition $\mathscr{E}(X_n^2) < \infty$ for each n, then we have also

$$(4) \qquad \mathscr{P}\{\max_{1\le j\le n} |X_j| \ge \lambda\} \le \frac{1}{\lambda^2}\mathscr{E}(X_n^2).$$

These are obtained by applying the theorem to the submartingales $\{|X_n|\}$ and $\{X_n^2\}$. In case X_n is the S_n in Theorem 5.3.1, (4) is precisely the Kolmogorov inequality there.

Corollary 2. Let $1 \le m \le n$, $\Lambda_m \in \mathscr{F}_m$ and $\mathrm{M} = \{\max_{m\le j\le n} X_j \ge \lambda\}$, then

$$\lambda\mathscr{P}\{\Lambda_m \cap \mathrm{M}\} \le \int_{\Lambda_m\cap\mathrm{M}} X_n \, d\mathscr{P}.$$

This is proved just as (1) and will be needed later.

We now come to a new kind of inequality, which will be the tool for proving the main convergence theorem below. Given any sequence of r.v.'s $\{X_j\}$, for each sample point ω, the convergence properties of the numerical sequence $\{X_j(\omega)\}$ hinge on the oscillation of the finite segments $\{X_j(\omega), j \in N_n\}$ as $n \to \infty$. In particular the sequence will have a limit, finite or infinite, if and only if the number of its oscillations between any two [rational] numbers a and b is finite (depending on a, b and ω). This is a standard type of argument used in measure and integration theory (cf. Exercise 10 of Sec. 4.2). The interesting thing is that for a smartingale, a sharp estimate of the expected number of oscillations is obtainable.

Let $a < b$. The number ν of "upcrossings" of the interval $[a, b]$ by a numerical sequence $\{x_1, \dots, x_n\}$ is defined as follows. Set

$$\alpha_1 = \min\{j: 1 \le j \le n, x_j \le a\},$$
$$\alpha_2 = \min\{j: \alpha_1 < j \le n, x_j \ge b\};$$

if either α_1 or α_2 is not defined because no such j exists, we define $\nu = 0$. In general, for $k \ge 2$ we set

$$\alpha_{2k-1} = \min\{j: \alpha_{2k-2} < j \le n, x_j \le a\},$$
$$\alpha_{2k} = \min\{j: \alpha_{2k-1} < j \le n, x_j \ge b\};$$

if any one of these is undefined, then all the subsequent ones will be undefined. Let α_ℓ be the last defined one, with $\ell = 0$ if α_1 is undefined, then ν is defined to be $[\ell/2]$. Thus ν is the actual number of successive times that the sequence crosses from $\le a$ to $\ge b$. Although the exact number is not essential, since a couple of crossings more or less would make no difference, we must adhere to a rigid way of counting in order to be accurate below.

Theorem 9.4.2. Let $\{X_j, \mathscr{F}_j, j \in N_n\}$ be a submartingale and $-\infty < a < b < \infty$. Let $\nu_{[a,b]}^{(n)}(\omega)$ denote the number of upcrossings of $[a, b]$ by the sample sequence $\{X_j(\omega); j \in N_n\}$. We have then

$$(5) \qquad \mathscr{E}\{\nu_{[a,b]}^{(n)}\} \le \frac{\mathscr{E}\{(X_n - a)^+\} - \mathscr{E}\{(X_1 - a)^+\}}{b - a} \le \frac{\mathscr{E}\{X_n^+\} + |a|}{b - a}.$$

PROOF. Consider first the case where $X_j \ge 0$ for every j and $0 = a < b$, so that $\nu_{[a,b]}^{(n)}(\omega)$ becomes $\nu_{[0,b]}^{(n)}(\omega)$, and $X_{\alpha_j}(\omega) = 0$ if j is odd, where $\alpha_j = \alpha_j(\omega)$ is defined as above with $x_j = X_j(\omega)$. For each ω, the sequence $\alpha_j(\omega)$ is defined only up to $\ell(\omega)$, where $0 \le \ell(\omega) \le n$. But now we modify the definition so that $\alpha_j(\omega)$ is defined for $1 \le j \le n$ by setting it to be equal to n wherever it was previously undefined. Since for some ω, a previously defined $\alpha_j(\omega)$ may also be equal to n, this apparent confusion will actually simplify the formulas below. In the same vein we set $\alpha_0 \equiv 1$. Observe that $\alpha_n = n$ in any case, so that

$$X_n - X_1 = X_{\alpha_n} - X_{\alpha_0} = \sum_{j=0}^{n-1}(X_{\alpha_{j+1}} - X_{\alpha_j}) = \sum_{j \text{ even}} + \sum_{j \text{ odd}}.$$

If j is odd and $j+1 \leq \ell(\omega)$, then

$$X_{\alpha_{j+1}}(\omega) \geq b>0 = X_{\alpha_j}(\omega);$$

If j is odd and $j = \ell(\omega)$, then

$$X_{\alpha_{j+1}}(\omega) = X_n(\omega) \geq 0 = X_{\alpha_j}(\omega);$$

if j is odd and $\ell(\omega) < j$, then

$$X_{\alpha_{j+1}}(\omega) = X_n(\omega) = X_{\alpha_j}(\omega).$$

Hence in all cases we have

$$\sum_{j \text{ odd}} (X_{\alpha_{j+1}}(\omega) - X_{\alpha_j}(\omega)) \geq \sum_{\substack{j \text{ odd} \\ j+1 \leq \ell(\omega)}} (X_{\alpha_{j+1}}(\omega) - X_{\alpha_j}(\omega)) \geq \left[\frac{\ell(\omega)}{2}\right] b = \nu^{(n)}_{[0,b]}(\omega) b. \tag{6}$$

Next, observe that $\{\alpha_j, 0 \leq j \leq n\}$ as modified above is in general of the form $1 = \alpha_0 \leq \alpha_1 < \alpha_2 < \cdots < \alpha_\ell \leq \alpha_{\ell+1} = \cdots = \alpha_n = n$, and since constants are optional, this is an increasing sequence of optional r.v.'s. Hence by Theorem 9.3.4, $\{X_{\alpha_j}, 0 \leq j \leq n\}$ is a submartingale so that for each $j, 0 \leq j \leq n-1$, we have $\mathscr{E}\{X_{\alpha_j+1} - X_{\alpha_j}\} \geq 0$ and consequently

$$\mathscr{E}\left\{\sum_{j \text{ even}} (X_{\alpha_{j+1}} - X_{\alpha_j})\right\} \geq 0.$$

Adding to this the expectations of the extreme terms in (6), we obtain

$$\mathscr{E}(X_n - X_1) \geq \mathscr{E}(\nu^{(n)}_{[0,b]}) b, \tag{7}$$

which is the particular case of (5) under consideration.

In the general case we apply the case just proved to $\{(X_j - a)^+, j \in N_n\}$, which is a submartingale by Corollary 1 to Theorem 9.3.1. It is clear that the number of upcrossings of $[a, b]$ by the given submartingale is exactly that of $[0, b-a]$ by the modified one. The inequality (7) becomes the first inequality in (5) after the substitutions, and the second one follows since $(X_n - a)^+ \leq X_n^+ + |a|$. Theorem 9.4.2 is proved.

The corresponding result for a supermartingale will be given below; but after such a painstaking definition of *upcrossing*, we may leave the dual definition of *downcrossing* to the reader.

Theorem 9.4.3. Let $\{X_j, \mathscr{F}_j; j \in N_n\}$ be a supermartingale and let $-\infty < ab < \infty$. Let $\tilde{\nu}^{(n)}_{[a,b]}$ be the number of downcrossings of $[a, b]$ by the sample sequence $\{X_j(\omega), j \in N_n\}$. We have then

$$(8) \qquad \mathscr{E}\{\tilde{\nu}^{(n)}_{[a,b]}\} \le \frac{\mathscr{E}\{X_1 \wedge b\} - \mathscr{E}\{X_n \wedge b\}}{b-a}$$

PROOF. $\{-X_j, j \in N_n\}$ is a submartingale and $\tilde{\nu}^{(n)}_{[a,b]}$ is $\nu^{(n)}_{[-b,-a]}$ for this submartingale. Hence the first part of (5) becomes

$$\mathscr{E}\{\tilde{\nu}^{(n)}_{[a,b]}\} \le \frac{\mathscr{E}\{(-X_n + b)^+ - (-X_1 + b)^+\}}{-a-(-b)} = \frac{\mathscr{E}\{(b - X_n)^+ - (b - X_1)^+\}}{b-a}.$$

Since $(b - x)^+ = b - (b \wedge x)$ this is the same as in (8).

Corollary. For a positive supermartingale we have for $0 \le a < b < \infty$

$$\mathscr{E}\{\tilde{\nu}^{(n)}_{[a,b]}\} \le \frac{b}{b-a}.$$

G. Letta proved the sharper "dual":

$$\mathscr{E}\{\nu^{(n)}_{[a,b]}\} \le \frac{a}{b-a}.$$

(*Martingales et intégration stochastique*, Quaderni, Pisa, 1984, 48–49.)

The basic convergence theorem is an immediate consequence of the upcrossing inequality.

Theorem 9.4.4. If $\{X_n, \mathscr{F}_n; n \in N\}$ is an L^1-bounded submartingale, then $\{X_n\}$ converges a.e. to a finite limit.

Remark. Since

$$\mathscr{E}(|X_n|) = 2\mathscr{E}(X_n^+) - \mathscr{E}(X_n) \le 2\mathscr{E}(X_n^+) - \mathscr{E}(X_1),$$

the condition of L^1-boundedness is equivalent to the apparently weaker one below:

$$(9) \qquad \sup_n \mathscr{E}(X_n^+) < \infty.$$

PROOF. Let $\nu_{[a,b]} = \lim_n \nu^{(n)}_{[a,b]}$. Our hypothesis implies that the last term in (5) is bounded in n; letting $n \to \infty$, we obtain $\mathscr{E}\{\nu_{[a,b]}\} < \infty$ for every a and b, and consequently $\nu_{[a,b]}$ is finite with probability one. Hence, for each pair of rational numbers $a < b$, the set

$$\Lambda_{[a,b]} = \{\underline{\lim}_n X_n < a < b < \overline{\lim}_n X_n\}$$

is a null set; and so is the union over all such pairs. Since this union contains the set where $\underline{\lim}_n X_n < \overline{\lim}_n X_n$, the limit exists a.e. It must be finite a.e. by Fatou's lemma applied to the sequence $|X_n|$.

Corollary. Every uniformly bounded smartingale converges a.e. Every positive supermartingale and every negative submartingale converge a.e.

It may be instructive to sketch a direct proof of Theorem 9.4.4 which is done "by hand", so to speak. This is the original proof given by Doob (1940) for a martingale.

Suppose that the set $\Lambda_{[a,b]}$ above has probability $> \eta > 0$. For each ω in $\Lambda_{[a,b]}$, the sequence $\{X_n(\omega), n \in N\}$ takes an infinite number of values $< a$ and an infinite number of values $> b$. Let $1 = n_0 < n_1 < \dots$ and put

$$\Lambda_{2j-1} = \{\min_{n_{2j-2} \le i \le n_{2j-1}} X_i < a\}, \quad \Lambda_{2j} = \{\max_{n_{2j-1} < i \le n_{2j}} X_i > b\}.$$

Then for each k it is possible to choose the n_i's successively so that the differences $n_i - n_{i-1}$ for $1 \le i \le 2k$ are so large that "most" of $\Lambda_{[a,b]}$ is contained in $\bigcap_{i=1}^{2k} \Lambda_i$, so that

$$\mathscr{P}\left\{\bigcap_{i=1}^{2k} \Lambda_i\right\} > \eta.$$

Fixing an $n > n_{2k}$ and applying Corollary 2 to Theorem 9.4.1 to $\{-X_i\}$ as well as $\{X_i\}$, we have

$$a\mathscr{P}\left(\bigcap_{i=1}^{2j-1} \Lambda_i\right) \ge \int_{\bigcap_{i=1}^{2j-1}\Lambda_i} X_{n_{2j-1}}\, d\mathscr{P} = \int_{\bigcap_{i=1}^{2j-1}\Lambda_i} X_n\, d\mathscr{P},$$

$$b\mathscr{P}\left(\bigcap_{i=1}^{2j} \Lambda_i\right) \le \int_{\bigcap_{i=1}^{2j}\Lambda_i} X_{n_{2j}}\, d\mathscr{P} = \int_{\bigcap_{i=1}^{2j}\Lambda_i} X_n\, d\mathscr{P},$$

where the equalities follow from the martingale property. Upon subtraction we obtain

$$(b-a)\mathscr{P}\left(\bigcap_{i=1}^{2j} \Lambda_i\right) - a\mathscr{P}\left(\bigcap_{i=1}^{2j-1} \Lambda_i\Lambda_{2j}^c\right) \le -\int_{\bigcap_{i=1}^{2j-1}\Lambda_i\Lambda_{2j}^c} X_n\, d\mathscr{P},$$

and consequently, upon summing over $1 \le j \le k$:

$$k(b-a)\eta - |a| \le \mathscr{E}(|X_n|).$$

This is impossible if k is large enough, since $\{X_n\}$ is L^1-bounded.

Once Theorem 9.4.4 has been proved for a martingale, we can extend it easily to a positive or uniformly integrable supermartingale by using Doob's decomposition. Suppose $\{X_n\}$ is a positive supermartingale and $X_n = Y_n - Z_n$ as in Theorem 9.3.2. Then $0 \le Z_n \le Y_n$ and consequently

$$\mathscr{E}(Z_\infty) = \lim_{n\to\infty} \mathscr{E}(Z_n) \le \mathscr{E}(Y_1);$$

next we have

$$\mathscr{E}(Y_n) = \mathscr{E}(X_n) + \mathscr{E}(Z_n) \le \mathscr{E}(X_1) + \mathscr{E}(Z_\infty).$$

Hence $\{Y_n\}$ is an L^1-bounded martingale and so converges to a finite limit as $n \to \infty$. Since $Z_n \uparrow Z_\infty < \infty$ a.e., the convergence of $\{X_n\}$ follows. The case of a uniformly integrable supermartingale is just as easy by the corollary to Theorem 9.3.2.

It is trivial that a positive submartingale need not converge, since the sequence $\{n\}$ is such a one. The classical random walk $\{S_n\}$ (coin-tossing game) is an example of a martingale that does not converge (why?). An interesting and not so trivial consequence is that both $\mathscr{E}(S_n^+)$ and $\mathscr{E}(|S_n|)$ must diverge to $+\infty$! (Cf. Exercise 2 of Sec. 6.4.) Further examples are furnished by "stopped random walk". For the sake of concreteness, let us stay with the classical case and define γ to be the first time the walk reaches $+1$. As in our previous discussion of the gambler's-ruin problem, the modified random walk $\{\tilde{S}_n\}$, where $\tilde{S}_n = S_{\gamma\wedge n}$, is still a martingale, hence in particular we have for each n:

$$\mathscr{E}(\tilde{S}_n) = \mathscr{E}(\tilde{S}_1) = \int_{\{\gamma=1\}} S_1\, d\mathscr{P} + \int_{\{\gamma>1\}} S_1\, d\mathscr{P} = \mathscr{E}(S_1) = 0.$$

As in (21) of Sec. 9.3 we have, writing $\tilde{S}_\infty = S_\gamma = 1$,

$$\lim_n \tilde{S}_n = \tilde{S}_\infty \quad \text{a.e.},$$

since $\gamma < \infty$ a.e., but this convergence now also follows from Theorem 9.4.4, since $\tilde{S}_n^+ \le 1$. Observe, however, that

$$\mathscr{E}(\tilde{S}_n) = 0 < 1 = \mathscr{E}(\tilde{S}_\infty).$$

Next, we change the definition of γ to be the first time (≥ 1) the walk "returns" to 0, as usual supposing $S_0 \equiv 0$. Then $\tilde{S}_\infty = 0$ and we have indeed $\mathscr{E}(\tilde{S}_n) = \mathscr{E}(\tilde{S}_\infty)$. But for each n,

$$\int_{\{\tilde{S}_n>0\}} \tilde{S}_n\, d\mathscr{P} > 0 = \int_{\{\tilde{S}_n>0\}} \tilde{S}_\infty\, d\mathscr{P},$$

so that the "extended sequence" $\{\tilde{S}_1, \dots, \tilde{S}_n, \dots, \tilde{S}_\infty\}$ is no longer a martingale. These diverse circumstances will be dealt with below.

Theorem 9.4.5. The three propositions below are equivalent for a submartingale $\{X_n, \mathscr{F}_n; n \in N\}$:

(a) it is a uniformly integrable sequence;
(b) it converges in L^1;
(c) it converges a.e. to an integrable X_∞ such that $\{X_n, \mathscr{F}_n; n \in N_\infty\}$ is a submartingale and $\mathscr{E}(X_n)$ converges to $\mathscr{E}(X_\infty)$.

PROOF. (a) $\Rightarrow$ (b): under (a) the condition in Theorem 9.4.4 is satisfied so that $X_n \to X_\infty$ a.e. This together with uniform integrability implies $X_n \to X_\infty$ in L^1 by Theorem 4.5.4 with $r = 1$.

(b) $\Rightarrow$ (c): under (b) let $X_n \to X_\infty$ in L^1, then $\mathscr{E}(|X_n|) \to \mathscr{E}(|X_\infty|) < \infty$ and so $X_n \to X_\infty$ a.e. by Theorem 9.4.4. For each $\Lambda \in \mathscr{F}_n$ and $n < n'$, we have

$$\int_\Lambda X_n \, d\mathscr{P} \leq \int_\Lambda X_{n'} \, d\mathscr{P}$$

by the defining relation. The right member converges to $\int_\Lambda X_\infty \, d\mathscr{P}$ by L^1-convergence and the resulting inequality shows that $\{X_n, \mathscr{F}_n; n \in N_\infty\}$ is a submartingale. Since L^1-convergence also implies convergence of expectations, all three conditions in (c) are proved.

(c) $\Rightarrow$ (a); under (c), $\{X_n^+, \mathscr{F}_n; n \in N_\infty\}$ is a submartingale; hence we have for every $\lambda > 0$:

$$(10) \qquad \int_{\{X_n^+ > \lambda\}} X_n^+ \, d\mathscr{P} \leq \int_{\{X_n^+ > \lambda\}} X_\infty^+ d\mathscr{P},$$

which shows that $\{X_n^+, n \in N\}$ is uniformly integrable. Since $X_n^+ \to X_\infty^+$ a.e., this implies $\mathscr{E}(X_n^+) \to \mathscr{E}(X_\infty^+)$. Since by hypothesis $\mathscr{E}(X_n) \to \mathscr{E}(X_\infty)$, it follows that $\mathscr{E}(X_n^-) \to \mathscr{E}(X_\infty^-)$. This and $X_n^- \to X_\infty^-$ a.e. imply that $\{X_n^-\}$ is uniformly integrable by Theorem 4.5.4 for $r = 1$. Hence so is $\{X_n\}$.

Theorem 9.4.6. In the case of a martingale, propositions (a) and (b) above are equivalent to (c′) or (d) below:

(c′) it converges a.e. to an integrable X_∞ such that $\{X_n, \mathscr{F}_n; n \in N_\infty\}$ is a martingale;
(d) there exists an integrable r.v. Y such that $X_n = \mathscr{E}(Y \mid \mathscr{F}_n)$ for each $n \in N$.

PROOF. (b) $\Rightarrow$ (c′) as before; (c′) $\Rightarrow$ (a) as before if we observe that $\mathscr{E}(X_n) = \mathscr{E}(X_\infty)$ for every n in the present case, or more rapidly by considering $|X_n|$ instead of X_n^+ as below. (c′) $\Rightarrow$ (d) is trivial, since we may take the Y in (d) to be the X_∞ in (c′). To prove (d) $\Rightarrow$ (a), let $n < n'$, then by

Theorem 9.1.5:

$$\mathscr{E}(X_{n'} \mid \mathscr{F}_n) = \mathscr{E}(\mathscr{E}(Y \mid \mathscr{F}_{n'}) \mid \mathscr{F}_n) = \mathscr{E}(Y \mid \mathscr{F}_n) = X_n,$$

hence $\{X_n, \mathscr{F}_n, n \in N; Y, \mathscr{F}\}$ is a martingale by definition. Consequently $\{|X_n|, \mathscr{F}_n, n \in N; |Y|, \mathscr{F}\}$ is a submartingale, and we have for each $\lambda>0$:

$$\int_{\{|X_n|>\lambda\}} |X_n| \, d\mathscr{P} \leq \int_{\{|X_n|>\lambda\}} |Y| \, d\mathscr{P},$$

$$\mathscr{P}\{|X_n| > \lambda\} \leq \frac{1}{\lambda}\mathscr{E}(|X_n|) \leq \frac{1}{\lambda}\mathscr{E}(|Y|),$$

which together imply (a).

Corollary. Under (d), $\{X_n, \mathscr{F}_n, n \in N; X_\infty, \mathscr{F}_\infty; Y, \mathscr{F}\}$ is a martingale, where X_∞ is given in (c′).

Recall that we have introduced martingales of the form in (d) earlier in (13) in Sec. 9.3. Now we know this class coincides with the class of uniformly integrable martingales.

We have already observed that the defining relation for a smartingale is meaningful on any linearly ordered (or even partially ordered) index set. The idea of extending the latter to a limit index is useful in applications to continuous-time stochastic processes, where, for example, a martingale may be defined on a dense set of real numbers in (t_1, t_2) and extended to t_2. This corresponds to the case of extension from N to N_∞. The dual extension corresponding to that to t_1 will now be considered. Let $-N$ denote the set of strictly negative integers in their natural order, let $-\infty$ precede every element in $-N$, and denote by $-N_\infty$ the set $\{-\infty\} \cup (-N)$ in the prescribed order. If $\{\mathscr{F}_n, n \in -N\}$ is a decreasing (with decreasing n) sequence of Borel fields, their intersection $\bigcap_{n\in -N} \mathscr{F}_n$ will be denoted by $\mathscr{F}_{-\infty}$.

The convergence results for a submartingale on $-N$ are simpler because the right side of the upcrossing inequality (5) involves the expectation of the r.v. with the largest index, which in this case is the fixed -1 rather than the previous varying n. Hence for mere convergence there is no need for an extra condition such as (9).

Theorem 9.4.7. Let $\{X_n, n \in -N\}$ be a submartingale. Then

(11) $$\lim_{n\to-\infty} X_n = X_{-\infty}, \quad \text{where} \quad -\infty \leq X_{-\infty} < \infty \quad \text{a.e.}$$

The following conditions are equivalent, and they are automatically satisfied in case of a martingale with "submartingale" replaced by "martingale" in (c):

(a) $\{X_n\}$ is uniformly integrable;
(b) $X_n \to X_{-\infty}$ in L^1;
(c) $\{X_n, n \in -N_\infty\}$ is a submartingale;
(d) $\lim_{n\to-\infty} \downarrow \mathscr{E}(X_n) > -\infty$.

PROOF. Let $\nu^{(n)}_{[a,b]}$ be the number of upcrossings of $[a, b]$ by the sequence $\{X_{-n,\dots,}X_{-1}\}$. We have from Theorem 9.4.2:

$$\mathscr{E}\{\nu^{(n)}_{[a,b]}\} \le \frac{\mathscr{E}(X^+_{-1}) + |a|}{b-a}.$$

Letting $n \to \infty$ and arguing as the proof of Theorem 9.4.4, we conclude (11) by observing that

$$\mathscr{E}(X^+_{-\infty}) \le \varliminf_n \mathscr{E}(X^+_{-n}) \le \mathscr{E}(X^+_{-1}) < \infty.$$

The proofs of (a) $\Rightarrow$ (b) $\Rightarrow$ (c) are entirely similar to those in Theorem 9.4.5. (c) $\Rightarrow$ (d) is trivial, since $-\infty < \mathscr{E}(X_{-\infty}) \le \mathscr{E}(X_{-n})$ for each n. It remains to prove (d) $\Rightarrow$ (a). Letting C denote the limit in (d), we have for each $\lambda>0$:

$$(12) \quad \lambda\mathscr{P}\{|X_n| > \lambda\} \le \mathscr{E}(|X_n|) = 2\mathscr{E}(X_n^+) - \mathscr{E}(X_n) \le 2\mathscr{E}(X^+_{-1}) - C < \infty.$$

It follows that $\mathscr{P}\{|X_n| > \lambda\}$ converges to zero uniformly in n as $\lambda \to \infty$. Since

$$\int_{\{X_n^+>\lambda\}} X_n^+ \, d\mathscr{P} \le \int_{\{X_n^+>\lambda\}} X^+_{-1} \, d\mathscr{P},$$

this implies that $\{X_n^+\}$ is uniformly integrable. Next if $n < m$, then

$$\begin{aligned} 0 \ge \int_{\{X_n<-\lambda\}} X_n \, d\mathscr{P} &= \mathscr{E}(X_n) - \int_{\{X_n\ge-\lambda\}} X_n \, d\mathscr{P} \\ &\ge \mathscr{E}(X_n) - \int_{\{X_n\ge-\lambda\}} X_m \, d\mathscr{P} \\ &= \mathscr{E}(X_n - X_m) + \mathscr{E}(X_m) - \int_{\{X_n\ge-\lambda\}} X_m \, d\mathscr{P} \\ &= \mathscr{E}(X_n - X_m) + \int_{\{X_n<-\lambda\}} X_m \, d\mathscr{P}. \end{aligned}$$

By (d), we may choose $-m$ so large that $\mathscr{E}(X_n - X_m) > -\epsilon$ for any given $\epsilon>0$ and for every $n < m$. Having fixed such an m, we may choose λ so large that

$$\sup_n \int_{\{X_n<-\lambda\}} |X_m| \, d\mathscr{P} < \epsilon$$

by the remark after (12). It follows that $\{X_n^-\}$ is also uniformly integrable, and therefore (a) is proved.

The next result will be stated for the index set $\overline{N}$ of all integers in their natural order:

$$\overline{N} = \{\dots, -n, \dots, -2, -1, 0, 1, 2, \dots, n, \dots\}.$$

Let $\{\mathscr{F}_n\}$ be increasing B.F.'s on $\overline{N}$, namely: $\mathscr{F}_n \subset \mathscr{F}_m$ if $n \leq m$. We may "close" them at both ends by adjoining the B.F.'s below:

$$\mathscr{F}_{-\infty} = \bigwedge_n \mathscr{F}_n, \quad \mathscr{F}_\infty = \bigvee_n \mathscr{F}_n.$$

Let $\{Y_n\}$ be r.v.'s indexed by $\overline{N}$. If the B.F.'s and r.v.'s are only given on N or $-N$, they can be trivially extended to $\overline{N}$ by putting $\mathscr{F}_n = \mathscr{F}_1$, $Y_n = Y_1$ for all $n \leq 0$, or $\mathscr{F}_n = \mathscr{F}_{-1}$, $Y_n = Y_{-1}$ for all $n \geq 0$. The following convergence theorem is very useful.

Theorem 9.4.8. Suppose that the Y_n's are dominated by an integrable r.v.Z:

$$\text{(13)} \qquad \sup_n |Y_n| \leq Z;$$

and $\lim_n Y_n = Y_\infty$ or $Y_{-\infty}$ as $n \to \infty$ or $-\infty$. Then we have

$$\text{(14a)} \qquad \lim_{n\to\infty} \mathscr{E}\{Y_n \mid \mathscr{F}_n\} = \mathscr{E}\{Y_\infty \mid \mathscr{F}_\infty\};$$

$$\text{(14b)} \qquad \lim_{n\to-\infty} \mathscr{E}\{Y_n \mid \mathscr{F}_n\} = \mathscr{E}\{Y_{-\infty} \mid \mathscr{F}_{-\infty}\}.$$

In particular for a fixed integrable r.v. Y, we have

$$\text{(15a)} \qquad \lim_{n\to\infty} \mathscr{E}\{Y \mid \mathscr{F}_n\} = \mathscr{E}\{Y \mid \mathscr{F}_\infty\};$$

$$\text{(15b)} \qquad \lim_{n\to-\infty} \mathscr{E}\{Y \mid \mathscr{F}_n\} = \mathscr{E}\{Y \mid \mathscr{F}_{-\infty}\}.$$

where the convergence holds also in L^1 in both cases.

PROOF. We prove (15) first. Let $X_n = \mathscr{E}\{Y \mid \mathscr{F}_n\}$. For $n \in N$, $\{X_n, \mathscr{F}_n\}$ is a martingale already introduced in (13) of Sec. 9.3; the same is true for $n \in -N$. To prove (15a), we apply Theorem 9.4.6 to deduce (c′) there. It remains to identify the limit X_∞ with the right member of (15a). For each $\Lambda \in \mathscr{F}_n$, we have

$$\int_\Lambda Y \, d\mathscr{P} = \int_\Lambda X_n \, d\mathscr{P} = \int_\Lambda X_\infty \, d\mathscr{P}.$$

Hence the equations hold also for $\Lambda \in \mathscr{F}_\infty$ (why?), and this shows that X_∞ has the defining property of $\mathscr{E}(Y \mid \mathscr{F}_\infty$, since $X_\infty \in \mathscr{F}_\infty$. Similarly, the limit $X_{-\infty}$ in (15b) exists by Theorem 9.4.7; to identify it, we have by (c) there, for each $\Lambda \in \mathscr{F}_{-\infty}$:

$$\int_\Lambda X_{-\infty}\, d\mathscr{P} = \int_\Lambda X_n\, d\mathscr{P} = \int_\Lambda Y\, d\mathscr{P}.$$

This shows that $X_{-\infty}$ is equal to the right member of (15b).

We can now prove (14a). Put for $m \in N$:

$$W_m = \sup_{n \ge m} |Y_n - Y_\infty|;$$

then $|W_m| \le 2Z$ and $\lim_{m\to\infty} W_m = 0$ a.e. Applying (15a) to W_m we obtain

$$\overline{\lim_{n\to\infty}}\ \mathscr{E}\{|Y_n - Y_\infty| | \mathscr{F}_n\} \le \lim_{n\to\infty} \mathscr{E}\{W_m \mid \mathscr{F}_n\} = \mathscr{E}\{W_m \mid \mathscr{F}_\infty\}.$$

As $m \to \infty$, the last term above converges to zero by dominated convergence (see (vii) of Sec. 9.1). Hence the first term must be zero and this clearly implies (14a). The proof of (14b) is completely similar.

Although the corollary below is a very special case we give it for historical interest. It is called Paul Lévy's zero-or-one law (1935) and includes Theorem 8.1.1 as a particular case.

Corollary. If $\Lambda \in \mathscr{F}_\infty$, then

(16) $$\lim_{n\to\infty} \mathscr{P}(\Lambda \mid \mathscr{F}_n) = 1_\Lambda \quad \text{a.e.}$$

The reader is urged to ponder over the intuitive meaning of this result and judge for himself whether it is "obvious" or "incredible".

EXERCISES

*1. Prove that for any smartingale, we have for each $\lambda>0$:

$$\lambda \mathscr{P}\{\sup_n |X_n| \ge \lambda\} \le 3 \sup_n \mathscr{E}(|X_n|).$$

For a martingale or a positive or negative smartingale the constant 3 may be replaced by 1.

2. Let $\{X_n\}$ be a positive supermartingale. Then for almost every ω, $X_k(\omega) = 0$ implies $X_n(\omega) = 0$ for all $n \ge k$. [This is the analogue of a minimum principle in potential theory.]

3. Generalize the upcrossing inequality for a submartingale $\{X_n, \mathscr{F}_n\}$ as follows:

$$\mathscr{E}\{\nu_{[a,b]}^{(n)} \mid \mathscr{F}_1\} \le \frac{\mathscr{E}\{(X_n - a)^+ \mid \mathscr{F}_1\} - (X_1 - a)^+}{b - a}.$$

Similarly, generalize the downcrossing inequality for a positive supermartingale $\{X_n, \mathscr{F}_n\}$ as follows:

$$\mathscr{E}\{\tilde{\nu}_{[a,b]}^{(n)} \mid \mathscr{F}_1\} \le \frac{X_1 \wedge b}{b-a}.$$

***4.** As a sharpening of Theorems 9.4.2 and 9.4.3 we have, for a positive supermartingale $\{X_n, \mathscr{F}_n, n \in N\}$:

$$\mathscr{P}\{\nu_{[a,b]}^{(n)} \ge k\} \le \frac{\mathscr{E}(X_1 \wedge a)}{b}\left(\frac{a}{b}\right)^{k-1},$$

$$\mathscr{P}\{\tilde{\nu}_{[a,b]}^{(n)} \ge k\} \le \frac{\mathscr{E}(X_1 \wedge b)}{b}\left(\frac{a}{b}\right)^{k-1}.$$

These inequalities are due to Dubins. Derive Theorems 9.3.6 and 9.3.7 from them. [HINT:

$$b\mathscr{P}\{\alpha_{2j} < n\} \le \int_{\{\alpha_{2j}<n\}} X_{\alpha_{2j}} d\mathscr{P} \le \int_{\{\alpha_{2j-1}<n\}} X_{\alpha_{2j}}\, d\mathscr{P}$$

$$\le \int_{\{\alpha_{2j-1}<n\}} X_{\alpha_{2j-1}}\, d\mathscr{P} \le a\mathscr{P}\{\alpha_{2j-1} < n\}$$

since $\{\alpha_{2j-1} < n\} \in \mathscr{F}_{\alpha_{2j-1}}$.]

***5.** Every L^1-bounded martingale is the difference of two positive L^1-bounded martingales. This is due to Krickeberg. [HINT: Take one of them to be $\lim_{k\to\infty} \mathscr{E}\{X_k^+ \mid \mathscr{F}_n\}$.]

***6.** A smartingale $\{X_n, \mathscr{F}_n; n \in N\}$ is said to be *closable* [on the right] iff there exists a r.v. X_∞ such that $\{X_n, \mathscr{F}_n; n \in N_\infty\}$ is a smartingale of the same kind. Prove that if so then we can always take $X_\infty = \lim_{n\to\infty} X_n$. This supplies a missing link in the literature. [HINT: For a supermartingale consider $X_n = \mathscr{E}(X_\infty \mid \mathscr{F}_n) + Y_n$, then $\{Y_n, \mathscr{F}_n\}$ is a positive supermartingale so we may apply the convergence theorems to both terms of the decomposition.]

7. Prove a result for closability [on the left] which is similar to Exercise 6 but for the index set $-N$. Give an example to show that in case of N we may have $\lim_{n\to\infty} \mathscr{E}(X_n) \ne \mathscr{E}(X_\infty)$, whereas in case of $-N$ closability implies $\lim_{n\to-\infty} \mathscr{E}(X_n) = \mathscr{E}(X_{-\infty})$.

8. Let $\{X_n, \mathscr{F}_n, n \in N\}$ be a submartingale and let α be a finite optional r.v. satisfying the conditions: (a) $\mathscr{E}(|X_\alpha|) < \infty$, and (b)

$$\varliminf_{n\to\infty} \int_{\{\alpha>n\}} |X_n|\, d\mathscr{P} = 0.$$

Then $\{X_{\alpha\wedge n}, \mathscr{F}_{\alpha\wedge n}; n \in N_\infty\}$ is a submartingale. [HINT: for $\Lambda \in \mathscr{F}_{\alpha\wedge n}$ bound $\int_\Lambda (X_\alpha - X_{\alpha\wedge n})\, d\mathscr{P}$ below by interposing $X_{\alpha\wedge m}$ where $n < m$.]

9. Let $\{X_n, \mathscr{F}_n; n \in N\}$ be a supermartingale satisfying the condition $\lim_{n\to\infty} \mathscr{E}(X_n) > -\infty$. Then we have the representation $X_n = X'_n + X''_n$ where $\{X'_n, \mathscr{F}_n\}$ is a martingale and $\{X''_n, \mathscr{F}_n\}$ is a positive supermartingale such that $\lim_{n\to\infty} X''_n = 0$ in L^1 as well as a.e. This is the analogue of F. Riesz's decomposition of a superharmonic function, X'_n being the *harmonic* part and X''_n the *potential* part. [HINT: Use Doob's decomposition $X_n = Y_n - Z_n$ and put $X''_n = Y_n - \mathscr{E}(Z_\infty \mid \mathscr{F}_n)$.]

10. Let $\{X_n, \mathscr{F}_n\}$ be a *potential*; namely a positive supermartingale such that $\lim_{n\to\infty} \mathscr{E}(X_n) = 0$; and let $X_n = Y_n - Z_n$ be the Doob decomposition [cf. (6) of Sec. 9.3]. Show that

$$X_n = \mathscr{E}(Z_\infty \mid \mathscr{F}_n) - Z_n.$$

***11.** If $\{X_n\}$ is a martingale or positive submartingale such that $\sup_n \mathscr{E}(X_n^2) < \infty$, then $\{X_n\}$ converges in L^2 as well as a.e.

12. Let $\{\xi_n, n \in N\}$ be a sequence of independent and identically distributed r.v.'s with zero mean and unit variance; and $S_n = \sum_{j=1}^n \xi_j$. Then for any optional r.v. α relative to $\{\xi_n\}$ such that $\mathscr{E}(\sqrt{\alpha}) < \infty$, we have $\mathscr{E}(|S_\alpha|) \le \sqrt{2}\mathscr{E}(\sqrt{\alpha})$ and $\mathscr{E}(S_\alpha) = 0$. This is an extension of Wald's equation due to Louis Gordon. [HINT: Truncate α and put $\eta_k = (S_k^2/\sqrt{k}) - (S_{k-1}^2/\sqrt{k-1})$; then

$$\mathscr{E}\{S_\alpha^2/\sqrt{\alpha}\} = \sum_{k=1}^{\infty} \int_{\{\alpha \ge k\}} \eta_k \, d\mathscr{P} \le \sum_{k=1}^{\infty} \mathscr{P}\{\alpha \ge k\}/\sqrt{k} \le 2\mathscr{E}(\sqrt{\alpha});$$

now use Schwarz's inequality followed by Fatou's lemma.]

The next two problems are meant to give an idea of the passage from discrete parameter martingale theory to the continuous parameter theory.

13. Let $\{X_t, \mathscr{F}_t; t \in [0, 1]\}$ be a continuous parameter supermartingale. For each $t \in [0, 1]$ and sequence $\{t_n\}$ decreasing to t, $\{X_{t_n}\}$ converges a.e. and in L^1. For each $t \in [0, 1]$ and sequence $\{t_n\}$ increasing to t, $\{X_{t_n}\}$ converges a.e. but not necessarily in L^1. [HINT: In the second case consider $X_{t_n} - \mathscr{E}\{X_t \mid \mathscr{F}_{t_n}\}$.]

***14.** In Exercise 13 let Q be the set of rational numbers in [0, 1]. For each $t \in (0, 1)$ both limits below exist a.e.:

$$\lim_{\substack{s \uparrow t \\ s \in Q}} X_s, \quad \lim_{\substack{s \downarrow t \\ s \in Q}} X_s.$$

[HINT: Let $\{Q_n, n \ge 1\}$ be finite subsets of Q such that $Q_n \uparrow Q$; and apply the upcrossing inequality to $\{X_s, s \in Q_n\}$, then let $n \to \infty$.]