SINGULAR SET ESTIMATES IN QUASILINEAR ALT-CAFFARELLI PROBLEMS

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Abstract. For a nonnegative, uniformly convex function Φ, we study nonnegative minimizers of the quasilinear Alt-Caffarelli energy $\int_\Omega \Phi(\nabla u)+\chi_{\{u>0\}}$. Under mild regularity assumptions on Φ, we show that the codimension of the singular set of the free boundary is at least $2+\eta$ for a positive η. With a singular minimizer in three dimensions in a forthcoming work, this result is sharp as long as integer dimensions are considered. Our proof method is flexible, and we use it to also prove a similar result for minimizers of the p-Laplacian counterpart of the Alt-Caffarelli energy.

1. Introduction

For a nonnegative and uniformly convex function Φ, we consider the *quasilinear Alt-Caffarelli energy* on a bounded domain Ω in $\mathbb{R}^n$, defined as

$$\mathcal{E}^\Phi(u;\Omega) := \int_\Omega \Phi(\nabla u) + \chi_{\{u>0\}}. \tag{1.1}$$

This functional appears in a variety of applications, for instance, fluid mechanics [4, 48, 58, 57, 63, 64], combustion theory [12, 68, 81], electrostatics [72, 83] as well as material science [38, 39, 82]. For these, it is natural to consider the *one-phase* problem, that is, when the minimizer satisfies

$$u \geq 0 \quad \text{in } \Omega.$$

Such minimizers are obtained when we impose nonnegative boundary data.

When the integrand Φ is taken as $\Phi(p) = |p|^2$, our problem reduces to the classical Alt-Caffarelli problem, one of the most well-studied elliptic free boundary problems. For the classical theory, see the monographs [6, 85] and Section 2.4 below. For some recent developments, see [10, 27, 28, 34, 36, 35, 37, 40, 41, 45, 47, 46, 51, 53]. Motivated by applications in inhomogeneous media, the quasilinear version in (1.1) has also received extensive attention, for instance, [5, 9, 11, 59, 69, 68, 71, 11, 81, 82, 61, 60, 31, 55, 44].

Beyond its applications, the mathematical interest of (1.1) lies in the regularity properties of the minimizer u as well as its free boundary

$$\Gamma(u) := \partial\{u>0\} \cap \Omega. \tag{1.2}$$

This is the interface between the positive phase $\{u>0\}$ and the zero phase $\{u=0\}$.

Local Lipschitz regularity of minimizers and ε-regularity/improvement-of-flatness of the free boundary $\Gamma(u)$ are classical for the Alt-Caffarelli problem [3, 6, 85] and extend, under appropriate structural assumptions, to quasilinear problems [5, 9, 11, 71, 61, 60, 69, 81, 44]. Consequently, the free boundary decomposes into two pieces, the *regular part*, i.e., the set of points x where $\Gamma(u)$ is $C^{1,\alpha}$ in a neighborhood of x, and the *singular part*:

$$\Gamma(u) = \mathrm{Reg}(u) \cup \mathrm{Sing}(u); \tag{1.3}$$

see Definition 2.3. For a nice integrand Φ, the regular part is locally a smooth hypersurface [56].

The next stage is to estimate the size of the singular part $\mathrm{Sing}(u)$, for instance, in terms of its Hausdorff dimension. Here the theory in the quasilinear setting (1.1) diverges substantially from its classical counterpart.

In the classical setting (when $\Phi(p) = |p|^2$), with the monotonicity formula by Weiss [87], an estimate on the dimension of $\mathrm{Sing}(u)$ is equivalent to the classification of homogeneous minimizers in each dimension. Thanks to [8, 52], and very recently, [70] we know that there are no singular homogeneous minimizers in $\mathbb{R}^n$ for $n \leq 6$. As a result, the singular set has dimension at most $n-7$ for a minimizer u of the classical Alt-Caffarelli functional. This is sharp thanks to [27].

For the quasilinear problem (1.1), however, almost nothing is known about the size of $\mathrm{Sing}(u)$.[1] In Theorem 3.1, we show that a Weiss-type monotonicity formula does not hold in this general class. Even for a specialized class of integrands for which a monotonicity formula is known [54], it is not straightforward to reduce singularity analysis to the classification of homogeneous minimizers as in the classical setting. As a result, estimates on the size of $\mathrm{Sing}(u)$ require new ideas and techniques.

In this article, we estimate the Hausdorff dimension of the singular set of a minimizer of the quasilinear functional (1.1).

To be precise, for given constants $\Lambda \geq 1$, $\Upsilon > 0$ and $\alpha \in (0,1)$, the class of integrands we consider satisfy the following assumptions:

(1) $\Phi \geq 0$ in $\mathbb{R}^n$ and $\Phi(0) = 0$;

(2) $\Phi \in C^{1,1}(\mathbb{R}^n)$ with

$$\frac{1}{\Lambda}|\xi|^2 \leq D^2\Phi(p)[\xi,\xi] \leq \Lambda|\xi|^2 \qquad \text{for a.e. } p \in \mathbb{R}^n \text{ and for all } \xi \in \mathbb{R}^n\,. \tag{1.4}$$

(3) For $R_\Lambda > 1$ and $0 < r_\Lambda < 1$, depending only on Λ and the dimension n, we have $\Phi \in C^{2,\alpha}(\mathcal{A}(\Lambda))$ with

$$[D^2\Phi]_{C^\alpha(\mathcal{A}(\Lambda))} \leq \Upsilon \qquad \text{where} \qquad \mathcal{A}(\Lambda) := B_{R_\Lambda} \setminus \overline{B_{r_\Lambda}}. \tag{1.5}$$

For an integrand Φ satisfying all these assumptions, we write

$$\Phi \in I_{\Lambda,\alpha,\Upsilon}. \tag{1.6}$$

If a quantity depends only on the constants $\Lambda, \Upsilon, \alpha$ and the dimension n, then it is referred to as a *universal quantity*.

Remark 1.1. The two universal radii R_Λ and r_Λ arise from the Lipschitz estimate and the non-degeneracy of minimizers of (1.1) and are specified in Definition 4.6.

Remark 1.2. The class $I_{\Lambda,\alpha,\Upsilon}$ contains the following special cases:

(1) A uniformly convex integrand Φ in $C^{2,\alpha}(\mathbb{R}^n)$ that is nonnegative and vanishes at 0;

(2) A positively 2-homogeneous integrand of the form

$$\Phi(p) := |p|^2\varphi(p/|p|),$$

where $\varphi \in C^{2,\alpha}(\mathbb{S}^{n-1})$ is strictly positive and uniformly convex on the sphere.

In the second class, $D^2\Phi$ is 0-homogeneous away from the origin. It extends continuously to the origin if and only if Φ is a quadratic form, equivalently $\varphi(\theta) = \theta \cdot A\theta$ for a symmetric positive definite matrix A. For this reason, we do not pose continuity on $D^2\Phi$ near the origin in our definition of $I_{\Lambda,\alpha,\Upsilon}$ from (1.6). This leads to challenges regarding the Euler-Lagrange equation in a neighborhood of the critical set $\{\nabla u = 0\}$ of a minimizer u. See, for instance, (1.9), Lemma 4.5, Lemma 6.1, and Corollary 6.2.

Our main result is the following:

Theorem 1.3. *Let $n \geq 2$, let $\Phi \in I_{\Lambda,\alpha,\Upsilon}$, and let u be a nonnegative minimizer of $\mathcal{E}^\Phi(\cdot)$ in Ω. If $n \geq 3$, there exists a universal constant $\eta \in (0, n-2)$ such that*

$$\mathcal{H}^{n-2-\eta}(\mathrm{Sing}(u)) = 0. \tag{1.7}$$

If $n = 2$, then $\mathrm{Sing}(u) = \varnothing$.

Here and in the remainder of the paper, we denote the d-dimensional Hausdorff measure by $\mathcal{H}^d(\cdot)$. The energy $\mathcal{E}^\Phi(\cdot) = \mathcal{E}^\Phi(\cdot;\Omega)$ is as defined in (1.1), the class of integrands I is as defined in (1.6), and the singular part of the free boundary $\mathrm{Sing}(u)$ is defined in (1.3).

Remark 1.4. This result is new even without $\eta > 0$. Considering integer dimensions, this estimate is sharp since there is a singular minimizer in three dimensions [20].

Remark 1.5. For a similar estimate with $\eta = 0$, the third condition in the definition of I in (1.6) can be relaxed to merely $\Phi \in C^2(\overline{B_{R_\Lambda}} \setminus B_{r_\Lambda})$ without any quantified upper bound; see Remark 6.3.

[1] By [69], we have $\mathcal{H}^{n-1}(\Gamma(u)) < +\infty$. The theory of sets of finite perimeter implies $\mathcal{H}^{n-1}(\mathrm{Sing}(u)) = 0$.

The method is robust enough to cover a larger class of integrands that allow degeneracies in the convexity. To demonstrate the strategy, we estimate the singular set of minimizers of (1.1) when $\Phi(\cdot) = |\cdot|^p$, i.e. minimizers of

$$\mathcal{E}_p(u) = \int_\Omega |\nabla u|^p + \chi_{\{u>0\}}\,.$$

Theorem 1.6. *Let $I \subset (1,\infty)$ be a compact interval and fix $p \in I$. There exists $\eta \in (0, n-2)$ depending on n and I such that, for any nonnegative minimizer u of $\mathcal{E}_p(\cdot)$, we have*

$$\mathcal{H}^{n-2-\eta}(\mathrm{Sing}(u)) = 0.$$

For $n = 2$, this should be interpreted as $\mathrm{Sing}(u) = \varnothing$.

We refer the reader to Subsection 7.4 for a discussion of open problems and further settings where we believe the method may be applicable.

With ε-regularity (see Lemma 6.1 and Corollary 6.2), Theorem 1.3 follows from the following curvature estimates, which are of independent interest. Theorem 1.6 is proven through an analogous scheme and rests upon analogous curvature estimates; see Proposition 7.1.

Theorem 1.7. *Let u be a nonnegative minimizer of $\mathcal{E}^\Phi(\cdot)$ in B_2 with $\Phi \in I_{\Lambda,\alpha,\Upsilon}$, then there exist universal constants $\eta > 0$ and $C < \infty$ such that if $0 \in \Gamma(u)$, then*

$$\int_{B_1 \cap \{u>0\}} |D^2 u|^2 \le C, \tag{1.8}$$

and

$$\int_{B_1 \cap \{u>0\} \cap \{|\nabla u| \ge 2r_\Lambda\}} |D^2 u|^{2+\eta} \le C. \tag{1.9}$$

Here r_Λ is the lower radius introduced in (1.5), which is explicitly defined in Definition 4.6.

To explain the strategies behind these curvature estimates, we work in the setting when the *free boundary is smooth* and Φ is smooth.

Under this assumption, the $W^{2,2}$-estimate in (1.8) can be established for *stable solutions* to the Euler-Lagrange equation associated to (1.1). For this, we prove a Bochner-type inequality (Lemma 5.2), showing that a function of the gradient is a subsolution with right-hand side involving $|D^2 u|^2$. Then, the basic idea to prove (1.8) is to multiply this function by a cutoff and use this as a test function for the stability inequality (see Lemma 2.2 and Proposition 5.1). Note that this stability inequality holds in the classical sense only when the free boundary is regular. For the classical Alt-Caffarelli problem (when $\Phi(p) = |p|^2$), a similar estimate has been established by [10] and [43] when the free boundary is smooth.

To improve (1.8) to (1.9), we follow a strategy from [74]. The key is to rule out the concentration of $|D^2 u|$ in its large level sets as in Lemma 5.8. Such a non-concentration estimate requires a Hölder estimate of $D^2 u$ in the positive phase $\{u > 0\}$, which is false since our integrand Φ is not C^2 near the origin. To deal with this difficulty, we show that there is always a significant portion of $\{u > 0\}$ that is quantitatively separated from the critical set $\{\nabla u = 0\}$; see Lemma 4.5. As a result, it suffices to establish estimates in the set where $\nabla u(x)$ lies in the annulus $\mathcal{A}(\Lambda)$ from (1.5) where the integrand is assumed to be $C^{2,\alpha}$. See (1.9) above, as well as Proposition 5.5, Lemma 6.1, Corollary 6.2.

With this, we have the desired curvature estimates in Theorem 1.7 when the free boundary is smooth. Similar estimates can be established when $\mathcal{H}^{n-2}(\mathrm{Sing}(u)) = 0$, see Proposition 5.1. Unfortunately, we cannot afford this assumption without running a circular argument, as this is part of our main result in Theorem 1.3.

To overcome this difficulty, we adapt a topological argument by Schoen-Simon-Almgren [76] and De Philippis-Maggi [17]. We show that the collection of integrands satisfying the desired estimates in Theorem 1.3 is non-empty, open and closed in $I_{\Lambda,\alpha,\Upsilon}$ from (1.6) and conclude by connectedness of $I_{\Lambda,\alpha,\Upsilon}$.

More concretely, we let

$$I^*_{\Lambda,\alpha,\Upsilon} = \{\Phi \in I_{\Lambda,\alpha,\Upsilon} \ : \ \text{any minimizer } u \text{ of } \mathcal{E}^\Phi(\cdot) \text{ has } \mathcal{H}^{n-2}(\mathrm{Sing}(u)) = 0\} \tag{1.10}$$

denote the collection of "good integrands" such that Theorem 1.3 holds with $\eta = 0$.

The non-emptiness and openness of $I^*_{\Lambda,\alpha,\Upsilon}$ follow from known results (Theorem 2.4, Theorem 4.12, and Corollary 4.14). To show the closedness of this collection is much more involved. We need to approximate a minimizer of a potentially bad integrand with minimizers of energies with good integrands, using a selection principle. For this, we need to perturb the functional in (1.1) by adding a lower order term (see (2.4)). For minimizers of this perturbed functional, we establish a uniform curvature estimate as in (1.8) of Theorem 1.7. This bound passes to the limit, yielding an analogous curvature estimate for the original minimizer of the energy with potentially bad integrand. The estimate holds at all points and scales, hence a covering and scaling argument (Corollary 6.2) leads to the desired bound $\mathcal{H}^{n-2}(\mathrm{Sing}(u)) = 0$.

Before we conclude this introduction, we note that the estimate in (1.8) bears resemblance to Sternberg-Zumbrun estimates for stable solutions to the Allen-Cahn equation [79] and curvature estimates for (anisotropic) stable minimal surfaces of Schoen-Simon-Yau, Schoen-Simon, and Allard [77, 75, 2]. They are the key tools to prove compactness of stable minimal hypersurfaces, which is foundational in the regularity theory in min-max constructions [73, 15, 22, 21, 24]. In the context of anisotropic minimal surfaces, White [88] gives an argument, which he attributes to Almgren, to obtain an η-improvement of the dimension of the singular set. Our proof strategy is independent and gives a more quantified bound on the improvement.

The paper is organized as follows. In Section 2, we introduce some necessary background. Section 3 is dedicated to a rigidity result for Weiss-type monotonicity formulas. In Section 4, we give some regularity and convergence results for minimizers of $\mathcal{E}^\Phi$ and of the penalized functionals. Section 5 is dedicated to proving the curvature estimates (1.8) and (1.9) for minimizers when the integrand Φ is smooth and belongs to the class of good integrands $\Phi \in I^*_{\Lambda,\alpha,\Upsilon}$ defined in (1.10) above. In Section 6, we carry out the topological argument described above to prove Theorems 1.3 and 1.7. Finally, Section 7 extends this analysis to the degenerate elliptic and singular settings to establish Theorem 1.6.

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AI Usage Statement. Claude and Chat GPT were used to aid in the computation of variations in Appendix A, to help choose the constants in the barrier construction of Lemma C.1 to ensure some finicky set containments, and to obtain the Kato-type inequality with right-hand side for $p < 2$ in Proposition 7.1. AI did not make any serious contributions to the ideas or writing of this paper.

2. Preliminaries

Throughout the paper, $n \geq 2$, $\Lambda \geq 1$, $\alpha \in (0,1)$, and $\Upsilon > 0$ are fixed. We refer to these as universal parameters and call a constant universal if it depends only on these parameters.

2.1. **Facts about integrands.** Recall the class of integrands $I_{\Lambda,\alpha,\Upsilon}$ defined in (1.6) and suppose $\Phi \in I_{\Lambda,\alpha,\Upsilon}$. Observe that $D\Phi(0) = 0$ thanks to the assumptions that Φ is nonnegative, $C^{1,1}$, and vanishes at the origin. The Hessian bounds (1.4) and the fundamental theorem of calculus yield the upper and lower bounds

$$\frac{1}{2\Lambda}|p|^2 \leq \Phi(p) \leq \frac{\Lambda}{2}|p|^2, \qquad \frac{1}{\Lambda}|p| \leq |D\Phi(p)| \leq \Lambda|p|, \qquad \frac{|p|^2}{2\Lambda} \leq D\Phi(p)\cdot p - \Phi(p) \leq \frac{\Lambda|p|^2}{2}. \tag{2.1}$$

One consequence of the third estimate in (2.1) is that $D\Phi(p)\cdot p \geq 0$ for all $p \in \mathbb{R}^n$. A second is that for any $p \in \mathbb{R}^n$,

$$\mathcal{G}(p) := D\Phi(p)\cdot p - \Phi(p) = 1 \qquad \Longrightarrow \qquad \sqrt{\frac{2}{\Lambda}} \leq |p| \leq \sqrt{2\Lambda}. \tag{2.2}$$

Similarly, the uniform convexity (1.4) of Φ implies that $t \mapsto \mathcal{G}(tp)$ is strictly increasing for all $p \neq 0$. Hence, for any direction $e \in S^{n-1}$, there is a unique $a > 0$ such that the function $v_a(x) := a(x\cdot e)_+$ satisfies the free boundary condition $\mathcal{G}(\nabla v_a) = \mathcal{G}(ae) = 1$ on $\partial\{v_a > 0\}$; c.f. (2.3) below.

Next, for a sequence $\{\Phi_k\} \subset I_{\Lambda,\alpha,\Upsilon}$, we write

$$\Phi_k \to \Phi$$

to mean $\Phi_k \to \Phi$ locally uniformly on $\mathbb{R}^n$. From the definition of $I_{\Lambda,\alpha,\Upsilon}$ and the considerations above, we see that if $\Phi_k \in I_{\Lambda,\alpha,\Upsilon}$ and $\Phi_k \to \Phi$, then the common $C^{1,1}$ bound implies $\|\Phi_k - \Phi\|_{C^1(\overline{B_M})} \to 0$ for every $M < \infty$. The

common Hölder bound on the Hessians (1.5) on $\mathcal{A}(\Lambda)$ also gives $\|D^2\Phi_k - D^2\Phi\|_{C^0(\mathcal{A}(\Lambda))} \to 0$. These assertions follow by Arzelà-Ascoli and identification of limits by distributional differentiation.

Remark 2.1. By the same reasoning, we see that $I_{\Lambda,\alpha,\Upsilon}$ is compact when endowed with the topology induced by this convergence. Moreover, note that $I_{\Lambda,\alpha,\Upsilon}$ is convex and in particular connected. Finally, a basic mollification and vertical translation argument shows that smooth integrands are dense in $I_{\Lambda,\alpha,\Upsilon}$ in this topology.

2.2. **The main functional and the penalized functional.** Consider the main energy functional $\mathcal{E}^\Phi(v;\Omega)$ defined in (1.1), with integrand $\Phi \in I_{\Lambda,\alpha,\Upsilon}$, where $I_{\Lambda,\alpha,\Upsilon}$ is the class of integrands defined in (1.6). When the domain Ω is understood or plays no important role, we write $\mathcal{E}^\Phi(u)$ in place of $\mathcal{E}^\Phi(u;\Omega)$.

We say a nonnegative function $u \in W^{1,2}(\Omega)$ is a minimizer of $\mathcal{E}^\Phi(\cdot)$ in Ω if $\mathcal{E}^\Phi(u;\Omega) \le \mathcal{E}^\Phi(u+\varphi;\Omega)$ for every $\varphi \in W^{1,2}_0(\Omega)$. Every time we refer to a minimizer of $\mathcal{E}^\Phi$ in Ω, we tacitly assume $u \ge 0$ and $u \in W^{1,2}(\Omega)$. Formally, a minimizer u of $\mathcal{E}^\Phi(\cdot,\Omega)$ satisfies the quasilinear Bernoulli free boundary problem

$$\begin{cases} \operatorname{div}(D\Phi(\nabla u)) = 0 & \text{in } \{u>0\}\cap\Omega, \\ D\Phi(\nabla u)\cdot\nabla u - \Phi(\nabla u) = 1 & \text{on } \Gamma(u) := \partial\{u>0\}\cap\Omega. \end{cases} \tag{2.3}$$

The proof of Theorem 1.3 calls for a functional $\mathcal{E}^\Phi_{f,\mu}$ that is a perturbation of $\mathcal{E}^\Phi$. Indeed, for a sequence Φ_k in $I_{\Lambda,\alpha,\Upsilon}$ with $\Phi_k \to \Phi$ and a minimizer $\bar u$ of $\mathcal{E}^\Phi$, we will approximate $\bar u$ by functions u_k that minimize an energy functional that is the sum of $\mathcal{E}^{\Phi_k}$ and a fidelity term that forces minimizers u_k to be close to $\bar u$.

More precisely, given a nonnegative function $f \in C^\infty(\Omega)$ (which will be taken to be a mollification of $\bar u$ in the application described above) and a number $\mu \in [0,1]$, consider the energy functional

$$\mathcal{E}^\Phi_{f,\mu}(v,\Omega) = \mathcal{E}^\Phi(v,\Omega) + \int_\Omega |\mu v - f|^2. \tag{2.4}$$

The parameter μ makes class of functionals $\mathcal{E}^\Phi_{f,\mu}$ invariant under the blow-up rescaling: if u is minimizer of $\mathcal{E}^\Phi_{f,\mu}(\cdot,\Omega)$, $x_0 \in \Omega$, and $0 < s \le 1$, then the Lipschitz rescaling $\tilde u(x) := u(x_0 + sx)/s$ is a minimizer of $\mathcal{E}^\Phi_{\tilde f,\tilde\mu}(\cdot,\Omega_s)$, where

$$\Omega_s := (\Omega - x_0)/s, \qquad \tilde\mu := s\mu \in [0,1], \qquad \tilde f := f(x_0 + s\,\cdot), \qquad \|\tilde f\|_{L^\infty(\Omega_s)} \le \|f\|_{L^\infty(\Omega)}. \tag{2.5}$$

Indeed, since $\nabla\tilde u(x) = \nabla u(x_0 + sx)$, all three terms of $\mathcal{E}^\Phi_{f,\mu}$ scale by the factor s^{-n} under this transformation. Thus estimates for minimizers of $\mathcal{E}^\Phi_{f,\mu}$ that are uniform in $\mu \in [0,1]$ can be applied to minimizers of $\mathcal{E}^\Phi_{f,1}$ at all scales $s \in (0,1]$.

Lemma 2.2. *With f, μ, and $\mathcal{E}^\Phi_{f,\mu}$ as above, let u be a minimizer of $\mathcal{E}^\Phi_{f,\mu}(\cdot)$ in B_1 and assume $\Gamma(u) = \partial\{u>0\}\cap B_1$ is $C^{2,\alpha}$. Then u satisfies the Euler-Lagrange equation*

$$\begin{cases} \operatorname{div}\left(D\Phi(\nabla u)\right) = 2\mu(\mu u - f) & \text{in } \{u>0\}\cap B_1, \\ D\Phi(\nabla u)\cdot\nabla u - \Phi(\nabla u) = 1 & \text{on } \Gamma(u)\,. \end{cases} \tag{2.6}$$

Provided $\Phi \in C^2(\mathbb{R}^n)$, such a minimizer satisfies the following stability inequality:

$$\int_{\{u>0\}\cap B_1} D^2\Phi(\nabla u)[\nabla\psi,\nabla\psi] + 2\mu^2\psi^2\,dx \ge \int_{\Gamma(u)} \left\langle D^2\Phi(\nabla u)D^2u[\nu],\nu\right\rangle \frac{(D\Phi(\nabla u)\cdot\nu)}{1+\Phi(\nabla u)}\psi^2\,\mathcal{H}^{n-1} \tag{2.7}$$

for all $\psi \in C^1_c(B_1)$.

Here ν denotes the outer unit normal to $\{u>0\}$. The proofs of (2.6) and (2.7) are given in appendix, in Lemmas A.1 and A.2 respectively. The interior equation in (2.6) holds in the weak sense, and holds classically on $\{x : \nabla u(x) \in \mathcal{A}(\Lambda)\}$ with $\mathcal{A}(\Lambda)$ as in (1.5). The free boundary condition in (2.6) holds classically under this assumed regularity of the free boundary.

In the important special case when $\Phi \in I_{\Lambda,\alpha,\Upsilon}$ is two homogeneous, the free boundary condition in (2.6) simplifies to $\Phi(\nabla u) = 1$. The stability inequality (2.7) takes a form that is analogous to that for the classical Alt-Caffarelli energy, involving the mean curvature of the free boundary; see Remark A.4 and in particular (A.24).

In general, the free boundary $\Gamma(u)$ of a minimizer u of $\mathcal{E}^\Phi_{f,\mu}(\cdot)$ in B_1 need not be everywhere $C^{2,\alpha}$ as assumed above. We show in Proposition 4.11 and Corollary 4.13 below that if, for a point $x_0 \in \Gamma(u)$, there is a scale $r > 0$ such that $\Gamma(u)$ is flat at x_0 at scale r in the sense of (4.4) and the right-hand side $g = 2\mu(\mu u - f)$ is small at scale r in the sense of (4.5), then $\Gamma(u)$ is a $C^{2,\alpha}$ graph in a neighborhood of x_0.

Definition 2.3. A point $x_0 \in \Gamma(u)$ is regular if, after a rotation, $\{u > 0\}$ is locally the epigraph of a $C^{1,\alpha}$ function near x_0. We denote the set of regular points by $\mathrm{Reg}(u)$ and define

$$\mathrm{Sing}(u) = \Gamma(u) \setminus \mathrm{Reg}(u). \tag{2.8}$$

For $\Phi \in I_{\Lambda,\alpha,\Upsilon}$ and smooth data, we improve the regularity of such a graph to $C^{2,\alpha}$, see Corollary 4.13 below.

2.3. **Viscosity solutions.** As above, let $\Phi \in I_{\Lambda,\alpha,\Upsilon}$ where $I_{\Lambda,\alpha,\Upsilon}$ is the class of integrands defined in (1.6). In this section we use the shorthand

$$\mathcal{L}w = \mathrm{div}(D\Phi(\nabla w)) \qquad \text{and} \qquad \mathcal{G}(\nabla u) = D\Phi(\nabla u) \cdot \nabla u - \Phi(\nabla u). \tag{2.9}$$

For $g \in C(B_1) \cap L^\infty(B_1)$, let us define viscosity solutions to the free boundary problem

$$\begin{cases} \mathcal{L}u = g & \text{in } \{u > 0\} \cap \Omega, \\ \mathcal{G}(\nabla u) = 1 & \text{on } \Gamma(u)\,. \end{cases} \tag{2.10}$$

For a nonnegative function $u \in C^0(\Omega)$, we say

$$u \text{ satisfies } \mathcal{G}(\nabla u) = 1 \text{ on } \Gamma(u) \text{ in the viscosity sense} \tag{2.11}$$

if the following holds for any $x_0 \in \Gamma(u)$ and $\phi \in C^\infty(\Omega)$: if ϕ touches u from below (resp., ϕ_+ touches u from above) in Ω at x_0, then $\mathcal{G}(\nabla\phi(x_0)) \le 1$ (resp., $\mathcal{G}(\nabla\phi(x_0)) \ge 1$.).

A subtlety arises in defining viscosity solutions to $\mathcal{L}u = g$ in $\{u > 0\}$ since $D^2\Phi(p)$ only exists a.e. outside of $\mathcal{A}(\Lambda)$, thus $\mathcal{L}\phi = \mathrm{div}(D\Phi(\nabla\phi(x)))$ may not be defined for a smooth function ϕ at a point x where $\nabla\phi(x)$ is in the set where Φ is not twice differentiable. To deal with this, we recall the Pucci extremal operators defined for a symmetric matrix M with eigenvalues $\{\mu_i\}$ by

$$\mathcal{M}^+(M,\lambda,\Lambda) = \Lambda \sum_{\mu_i>0} \mu_i + \lambda \sum_{\mu_i<0} \mu_i, \qquad \mathcal{M}^-(M,\lambda,\Lambda) = \lambda \sum_{\mu_i>0} \mu_i + \Lambda \sum_{\mu_i<0} \mu_i\,.$$

Recall that a function $w \in C^0(\Omega)$ satisfies $\mathcal{M}^-(D^2w(x_0),\lambda,\Lambda) \le g(x_0) \le \mathcal{M}^+(D^2w(x_0),\lambda,\Lambda)$ in the viscosity sense at $x_0 \in \Omega$ if we have $\mathcal{M}^-(D^2\phi,\lambda,\Lambda) \le g(x_0)$ for any function $\phi \in C^\infty(\Omega)$ that touches w from below at x_0, and $\mathcal{M}^+(D^2\phi,\lambda,\Lambda) \ge g(x_0)$ for any function $\phi \in C^\infty(\Omega)$ that touches w from above at x_0.

So, for a nonnegative function $u \in C^0(\Omega)$, we say

$$u \text{ satisfies } \mathcal{L}u = g \text{ in } \{u > 0\} \text{ in the viscosity sense} \tag{2.12}$$

if the following holds for any $x_0 \in \{u > 0\} \cap \Omega$ and $\phi \in C^\infty(\Omega)$:
– if ϕ touches u from below (resp., above) in Ω at x_0 and $\nabla\phi(x_0) \in \mathcal{A}(\Lambda)$, then $\mathcal{L}\phi(x_0) \le g(x_0)$ (resp., $\mathcal{L}\phi(x_0) \ge g(x_0)$).
– if ϕ touches u from below (resp., above) in Ω at x_0 and $\nabla\phi(x_0) \notin \mathcal{A}(\Lambda)$, then $\mathcal{M}^-(D^2\phi, 1/\Lambda, \Lambda) \le g(x_0)$ (resp., $\mathcal{M}^+(D^2\phi, 1/\Lambda, \Lambda) \ge g(x_0)$).

Finally, we say u is a viscosity solution to (2.10) in Ω if $u \in C^0(\Omega)$ is nonnegative and satisfies (2.11) and (2.12).

Note that if u satisfies (2.12) then $\mathcal{M}^-(D^2u, 1/\Lambda, \Lambda) \le g \le \mathcal{M}^+(D^2u, 1/\Lambda, \Lambda)$ in the viscosity sense in $\{u > 0\}$.

In our application, we will consider minimizers u of $\mathcal{E}^\Phi_{f,\mu}$ where f is smooth and bounded; we show u is bounded (Lemma 4.1) and a viscosity solution to (2.6) (Proposition 4.11), thus we treat $g = 2\mu(\mu u - f)$ as a bounded right-hand side and consider viscosity solutions to the problem (2.10).

2.4. **The classical Alt-Caffarelli problem.** When $\Phi(v) = |v|^2$, in their foundational work, Alt and Caffarelli [3] established existence, local Lipschitz continuity and nondegeneracy of minimizers, together with density estimates for the two phases and local finiteness of the $(n-1)$-dimensional measure of the free boundary. Their improvement-of-flatness theorem gives $\mathcal{H}^{n-1}(\mathrm{Sing}(u)) = 0$ and that $\mathrm{Reg}(u)$ is relatively open and locally a $C^{1,\alpha}$ hypersurface. Higher regularity theory then implies that $\mathrm{Reg}(u)$ is analytic; see Kinderlehrer-Nirenberg [56]. A different proof of improvement of flatness was developed by De Silva [25], combining a Harnack inequality with compactness and linearization around a half-plane solution.

The principal tool for improving the bound on $\mathrm{Sing}(u)$ is the monotonicity formula of Weiss [87]. For $x_0 \in \Gamma(u)$ and $B_r(x_0) \Subset \Omega$, set

$$W(u,x_0,r) := r^{-n} \int_{B_r(x_0)} \left(|\nabla u|^2 + \chi_{\{u>0\}}\right) dx - r^{-n-1} \int_{\partial B_r(x_0)} u^2 \, d\mathcal{H}^{n-1}. \tag{2.13}$$

For almost every r,

$$\frac{d}{dr}W(u,x_0,r) = 2r^{-n}\int_{\partial B_r(x_0)}\left(\partial_\nu u - \frac{u}{r}\right)^2 d\mathcal{H}^{n-1} \geq 0, \tag{2.14}$$

where ν is the outer unit normal to $\partial B_r(x_0)$. Consequently, every subsequential blow-up is a nonzero, one-homogeneous global minimizer. Moreover, a half-plane blow-up implies regularity of x_0 by improvement of flatness. Singular points are therefore detected by nonflat minimizing cones, and a Federer dimension-reduction argument converts low-dimensional cone classification into a Hausdorff dimension estimate.

Caffarelli-Jerison-Kenig [8] ruled out nonflat stable cones in dimension three, and Jerison-Savin [52] extended this conclusion to dimension four. Very recently OpenAI extended this though dimension six [70]. This implies free boundaries are analytic in dimensions $n \leq 6$, and the universal bound in higher dimensions is $\dim_{\mathcal{H}} \mathrm{Sing}(u) \leq n-7$. In [27], De Silva-Jerison constructed a singular minimizing cone in $\mathbb{R}^7$; see also [47, 46] for further minimizing cones in $\mathbb{R}^7$. All together, these works amount to the following:

Theorem 2.4. *Let u be a minimizer of the classical Alt-Caffarelli energy $\int_\Omega |\nabla u|^2 + \chi_{\{u>0\}}$. Then the free boundary $\Gamma(u)$ is analytic outside a singular set $\mathrm{Sing}(u)$ with*

$$\dim_{\mathcal{H}}(\mathrm{Sing}(u)) \leq n-7. \tag{2.15}$$

Fernández-Real-Yu [42, 43] establish improved bounds for the singular set for generic boundary data. Finer properties of the singular set are provided in the form of rectifiability and Minkowski estimates for the strata of the singular set (Edelen-Engelstein [33]); uniqueness of blow-ups at isolated singularities (Engelstein-Spolaor-Velichkov [37]), and Hardt-Simon type foliations for homogeneous global minimizers (De Silva-Jerison-Shahgholian [28] and Edelen-Spolaor-Velichkov [34]; see related recent work of Engelstein-Restrepo-Zhao [36]).

2.5. **PDE Estimates.** In the course of proving the main theorems, we will make use of the following interior and boundary regularity estimates for solutions to the interior equation in (2.6).

Lemma 2.5. *Let $\Phi \in I_{\Lambda,\alpha,\Upsilon}$. The following estimates hold.*

(1) (Interior estimates) Let $\psi \in L^\infty(B_r(x_0))$ and let $w \in H^1(B_r(x_0)) \cap L^\infty(B_r(x_0))$ be a weak solution to

$$div\big(D\Phi(\nabla w)\big) = \psi \qquad \text{in } B_r(x_0).$$

There are $\beta = \beta(n,\Lambda) \in (0,1)$ and $C = C(n,\Lambda)$ such that $w \in C^{1,\beta}_{\mathrm{loc}}(B_r(x_0)) \cap H^2_{\mathrm{loc}}(B_r(x_0))$ and

$$r\|\nabla w\|_{L^\infty(B_{r/2}(x_0))} + r^{1+\beta}[\nabla w]_{C^{0,\beta}(B_{r/2}(x_0))} \leq C\left(\|w\|_{L^\infty(B_r(x_0))} + r^2\|\psi\|_{L^\infty(B_r(x_0))}\right). \tag{2.16}$$

Moreover, if $\psi = 0$ and $\nabla w(x) \in \mathcal{A}(\Lambda)$ for all $x \in B_r(x_0)$, then $w \in C^{2,\alpha}_{\mathrm{loc}}\big(B_r(x_0)\big)$ with

$$r^2\|D^2 w\|_{L^\infty(B_{r/2}(x_0))} + r^{2+\alpha}[D^2 w]_{C^{0,\alpha}(B_{r/2}(x_0))} \leq C. \tag{2.17}$$

for a universal constant $C > 0$.

(2) (Boundary estimates) Let Ω be an open bounded set such that $\Gamma := \partial\Omega \cap B_r(x_0)$ is of class $C^{2,\alpha}$. Let $w \in H^1(\Omega \cap B_r(x_0)) \cap C^0(\overline{\Omega} \cap B_r(x_0))$ be a weak solution to

$$div\big(D\Phi(\nabla w)\big) = 0 \quad \text{in } \Omega \cap B_r(x_0), \qquad\qquad w = 0 \text{ on } \Gamma.$$

Assume that, after rescaling by r^{-1}, the boundary admits $C^{2,\alpha}$ graph charts of fixed radius, with their $C^{2,\alpha}$ norms bounded by M. There are $\beta = \beta(n,\Lambda) \in (0,1)$ and $C = C(n,\Lambda,M)$ such that $w \in C^{1,\beta}_{loc}(\overline{\Omega} \cap B_r(x_0)) \cap H^2_{loc}(\overline{\Omega} \cap B_r(x_0))$ with

$$r\|\nabla w\|_{L^\infty(\overline{\Omega}\cap B_{r/2}(x_0))} + r^{1+\beta}[\nabla w]_{C^{0,\beta}(\overline{\Omega}\cap B_{r/2}(x_0))} \leq C\|w\|_{L^\infty(\overline{\Omega}\cap B_r(x_0))}. \tag{2.18}$$

Moreover, if $\nabla w(x) \in \mathcal{A}(\Lambda)$ for all $x \in \Omega \cap B_r(x_0)$, then $w \in C^{2,\alpha}_{\mathrm{loc}}(\overline{\Omega} \cap B_r(x_0))$ and

$$r^2\|D^2 w\|_{L^\infty(\overline{\Omega}\cap B_{r/2}(x_0))} + r^{2+\alpha}[D^2 w]_{C^{0,\alpha}(\overline{\Omega}\cap B_{r/2}(x_0))} \leq C. \tag{2.19}$$

Note that the constants for the gradient estimates (2.16) and (2.18) depend only on n and Λ, not on α or Υ. These estimates hold by applying De Giorgi to the equation satisfied by $\partial_i u$; see [49, Ch. 8], first for a regularization $\Phi^\varepsilon \in I_{\Lambda,\alpha,\Upsilon}$ of Φ (recall Remark 2.1) and then passing the estimates to the limit. The Hessian estimates (2.17) and (2.19) follow from Schauder estimates; note that assumption $\nabla w \in \mathcal{A}(\Lambda)$ in particular implies $\|\nabla w\|_{L^\infty} \leq R_\Lambda$.

3. Rigidity of Weiss-type monotonicity formulas

As described in Section 2.4, the Weiss monotonicity formula (2.13) plays a central role in singularity analysis for the classical Alt-Caffarelli problem. In this section, we show that the specific structure of the integrand $p \mapsto |p|^2$ is quite special, and the space of integrands admitting a Weiss-type monotonicity formula is rigid. This is an analogue of the rigidity result in [1, Section 3] in the context of anisotropic minimal surfaces, where the lack of the monotonicity formula for the density ratio makes the anisotropic minimal surface theory sensibly harder [18, 19].

Let $\rho : \mathbb{R}^n \to [0,\infty)$ be positive on $\mathbb{R}^n \setminus \{0\}$, C^2 away from the origin, and positively one-homogeneous. Set

$$B_r^\rho(x_0) := \{x : \rho(x - x_0) < r\}, \qquad \nu_\rho(z) := \frac{D\rho(z)}{|D\rho(z)|}.$$

Thus ν_ρ is the outer unit normal to $\partial B_1^\rho(0)$ and $z \cdot \nu_\rho(z) > 0$. Let $h \in C^1(\partial B_1^\rho(0) \times [0,\infty))$, with $h(z,0) = 0$. The normalization at zero only removes an additive constant from the quantity below. For a nonnegative locally Lipschitz minimizer u of $\mathcal{E}^\Phi$ in Ω, define

$$W_{\rho,h}(u,x_0,r) := \frac{\mathcal{E}^\Phi(u; B_r^\rho(x_0))}{r^n} - \int_{\partial B_1^\rho(0)} h\left(z, \frac{u(x_0+rz)}{r}\right) \, \mathrm{d}\mathcal{H}^{n-1}(z), \tag{3.1}$$

whenever $B_r^\rho(x_0) \Subset \Omega$. Introduce the algebraic defect

$$\sigma_{\Phi,\rho,h}(p,z,s) := (p\cdot z - s)\big(D\Phi(p)\cdot \nu_\rho(z) - \partial_s h(z,s)\big), \qquad p \in \mathbb{R}^n, \quad z \in \partial B_1^\rho(0), \quad s \geq 0. \tag{3.2}$$

Theorem 3.1. *Let $n \geq 2$, $\Phi \in I_{\Lambda,\alpha,\Upsilon}$, and let ρ, h be as above. For every minimizer u as above, $W_{\rho,h}(u,x_0,\cdot)$ is locally absolutely continuous and, for almost every admissible r,*

$$\frac{d}{dr} W_{\rho,h}(u,x_0,r) = \frac{1}{r}\int_{\partial B_1^\rho(0)} \sigma_{\Phi,\rho,h}\left(\nabla u(x_0+rz), z, \frac{u(x_0+rz)}{r}\right) \, \mathrm{d}\mathcal{H}^{n-1}(z). \tag{3.3}$$

Moreover, the following are equivalent:

(i) *The pointwise sign condition*

$$\sigma_{\Phi,\rho,h}(p,z,s) \geq 0 \qquad \text{for every } p \in \mathbb{R}^n,\ z \in \partial B_1^\rho(0),\ s \geq 0 \tag{3.4}$$

holds.

(ii) *There exist an invertible linear map $L : \mathbb{R}^n \to \mathbb{R}^n$ and $c > 0$ such that*

$$\rho(x) = |L^{-1}x|, \qquad \Phi(p) = c|L^T p|^2, \qquad h(L\theta, s) = \frac{cs^2}{|L^{-T}\theta|} \tag{3.5}$$

for all $x, p \in \mathbb{R}^n$, $\theta \in S^{n-1}$, and $s \geq 0$.

In particular, for Euclidean balls the sign condition holds if and only if $\Phi(p) = c|p|^2$ and $h(\theta,s) = cs^2$. For a genuinely nonquadratic integrand, no choice of ρ and h in the above class gives an everywhere nonnegative defect in (3.3), even when Φ is two-homogeneous.

Remark 3.2. For $\rho(x) = |x|$ and $h(\theta,s) = cs^2$, the quantity (3.1) above is

$$W_c(u,x_0,r) = \frac{\mathcal{E}^\Phi(u; B_r(x_0))}{r^n} - \frac{c}{r^{n+1}}\int_{\partial B_r(x_0)} u^2 \, \mathrm{d}\mathcal{H}^{n-1},$$

and (3.3) becomes

$$\frac{d}{dr} W_c(u,x_0,r) = \frac{2c}{r^n}\int_{\partial B_r(x_0)} \left(\partial_\nu u - \frac{u}{r}\right)^2 \, \mathrm{d}\mathcal{H}^{n-1} + \frac{1}{r^n}\int_{\partial B_r(x_0)} \left(\partial_\nu u - \frac{u}{r}\right)\big(D\Phi(\nabla u) - 2c\nabla u\big)\cdot \nu \, \mathrm{d}\mathcal{H}^{n-1}.$$

The second term is the obstruction to the usual square-completion argument that yields monotonicity in the classical Alt-Caffarelli setting [87]. In the quadratic case in (3.5), the change of variables $v(y) := u(x_0 + Ly)$ gives instead

$$W_{\rho,h}(u,x_0,r) = |\det L|\left[\frac{1}{r^n}\int_{B_r}\big(c|\nabla v|^2 + \chi_{\{v>0\}}\big)\,dy - \frac{c}{r^{n+1}}\int_{\partial B_r} v^2 \, \mathrm{d}\mathcal{H}^{n-1}\right],$$

so that the classical Weiss formula is recovered on the corresponding ellipsoids.

The pointwise condition (3.4) is stronger than monotonicity of $W_{\rho,h}$ along minimizers. Our argument does not show that (3.4) is necessary for the integral in (3.3) to be nonnegative for every minimizer. In particular, the theorem does not rule out cancellations in the boundary integral.

Remark 3.3. It is important to remark that Theorem 3.1 does not exclude corrections involving gradients, nonlocal quantities, or auxiliary functions. The correction used for a class of radial integrands in [54] lies outside the class considered here.

Proof of Theorem 3.1. Step 1: The radial first-variation identity. By translation, assume $x_0 = 0$. Inner variations and the positivity-preserving variations $u_\varepsilon = (1+\varepsilon\psi)u$ give, respectively,

$$\int_\Omega \left((\Phi(\nabla u) + \chi_{\{u>0\}}) \operatorname{div} X - D\Phi(\nabla u) \cdot DX^T \nabla u \right) \, dx = 0, \tag{3.6}$$

$$\int_\Omega D\Phi(\nabla u) \cdot (\psi \nabla u + u \nabla \psi) \, dx = 0, \tag{3.7}$$

for $X \in C_c^1(\Omega;\mathbb{R}^n)$ and $\psi \in C_c^1(\Omega)$; see Appendix A. These identities only require $\Phi \in C^{1,1}$ and do not require regularity of $\Gamma(u)$.

Use radial cutoffs $X(x) = \eta(\rho(x))x$ and $\psi(x) = \eta(\rho(x))$, with η constant near zero, and let η approximate the characteristic function of $[0,r)$. For almost every r, the coarea formula yields

$$n\mathcal{E}^\Phi(u;B_r^\rho) - \int_{B_r^\rho} D\Phi(\nabla u)\cdot\nabla u \, dx = \int_{\partial B_r^\rho} \left((\Phi(\nabla u) + \chi_{\{u>0\}})\, x\cdot\nu - (x\cdot\nabla u) D\Phi(\nabla u)\cdot\nu \right) \, \mathrm{d}\mathcal{H}^{n-1}, \tag{3.8}$$

$$\int_{B_r^\rho} D\Phi(\nabla u)\cdot\nabla u \, dx = \int_{\partial B_r^\rho} u D\Phi(\nabla u)\cdot\nu \, \mathrm{d}\mathcal{H}^{n-1}, \tag{3.9}$$

where ν is the outer unit normal to ∂B_r^ρ. Euler's identity for ρ gives $x\cdot\nu = r/|D\rho(x)|$ on ∂B_r^ρ, and hence

$$r\frac{d}{dr}\big(\mathcal{E}^\Phi(u;B_r^\rho)\big) = \int_{\partial B_r^\rho} (\Phi(\nabla u) + \chi_{\{u>0\}})\, x\cdot\nu \, \mathrm{d}\mathcal{H}^{n-1}.$$

Combining these identities, we obtain

$$\frac{d}{dr}\big(r^{-n}\mathcal{E}^\Phi(u;B_r^\rho)\big) = \frac{1}{r^{n+1}} \int_{\partial B_r^\rho} (x\cdot\nabla u - u) D\Phi(\nabla u)\cdot\nu \, \mathrm{d}\mathcal{H}^{n-1}. \tag{3.10}$$

For almost every (r,z),

$$\partial_r\left(\frac{u(rz)}{r}\right) = \frac{z\cdot\nabla u(rz) - \frac{u(rz)}{r}}{r}.$$

Changing variables $x = rz$ in (3.10) and differentiating the boundary term in (3.1) proves (3.3). Local absolute continuity follows from the coarea formula and the local Lipschitz regularity of u. Notice that no homogeneity assumption on Φ was used.

Step 2: The sign condition and Euclidean rigidity. Assume (3.4). Fix $z \in \partial B_1^\rho(0)$ and $p \in \mathbb{R}^n$ with $p\cdot z > 0$. For every $t \in \mathbb{R}$ with $|t|$ small enough, we can test (3.4) at $s = p\cdot z + t$ and we get

$$-t\big(D\Phi(p)\cdot\nu_\rho(z) - \partial_s h(z, p\cdot z + t)\big) \geq 0.$$

Letting $t \downarrow 0^+$ and $t \uparrow 0^-$ yields

$$D\Phi(p)\cdot\nu_\rho(z) = \partial_s h(z, p\cdot z) \qquad \text{whenever } p\cdot z > 0. \tag{3.11}$$

First suppose $\rho(x) = |x|$, so that $z = \nu_\rho(z) = \theta$. For $q\cdot\theta = 0$ and $s > 0$, the preceding identity reads

$$D\Phi(q + s\theta)\cdot\theta = \partial_s h(\theta, s). \tag{3.12}$$

Taking $q = 0$ and letting $s \downarrow 0$, we obtain $\partial_s h(\theta,0) = 0$, since $D\Phi(0) = 0$. Letting $s \downarrow 0$ for arbitrary $q \perp \theta$ then gives

$$D\Phi(q)\cdot\theta = 0 \qquad \text{whenever } q\cdot\theta = 0.$$

Thus $D\Phi(q)$ is parallel to q for every $q \neq 0$. Since $n \geq 2$ and spheres are connected, there is a function F such that $\Phi(p) = F(|p|)$.

Given $0 < s \leq R$, choose $q \perp \theta$ with $|q|^2 = R^2 - s^2$. Equation (3.12) becomes

$$\partial_s h(\theta, s) = s\frac{F'(R)}{R}.$$

For any $R_1, R_2 > 0$, choosing $0 < s < \min\{R_1, R_2\}$ shows that $F'(R_1)/R_1 = F'(R_2)/R_2$. Consequently $F'(R) = 2cR$ for a constant $c > 0$. Since $\Phi(0) = 0$ and $h(\theta,0) = 0$, it follows that

$$\Phi(p) = c|p|^2, \qquad h(\theta,s) = cs^2.$$

This Euclidean argument uses only C^1 regularity, $\Phi(0) = D\Phi(0) = 0$, and strict convexity of Φ.

Step 3: Rigidity of the domains. Return to an arbitrary ρ. Write $Q(p) := D^2\Phi(p)$ for p in $\mathcal{A}(\Lambda)$, where Φ is C^2. Choose $p_0 \neq 0$ in this annulus. For $p_0 \cdot z > 0$, differentiate (3.11) with respect to p along any $v \perp z$. The right-hand side stays constant, so $Q(p_0)\nu_\rho(z) \cdot v = 0$. It follows that

$$Q(p_0)\nu_\rho(z) \in (0,\infty)z \qquad \text{if } p_0 \cdot z > 0. \tag{3.13}$$

The multiple is positive because $Q(p_0)$ is positive definite and $z \cdot \nu_\rho(z) > 0$.

Choose q_0 in $\mathcal{A}(\Lambda)$ which is not collinear with p_0. On the nonempty relatively open set

$$\{z \in \partial B_1^\rho(0) : p_0 \cdot z > 0,\ q_0 \cdot z > 0\},$$

(3.13) implies that $Q(q_0)Q(p_0)^{-1}z$ is parallel to z. The positive rays through this set form an open subset of $\mathbb{R}^n$. A linear map for which an open set of vectors consists of eigenvectors is a scalar multiple of the identity. Thus $Q(q_0)$ is a positive scalar multiple of $Q(p_0)$. The same argument applied to q_0 and $-p_0$ shows that $Q(-p_0)$ is also a positive scalar multiple of $Q(p_0)$. Therefore, with $Q := Q(p_0)$, equation (3.13) and continuity imply

$$\nu_\rho(z) \in (0,\infty)Q^{-1}z \qquad \text{for every } z \in \partial B_1^\rho(0).$$

The tangential derivative of $z \mapsto z \cdot Q^{-1}z$ consequently vanishes on $\partial B_1^\rho(0)$. This hypersurface is connected, since it is a radial graph over S^{n-1}. Hence

$$z \cdot Q^{-1}z = R_0^2 \qquad \text{on } \partial B_1^\rho(0)$$

for some $R_0 > 0$. Since every positive ray meets $\partial B_1^\rho(0)$ exactly once, setting $L := R_0 Q^{1/2}$ gives

$$\partial B_1^\rho(0) = LS^{n-1}, \qquad \rho(x) = |L^{-1}x|.$$

Step 4: Reduction to the Euclidean case and the converse. For $\theta \in S^{n-1}$, set

$$\widetilde{\Phi}(q) := \Phi(L^{-T}q), \qquad \widetilde{h}(\theta, s) := |L^{-T}\theta| h(L\theta, s).$$

Since

$$\nu_\rho(L\theta) = \frac{L^{-T}\theta}{|L^{-T}\theta|},$$

a direct computation yields

$$(q \cdot \theta - s)\big(D\widetilde{\Phi}(q) \cdot \theta - \partial_s \widetilde{h}(\theta, s)\big) = |L^{-T}\theta|\, \sigma_{\Phi,\rho,h}(L^{-T}q, L\theta, s) \geq 0.$$

Applying the Euclidean argument from Step 2 gives $\widetilde{\Phi}(q) = c|q|^2$ and $\widetilde{h}(\theta, s) = cs^2$. These are precisely (3.5).

Conversely, if (3.5) holds, then

$$\sigma_{\Phi,\rho,h}(p, L\theta, s) = \frac{2c}{|L^{-T}\theta|}(p \cdot L\theta - s)^2 \geq 0.$$

This proves the equivalence and all the assertions. □

4. Regularity and compactness for minimizers of the penalized functional

Minimizers of the penalized functional $\mathcal{E}^\Phi_{f,\mu}$ introduced in (2.4) enjoy many of the regularity properties of minimizers of the classical Alt-Caffarelli functionals, and sequences of minimizers have good compactness properties.

4.1. Geometric properties of minimizers.

We start with the classical Lipschitz and non-degeneracy estimates and their consequences. Throughout this subsection, we let $\Phi \in C^{1,1}(\mathbb{R}^n)$ have $\Phi \geq 0$ in $\mathbb{R}^n$ and $\Phi(0) = 0$ and satisfy the ellipticity bound (1.4), without assuming any further regularity of Φ on an annulus. The estimates of this subsection depend only on Λ and n, not α or Υ, and will determine the annulus $\mathcal{A}(\Lambda)$ in Definition 4.6 below.

These results were established by Petrosyan and Valdinoci [71] for the non-perturbed functional $\mathcal{E}^\Phi$. In [71] the authors consider integrands Φ with more general p-Laplace type degeneracy, and hence assume Φ is C^2 away from the origin so they can apply interior gradient estimates of [84]. Instead, when $\Phi \in I_{\Lambda,\alpha,\Upsilon}$, and in particular satisfies the uniform ellipticity (1.4), estimate (2.16) of Lemma 2.5 takes the place of [84]. For minimizers of the perturbed functional $\mathcal{E}^\Phi_{f,\mu}$ rather than $\mathcal{E}^\Phi$, the proofs from [71] go through with minor modification and thus we do not repeat them here.

We first recall the Lipschitz bound, compare [71, Theorem 4.1].

Lemma 4.1. *Let u minimize $\mathcal{E}^{\Phi}_{f,\mu}$ in B_1, with $0 \le \mu \le 1$, $f \ge 0$, and $\|f\|_{L^\infty(B_1)} \le F$. For $0 < \rho < \rho' < 1$, if $\|u\|_{L^\infty(B_{\rho'})} \le K$, then*

$$\|\nabla u\|_{L^\infty(B_\rho)} \le C(n, \Lambda, F, \rho, \rho')(1 + K).$$

If $\{u = 0\} \cap B_{1/4} \neq \varnothing$, then

$$\|u\|_{L^\infty(B_\rho)} + \|\nabla u\|_{L^\infty(B_\rho)} \le C(n, \Lambda, F, \rho).$$

Next, we recall nondegeneracy of minimizers of $\mathcal{E}^{\Phi}_{f,\mu}$; compare [71, Lemma 5.2].

Lemma 4.2. *Let u be a minimizer of $\mathcal{E}^{\Phi}_{f,\mu}$ in Ω and let $x_0 \in \{u > 0\}$. For $0 < r < \min\{1, \operatorname{dist}(x_0, \partial\Omega)\}$, denote by U_r the connected component of $\{u > 0\} \cap B_r(x_0)$ containing x_0. Then*

$$\sup_{U_r \cap B_{r/2}(x_0)} u \ge c\big(n, \Lambda, \|f\|_{L^\infty(\Omega)}\big)\, r.$$

In particular, for every connected component U of $\{u > 0\} \cap \Omega$ and every $x_0 \in U$, we have $\sup_{U \cap B_{r/2}(x_0)} u \ge c\, r$ for such r.

By combining the Lipschitz and nondegeneracy estimates, we obtain volume density estimates for the positivity set of a minimizer u, compare [71, Theorem 2.2].

Corollary 4.3. *Let u be a minimizer of $\mathcal{E}^{\Phi}_{f,\mu}$ in Ω. Then for any $x_0 \in \Gamma(u)$ and $0 < r < \frac{1}{2}\min\{1, \operatorname{dist}(x_0, \partial\Omega)\}$, we have*

$$c\big(n, \Lambda, \|f\|_{L^\infty(\Omega)}\big) \le \frac{|\{u > 0\} \cap B_r(x_0)|}{|B_r(x_0)|} \le 1 - c\big(n, \Lambda, \|f\|_{L^\infty(\Omega)}\big).$$

Moreover, for any compact subset $K \Subset \Omega$, we have $\mathcal{H}^{n-1}(\Gamma(u) \cap K) < \infty$.

Remark 4.4. Suppose u is a minimizer of $\mathcal{E}^{\Phi}$ in B_1 such that $\{u > 0\} \cap B_1$ has k connected components $V_1, \dots, V_k$, and on each connected component,

$$u(x) = (x \cdot \xi_i + c_i)_+ \qquad \text{for } x \in V_i,$$

for some $c_i \ge 0$ and $\xi_i \in \mathbb{R}^n$. Suppose $c_1 = 0$ so $0 \in \Gamma(u)$. By Corollary 4.3 and disjointness of the connected components, there exists $\rho_0 = \rho_0(n, \Lambda) > 0$ such that V_1 is the only connected component intersecting B_{ρ_0}. In particular,

$$u = (x \cdot \xi_1)_+ \qquad \text{in } B_{\rho_0}.$$

The first variation (2.6) then guarantees that $D\Phi(\xi_1) \cdot \xi_1 - \Phi(\xi_1) = 1$.

The following lemma is the key tool that allows us to focus on regions where $|\nabla u|$ has a positive lower bound, since this always captures a definite ball in the positivity set.

Lemma 4.5. *Suppose that u minimizes $\mathcal{E}^{\Phi}$ in B_1 and $0 \in \Gamma(u)$. There is $\bar\delta = \bar\delta(n, \Lambda) > 0$ such that every connected component V of $B_1 \cap \{u > 0\}$ which intersects $B_{1/8}$ contains a ball*

$$B_{\bar\delta}(y_V) \subset V \cap B_{1/2} \cap \{|\nabla u| \ge 8\bar\delta\}.$$

We may require $\bar\delta < \min\{1/64, \sqrt{2/\Lambda}/16\}$.

Proof. Let $L = L(n, \Lambda)$ bound the Lipschitz constant of u on $B_{3/4}$. Choose $x \in V \cap B_{1/8}$. Componentwise nondegeneracy, applied in $B_{1/8}(x)$, gives $q \in V \cap B_{1/4}$ with $u(q) \ge c_0$, where $c_0 > 0$ depends only on n, Λ. Let z be a closest zero of u to q, and put $d = |q - z|$. Since $u(0) = 0$,

$$c_0/L \le d \le 1/4, \qquad B_d(q) \subset V \cap B_{1/2}.$$

Set $t_0 = c_0/(4L)$ and let $y = z + t_0(q - z)/d$. Then $u(y) \le c_0/4$. The mean value theorem on $[y, q]$ provides $a \in (y, q)$ with $|\nabla u(a)| \ge 3c_0$. Moreover $B_{t_0}(a) \subset B_d(q)$. The interior $C^{1,\beta}$ estimate (2.16) of Lemma 2.5 therefore gives a radius $\rho = \rho(n, \Lambda) > 0$, with $\rho < t_0/2$, such that $|\nabla u| \ge 2c_0$ on $B_\rho(a)$. Choose $\bar\delta \le \min\{\rho, c_0/4, 1/64, \sqrt{2/\Lambda}/16\}$, decreasing it slightly if necessary. The result follows. □

With these lemmas in hand, we can now specify the radii $0 < r_\Lambda < R_\Lambda$ that define the annulus $\mathcal{A}(\Lambda)$ in (1.5).

Definition 4.6. Let $\bar\delta$ be fixed as in Lemma 4.5. Choose $L_* = L_*(n, \Lambda) \ge 1$ so that every minimizer of $\mathcal{E}^{\Phi}$ in B_2 with $0 \in \Gamma(u)$ satisfies $\|\nabla u\|_{L^\infty(B_{15/8})} \le L_*$, and set

$$r_\Lambda = \bar\delta/2, \qquad R_\Lambda = 4\max\{L_*, \sqrt{2\Lambda}, 1\}, \qquad \mathcal{A}(\Lambda) = B_{R_\Lambda} \setminus \overline{B_{r_\Lambda}}.$$

Remark 4.7. Thanks to (2.2), any $p \in \mathbb{R}^n$ satisfying the identity $D\Phi(p) \cdot p - \Phi(p) = 1$ corresponding to the free boundary condition (2.6) lies in a compact subannulus of $\mathcal{A}(\Lambda)$. Moreover, for any minimizer of $\mathcal{E}^\Phi$ in B_2 with $0 \in \Gamma$, for each point x in each ball $B_{\bar\delta}(y_V)$ obtained in Lemma 4.5, $\nabla u(x)$ lies in a compact subset of $\mathcal{A}(\Lambda)$.

4.2. **Compactness properties of minimizers.** The Lipschitz and nondegeneracy estimates of the previous subsection imply various compactness properties for minimizers of our penalized functionals. All statements are formulated on balls, which is all that is needed in the sequel, but could of course be shown on general domains. The results of this section also hold for the more general class of integrands considered in Section 4.1, but now that the annulus $\mathcal{A}(\Lambda)$ has been specified in Definition 4.6, for the remainder of the paper we restrict our attention to $\Phi \in I_{\Lambda,\alpha,\Upsilon}$ for simplicity.

Lemma 4.8. *Let $\{\Phi_k\} \subset I_{\Lambda,\alpha,\Upsilon}$, $\{\mu_k\} \subset [0,1]$, and let $\{f_k\} \subset L^\infty(B_1)$ be nonnegative with $\sup_k \|f_k\|_{L^\infty(B_1)} \le F < \infty$. For each k, let u_k be a minimizer of $\mathcal{E}^{\Phi_k}_{f_k,\mu_k}(\cdot, B_1)$ with*

$$\{u_k = 0\} \cap B_{1/4} \neq \emptyset.$$

Then, for every $\rho \in (0,1)$,

$$\sup_k \Big(\sup_{B_\rho} u_k + \mathrm{Lip}(u_k; B_\rho) \Big) \le C(n, \Lambda, F, \rho), \tag{4.1}$$

and there is a nonnegative $u \in C^{0,1}_{\mathrm{loc}}(B_1)$ such that, up to a subsequence (unrelabeled),

$$u_k \to u \ \text{ locally uniformly in } B_1, \qquad u_k \rightharpoonup u \ \text{ weakly in } H^1(B_\rho) \ \text{ for every } \rho \in (0,1). \tag{4.2}$$

Proof. The bound (4.1) is Lemma 4.1. By the Arzelà-Ascoli theorem and a diagonal argument over an exhaustion $B_{\rho_m} \uparrow B_1$, a subsequence converges locally uniformly to a nonnegative u, which is locally Lipschitz with the same constants. For each $\rho < 1$, the sequence u_k is bounded in $H^1(B_\rho)$ by (4.1), so, after a further diagonal extraction, $u_k \rightharpoonup u$ weakly in $H^1(B_\rho)$ for every $\rho < 1$. □

Lemma 4.9. *Suppose $\Phi_k, \Phi \in I_{\Lambda,\alpha,\Upsilon}$ and $\Phi_k \to \Phi$ locally uniformly in $\mathbb{R}^n$. Suppose $\{f_k\} \subset C^0(\overline{B_1})$ are nonnegative with $f_k \to f$ uniformly in B_1, and $\{\mu_k\} \subset [0,1]$ with $\mu_k \to \mu$. For each k, let u_k be a nonnegative minimizer of $\mathcal{E}^{\Phi_k}_{f_k,\mu_k}$ in B_1, and suppose that $u_k \to u$ locally uniformly and weakly in $H^1_{\mathrm{loc}}(B_1)$ (by Lemma 4.8, this holds along a subsequence whenever $\{u_k = 0\} \cap B_{1/4} \neq \emptyset$). Then:*

1. *$u_k \to u$ strongly in $H^1_{\mathrm{loc}}(B_1)$;*
2. *u is a local minimizer of $\mathcal{E}^\Phi_{f,\mu}$ in B_1. If, in addition, $u \in H^1(B_1)$, then u is a minimizer with respect to all perturbations in $H^1_0(B_1)$;*
3. *$\{u_k > 0\} \to \{u > 0\}$ locally in the Hausdorff distance, and $\chi_{\{u_k>0\}} \to \chi_{\{u>0\}}$ in $L^1_{\mathrm{loc}}(B_1)$;*
4. *if $\{x_k\} \Subset B_1$ is a sequence with $x_k \in \partial\{u_k > 0\}$ for each k and $x_k \to x$, then $x \in \partial\{u > 0\}$. Conversely, every point of $\Gamma(u)$ is a limit of points of Γ_{u_k}. In particular, $\Gamma_{u_k} \to \Gamma(u)$ locally in the Hausdorff distance;*
5. *for any $r \in (0,1]$ and $p \in (1,\infty)$, $\liminf_k \int_{\{u_k>0\}\cap B_r} |D^2 u_k|^p \ge \int_{\{u>0\}\cap B_r} |D^2 u|^p$.*

The proof of Lemma 4.9 is postponed to Appendix B.

Remark 4.10. By item (2), all the results of the previous sections apply to the limit u: in particular, u satisfies the Lipschitz estimate, the nondegeneracy of Lemma 4.2, and the density estimates of Corollary 4.3; the latter imply $|\partial\{u > 0\} \cap B_1| = 0$.

4.3. **Minimizers are viscosity solutions.** Next we show that any minimizer of $\mathcal{E}^\Phi_{f,\mu}$ is a viscosity solution to the associated Euler-Lagrange equation. The following proof is inspired by an argument from anisotropic minimal surfaces; see [50, 16, 13, 14, 15, 23]. Throughout this section, $\Phi \in I_{\Lambda,\alpha,\Upsilon}$ is fixed, $f \in C^\infty(\Omega) \cap L^\infty(\Omega)$ is nonnegative, and $\mu \in [0,1]$. We again use the short-hand $\mathcal{L}w = \mathrm{div}(D\Phi(\nabla w))$ and $\mathcal{G}(p) = D\Phi(p) \cdot p - \Phi(p)$ from (2.9).

Proposition 4.11. *If u is a minimizer of $\mathcal{E}^\Phi_{f,\mu}$ in Ω, then u is a viscosity solution to (2.10) with $g = 2\mu(\mu u - f)$.*

Proof of Proposition 4.11. Step 1: We begin by showing u satisfies (2.12). Suppose $x_0 \in \{u > 0\} \cap \Omega$ and $\phi \in C^\infty(\mathbb{R}^n)$ touches u from below in $\{u > 0\}$ at x_0. Recall that $u \in C^{1,\alpha}(\{u > 0\}) \cap C^{2,\alpha}(\{x : \nabla u(x) \in \mathcal{A}(\Lambda)\})$.

Case 1: First suppose that $\nabla u(x_0) \in \mathcal{A}(\Lambda)$, and hence u is C^2 in a neighborhood of x_0 and $D^2u(x_0) - D^2\phi(x_0)$ is nonnegative definite. Since $\nabla\phi(x_0) = \nabla u(x_0) := p_0$, we see that $\mathcal{L}u(x_0) - \mathcal{L}\phi(x_0) = D^2\Phi(p_0) : (D^2u(x_0) - D^2\phi(x_0)) \ge 0$ by ellipticity. Since $\mathcal{L}u(x_0) = g(x_0)$, it follows that $\mathcal{L}\phi(x_0) \le g(x_0)$.

Case 2: Next suppose that $\nabla u(x_0) \notin \mathcal{A}(\Lambda)$. We assume by way of contradiction that $\mathcal{M}^-(D^2\phi(x_0), 1/\Lambda, \Lambda) > g(x_0)$. Since ϕ is smooth and g is continuous by Lemma 4.1, we may choose $r_0 > 0$ so that $\mathcal{M}^-(D^2\phi(x), 1/\Lambda, \Lambda) \geq g(x) + r_0$ for all x in $B_{r_0}(x_0)$. This also implies that $\mathcal{L}\phi(x) \geq g(x) + r_0$ weakly in $B_{r_0}(x_0)$. Indeed, letting Φ_ε be a scale-ε mollification of Φ (which satisfies the same ellipticity bounds), for any nonnegative test function $\psi \in C_c^1(B_{r_0}(x_0))$ we have

$$-\int D\Phi_\varepsilon(\nabla\phi)\cdot\nabla\psi = \int (D^2\Phi_\varepsilon(\nabla\phi) : D^2\phi)\,\psi \geq \int \mathcal{M}^-(D^2\phi, 1/\Lambda, \Lambda)\,\psi \geq \int (g + r_0)\psi.$$

Passing $\varepsilon \to 0$ shows that $\mathcal{L}\phi(x) \geq g(x) + r_0$ weakly in $B_{r_0}(x_0)$. By taking a small vertical translation of ϕ, which leaves the equation unchanged, this contradicts the comparison principle for weak solutions. Thus $\mathcal{M}^-(D^2\phi(x_0), 1/\Lambda, \Lambda) \leq g(x_0)$.

Pairing this with an analogous argument when $\phi \in C^\infty(\mathbb{R}^n)$ touches u from above in $\{u > 0\}$ at x_0 shows that u satisfies $\mathcal{L}u = g$ in $\{u > 0\}$ in the viscosity sense.

Step 2: Next we show that u satisfies (2.11). The sub and super solution arguments are analogous, so we just consider the case when $x_0 \in \Gamma(u)$ and ϕ touches u from below. Suppose by way of contradiction that $\mathcal{G}(\nabla\phi(x_0)) > 1$. Write $\nabla\phi(x_0) = a_* e$ with $a_* > 0$ and $e \in S^{n-1}$. Recall from Section 2.1 that $t \mapsto \mathcal{G}(te)$ is strictly increasing, so there is a unique $b > 0$ such that $\mathcal{G}(be) = 1$ and $b < a_*$.

Now, consider a blowup sequence $u_k = u(x_0 + r_k x)/r_k$ with $r_k \to 0$. For each k, u_k minimizes $\mathcal{E}^\Phi_{f_k,\mu_k}$ in $B_{r_k^{-1}}$, where μ_k, f_k are determined by the rescaling (2.5). So, by Lemmas 4.8 and 4.9, up to a subsequence, u_k converges to a function $u_\infty : \mathbb{R}^n \to \mathbb{R}$ with $0 \in \Gamma(u_\infty)$ such that

$$u_\infty(x) \geq a_*(x\cdot e)_+ \qquad \text{for all } x \in \mathbb{R}^n, \tag{4.3}$$

and u_∞ is an entire minimizer of $\mathcal{E}^\Phi$. Indeed, by Lemma 4.9, for each $R > 0$, u_∞ minimizes $\mathcal{E}^\Phi(\cdot; B_R) + |f(x_0)|^2|B_R|$ and thus minimizes $\mathcal{E}^\Phi(\cdot; B_R)$.

Let $\mathcal{B}(u_\infty)$ denote the collection of all blowups of u_∞ at 0. The argument above shows that every $v \in \mathcal{B}(u_\infty)$ is again an entire minimizer of $\mathcal{E}^\Phi$ with $0 \in \Gamma(v)$ that satisfies (4.3) with v in place of u_∞. In particular, $\{y\cdot e > 0\} \subset \{v > 0\}$ for every $v \in \mathcal{B}(u_\infty)$. A diagonal argument shows that $\mathcal{B}(u_\infty)$ is closed, and hence Lemma 4.8 shows $\mathcal{B}(u_\infty)$ is sequentially compact, with respect to the convergence of the lemma.

Claim: There exist $\bar v \in \mathcal{B}(u_\infty)$ and $\bar a > 0$ such that $\bar v(x) = \bar a(x\cdot e)_+$ for all $x \in \{y\cdot e > 0\}$.

Before proving the claim, let us see how it will yield a contradiction. On one hand, (4.3) guarantees that $\bar a \geq a_* > b$. On the other hand, the blowup $\bar v$ produced by the claim is an entire minimizer of $\mathcal{E}^\Phi$. Since the restriction of a minimizer to one component of its positivity set is again a minimizer, this means $\bar a(x\cdot e)_+$ is also a minimizer of $\mathcal{E}^\Phi$. But as noted above, the free boundary condition cannot be satisfied unless $\bar a = b$, a contradiction.

Let us now prove the claim. For each $v \in \mathcal{B}(u_\infty)$, let

$$a(v) = \sup\{a : a(x\cdot e)_+ \leq v(x) \text{ for all } x \in \mathbb{R}^n\}.$$

By (4.3), $a(v) \geq a_*$, while the Lipschitz bound of Lemma 4.1 guarantees that $a(v) \leq L$ for $L = L(n, \Lambda)$. Moreover, one easily checks that $v \mapsto a(v)$ is upper semicontinuous with respect to local uniform convergence on $\mathbb{R}^n$. So, $\bar a := \sup\{a(v) : v \in \mathcal{B}(u_\infty)\}$ is achieved by some $\bar v \in \mathcal{B}(u_\infty)$. It is for this $\bar v$ that we establish the claim.

Suppose by way of contradiction that $\bar v(x) - \bar a(x\cdot e)_+$ is not identically zero in $\{x\cdot e > 0\}$, so there exists some $\bar x \in \{x\cdot e > 0\}$ such that $\bar v(\bar x) > \bar a(\bar x\cdot e)_+$. Let $\bar r = |\bar x|$ and let $B^+ = \{x : |x| < \bar r, x\cdot e > 0\}$. Now, we can find a function $w \in C^{1,1}(B^+) \cap \mathrm{Lip}(\partial B^+)$ such that $w = 0$ on $\partial B^+ \cap \{x\cdot e = 0\}$ and $\bar a(x\cdot e)_+ \leq w(x) \leq \bar v(x)$ on $\partial B^+ \cap \{x\cdot e > 0\}$, with strict inequality at $\bar x \in \partial B^+ \cap \{x\cdot e > 0\}$.

Now, let $\tilde w$ solve $\mathrm{div}(D\Phi(\nabla\tilde w)) = 0$ in B^+ with $\tilde w = w$ on ∂B^+. Since $B^+ \subset \{\bar v > 0\}$, $\bar v$ solves the same equation on B^+ and by the maximum principle, $\tilde w \leq \bar v$ in B^+. Moreover, $\tilde w \in C^{1,\beta}(\bar B^+_{\bar r/2})$ by estimate (2.18) of Lemma 2.5, and in particular $\nabla\tilde w(0)$ exists. Note that $\nabla\tilde w(0) = \tilde a e$ for some $\tilde a > 0$ thanks to our choice of boundary data. By strong maximum principle, $\tilde w > \bar a(x\cdot e)_+$ in B^+ and then by Hopf lemma, $\tilde a > \bar a$. But then since $\hat v \geq \tilde w$ in B^+, this means that we can find a blowup $\hat v$ of $\bar v$ (which is also a blowup of u_∞) with $a(\hat v) \geq \tilde a > \bar a$. This contradicts the definition of $\bar a$ and proves the claim. □

4.4. **Epsilon regularity for viscosity solutions.** Next, viscosity solutions to (2.10), which we recall were defined in Section 2, enjoy an ε-regularity theorem. Throughout this section, $\Phi \in I_{\Lambda,\alpha,\Upsilon}$, and we continue to use the shorthand $\mathcal{L}w = \mathrm{div}(D\Phi(\nabla w))$ and $\mathcal{G}(p) = D\Phi(p)\cdot p - \Phi(p)$ as in (2.9). We consider viscosity solutions u to (2.10) in

B_r with $0 \in \Gamma_u$ that are flat in the sense that

$$(x \cdot e - \bar{\varepsilon} r)_+ \leq u(x) \leq (x \cdot e + \bar{\varepsilon} r)_+ \qquad \text{in } B_r \tag{4.4}$$

for some $e \in \mathbb{R}^n$ with $\mathcal{G}(e) = 1$ and such that

$$r \|g\|_{L^\infty} \leq \bar{\varepsilon}. \tag{4.5}$$

For such solutions, we have the following.

Theorem 4.12 (ε-Regularity)**.** *There are universal constants $\bar{\varepsilon} > 0$ and $\delta > 0$ such that the following holds. Fix $r > 0$, $g \in C(B_r) \cap L^\infty(B_r)$, and $e \in \mathbb{R}^n$ with $\mathcal{G}(e) = 1$.*

Suppose u is a viscosity solution to (2.10) in B_r with $0 \in \Gamma_u$ satisfying (4.4) and (4.5). Then $\Gamma(u)$ is $C^{1,\beta}$ for some universal $\beta \in (0, 1/2)$ at scale δr near the origin.

More precisely, let p, q be the projections onto $\{x \cdot e = 0\} \simeq \mathbb{R}^{n-1}$ and $\{x \cdot e = \pm |x||e|\} \simeq \mathbb{R}$ respectively and let $B'_r = p(B_r)$. The conclusion of Theorem 4.12 says there is a function $h : B'_{\delta r} \to [-\delta r, \delta r]$ such that

$$\{u > 0\} \cap C_e(\delta r) = \{x = (px, qx) \ : \ px \in B'_{\delta r},\ qx > h(px)\}$$

in the cylinder $C_e(\delta r) := \{(px, qx) : |px| < \delta r,\ |qx| < \delta r\}$ and $\|h(r\cdot)/r\|_{C^{1,\beta}(B'_\delta)} \leq C$ for a universal constant $C > 0$.

Theorem 4.12 is proven using De Silva's method [25]. This flexible approach has been used to prove ε-regularity theorems for viscosity solutions of one- and two-phase problems with various operators and right-hand sides. The equation (2.10) does not precisely fit into the existing frameworks since Φ is only $C^{1,1}$ outside of $\mathcal{A}(\Lambda)$, thus $D^2\Phi$ is not defined everywhere. However, Φ is $C^{2,\alpha}$ in the annulus $\mathcal{A}(\Lambda)$, which includes in particular a neighborhood of $\{p \in \mathbb{R}^n : \mathcal{G}(p) = 1\}$ as noted in Remark 4.7. Thus, the proof goes through without major modifications from, e.g. [25, 26, 9].

We sketch the proof in Appendix C, emphasizing the places where the proof differs from the standard one.

For minimizers u of $\mathcal{E}^\Phi_{f,\mu}$, the assumed $C^{2,\alpha}$ regularity of Φ on $\mathcal{A}(\Lambda)$ means that flat free boundaries are $C^{2,\alpha}$.

Corollary 4.13. *With the same assumptions and notation of Theorem 4.12, if $g \in C^{0,\alpha}(B_r)$, then the function h is $C^{2,\alpha}$ in $B'_{\delta r/2}$. In particular, this holds for minimizers of $\mathcal{E}^\Phi_{f,\mu}$ with $g = 2\mu(\mu u - f)$ for $\mu \in [0, 1]$ and $f \in C^\infty(B_r) \cap L^\infty(B_r)$ nonnegative.*

Proof of Corollary 4.13. By Theorem 4.12, the free boundary condition $\mathcal{G}(\nabla u) = 1$ is achieved classically in $B_{\delta r}(0)$. In particular, by Remark 4.7, $\{\nabla u(x) : x \in \Gamma(u) \cap B_{\delta r}(0)\}$ is contained in a compact subset of $\mathcal{A}(\Lambda)$. The proof then follows from a classical argument using the hodograph transform [56]. To see the "in particular," if u is a minimizer of $\mathcal{E}^\Phi_{f,\mu}$, then u is a viscosity solution to (2.10) with $g = 2\mu(\mu u - f)$ by Proposition 4.11. Since f is assumed to be (qualitatively) smooth and u is Lipchitz thanks to Lemma 4.1, this choice of g is Hölder continuous for any $\beta \in (0, 1)$.

For the unperturbed problem, the argument above is quantitative. More precisely, there are universal constants $c, C > 0$ such that $\|h(r\,\cdot)/r\|_{C^{2,\alpha}(B'_c)} \leq C$. Indeed, after normalizing $r = 1$, Theorem 4.12 gives a uniformly controlled $C^{1,\beta}$ graph. The boundary gradient estimates and the Bernoulli condition give a uniformly nondegenerate hodograph map on a fixed smaller neighborhood. The Schauder estimates in the hodograph argument are therefore uniform, since the ellipticity constants and the annular $C^{2,\alpha}$ bounds for Φ are uniform. □

A direct consequence of Theorem 4.12 is the following propagation of singularities statement; we omit its standard proof. In this statement, $\mathcal{H}^s_\infty$ is the Hausdorff content.

Corollary 4.14. *Suppose $\Phi_k, \Phi \in I_{\Lambda,\alpha,\Upsilon}$ with $\Phi_k \to \Phi$, $f_k, f \in C^\infty(B_1) \cap L^\infty(B_1)$ nonnegative with $f_k \to f$ uniformly, and take $\mu_k, \mu \in [0, 1]$ with $\mu_k \to \mu$. Let u_k, u be minimizers of $\mathcal{E}^{\Phi_k}_{f_k,\mu_k}$ and $\mathcal{E}^\Phi_{f,\mu}$ respectively in B_1. If $u_k \to u$ in the sense of Lemma 4.9 above and $x_k \in \mathrm{Sing}(u_k) \cap B_1$ with $x_k \to x \in B_1$, then $x \in \mathrm{Sing}(u) \cap B_1$. Moreover, for any compact set $K \Subset B_1$ and $s \geq 0$, if $\mathcal{H}^s_\infty(\mathrm{Sing}(u_k) \cap K) \geq c$, then $\mathcal{H}^s_\infty(\mathrm{Sing}(u) \cap K) \geq c$.*

5. L^q Hessian estimates in the space of good integrands

As described in the introduction, the key ingredients in the proof of Theorem 1.3 are a priori L^q estimates for the Hessian akin to those in Theorem 1.7. In this section, we establish these curvature estimates for minimizers of $\mathcal{E}^\Phi$ (and its perturbations) when Φ belongs to the restricted class of integrands $I^*_{\Lambda,\alpha,\Upsilon} \cap C^\infty(\mathbb{R}^n)$. Later on, in Section 6 we will extend these estimates to minimizers of $\mathcal{E}^\Phi$ for any $\Phi \in I_{\Lambda,\alpha,\Upsilon}$, establishing Theorem 1.7.

5.1. L^2 **Hessian Estimates.** We begin by proving an L^2 estimate for the Hessian; compare to (1.8). Note that this estimate holds for minimizers of the penalized functional $\mathcal{E}^{\Phi}_{f,\mu}$.

Proposition 5.1. *Fix $\Phi \in I^*_{\Lambda,\alpha,\Upsilon} \cap C^\infty(\mathbb{R}^n)$, a nonnegative function $f \in C^\infty(\Omega) \cap L^\infty(\Omega)$, and $\mu \in [0,1]$. There is $C > 0$ depending on the universal parameters and on an upper bound for $\|f\|_{L^\infty(\Omega)}$ such that if u is a minimizer of $\mathcal{E}^{\Phi}_{f,\mu}$ in Ω, then for all $p \in \Gamma(u)$ and $r \in (0,1]$ with $B_{2r}(p) \subset \Omega$, we have*

$$\int_{\{u>0\}\cap B_r(p)} |D^2 u|^2 \leq C r^{n-2}. \tag{5.1}$$

To prove Proposition 5.1, we need two preparatory lemmas. The first is a Bochner-type inequality for the function $G = \Phi(\nabla u) + 1$. Similar computations have been carried out in various settings; see e.g. [86, Theorem 4].

Lemma 5.2. *Let $\Phi \in I_{\Lambda,\alpha,\Upsilon} \cap C^\infty(\mathbb{R}^n)$. Let $\Omega \subset \mathbb{R}^n$ be an open bounded set, let $f \in C^\infty(\Omega) \cap L^\infty(\Omega)$ and let $\mu \in [0,1]$. Suppose $u \in W^{1,2}(\Omega)$ is a continuous nonnegative function satisfying*

$$\begin{cases} \operatorname{div}(D\Phi(\nabla u)) = 2\mu(\mu u - f) & \text{in } \{u>0\}\cap\Omega \\ D\Phi(\nabla u)\cdot\nabla u - \Phi(\nabla u) = 1 & \text{on } \operatorname{Reg}(u)\,. \end{cases} \tag{5.2}$$

Then the function $G = 1 + \Phi(\nabla u)$ satisfies

$$\begin{cases} \operatorname{div}(D^2\Phi(\nabla u)\nabla G) \geq \Lambda^{-2}|D^2 u|^2 - 2\mu\nabla\Phi(\nabla u)\cdot\nabla f & \text{on } \{u>0\}\cap\Omega \\ D^2\Phi(\nabla u)[\nabla G,\nu] = (D\Phi(\nabla u)\cdot\nu)\,(D^2\Phi(\nabla u)D^2 u[\nu]\cdot\nu) & \text{on } \operatorname{Reg}(u)\,. \end{cases} \tag{5.3}$$

Here ν is the outer unit normal to $\{u>0\}\cap\Omega$ at x.

Proof. We let subscripts denote partial derivatives. Thanks to the smoothness assumptions on Φ and f, all derivatives are classical. Keeing in mind that $\partial_i G = D\Phi(\nabla u)\cdot\nabla u_i$, we directly compute in $\{u>0\}\cap\Omega$

$$\begin{aligned} \operatorname{div}(D^2\Phi(\nabla u)\nabla G) &= \sum_k \partial_k\Big(\sum_j \Phi_{kj}(\nabla u)G_j\Big) = \sum_k \partial_k\Big(\sum_{j,m}\Phi_{k,j}(\nabla u)\Phi_m(\nabla u)u_{mj}\Big) \\ &= \sum_{k,j,m}\Phi_m(\nabla u)\partial_k\big(\Phi_{kj}(\nabla u)u_{mj}\big) + \sum_{k,j,m}\Phi_{kj}(\nabla u)u_{mj}\partial_k\big(\Phi_m(\nabla u)\big) =: I + II. \end{aligned}$$

For term I, we use the interior equation (5.2) satisfied by u. Differentiating (5.2) in the m-direction, we find that $\operatorname{div}(D^2\Phi(\nabla u)\nabla u_m) = 2\mu(\mu\, u_m - f_m)$ in $\{u>0\}\cap\Omega$. Recalling from Section 2.1 that $D\Phi(p)\cdot p \geq 0$, we thus find

$$I = \sum_m \Phi_m(\nabla u)\operatorname{div}(D^2\Phi(\nabla u)\nabla u_m) = 2\mu(\mu\, D\Phi(\nabla u)\cdot\nabla u - D\Phi(\nabla u)\cdot\nabla f) \geq -2\mu D\Phi(\nabla u)\cdot\nabla f. \tag{5.4}$$

Next, for term II, we note that $\partial_k\Phi_m(\nabla u) = \sum_l \Phi_{ml}(\nabla u)u_{lk}$ in U. Thus,

$$II = \sum_{kjml}\Phi_{kj}(\nabla u)u_{mj}\Phi_{ml}(\nabla u)u_{lk} = |D^2\Phi(\nabla u)^{1/2}D^2 u D^2\Phi(\nabla u)^{1/2}|^2 \geq \Lambda^{-2}|D^2 u|^2.$$

Combining this with (5.4) gives the first part of (5.3).

Now, for $x \in \operatorname{Reg}(u)$, write $D\Phi(\nabla u) = a\nu + b\tau$ where $a = D\Phi(\nabla u)\cdot\nu$ and $|\tau| = 1$ with $\tau\cdot\nu = 0$. Then, differentiating the free boundary condition $D\Phi(\nabla u)\cdot\nabla u - \Phi(\nabla u) = 1$ in the τ direction, we find that $D^2\Phi(\nabla u)D^2 u[\tau]\cdot\nabla u = 0$. So, keeping in mind that $\nabla u = \partial_\nu u\nu$ since u is constant on Γ, we have

$$D^2\Phi(\nabla u)[\nabla G,\nu] = (D\Phi(\nabla u)\cdot\nu)\,(D^2\Phi(\nabla u)D^2 u[\nu]\cdot\nu)\,.$$

This gives the boundary term in (5.3) and completes the proof. □

Next, for $\Phi \in I^*_{\Lambda,\alpha,\Upsilon}$, the assumed bound $\mathcal{H}^{n-2}(\operatorname{Sing}(u)) = 0$ for minimizers u of $\mathcal{E}^\Phi$ also holds for minimizers of the perturbed functional $\mathcal{E}^{\Phi}_{f,\mu}$.

Lemma 5.3. *Let $\Phi \in I^*_{\Lambda,\alpha,\Upsilon}$. For a nonnegative function $f \in C^\infty(B_2)\cap L^\infty(B_2)$ and $\mu \in [0,1]$, any minimizer u of $\mathcal{E}^{\Phi}_{f,\mu}$ in B_2 satisfies $\mathcal{H}^{n-2}(\operatorname{Sing}(u)\cap B_2) = 0$.*

Proof. Let u be a minimizer as in the lemma and suppose by way of contradiction that $\mathcal{H}^{n-2}(\operatorname{Sing}(u)\cap B_2) > 0$. By a general measure theory fact, see [78, Theorem 3.26(2)], for $\mathcal{H}^{n-2}$-a.e. $x \in \operatorname{Sing}(u)\cap B_2$ we have

$$\limsup_{r\to 0}\frac{\mathcal{H}^{n-2}_\infty(\operatorname{Sing}(u)\cap B_2\cap B_r(x))}{\omega_{n-2}r^{n-2}} \geq \frac{1}{2^{n-2}}\,. \tag{5.5}$$

Choose one such point $\bar{x} \in \mathrm{Sing}(u) \cap B_2$, and take a sequence $r_k \to 0$ such that $u_k = u(\bar{x} + r_k x)/r_k$ satisfies $\mathcal{H}^{n-2}_\infty(\mathrm{Sing}(u_k) \cap B_1) \geq \omega_{n-2} 2^{-n}$. By scaling, u_k is a minimizer of $\mathcal{E}^\Phi_{f_k, r_k \mu}$ in $B_{r_k^{-1}}$ where $f_k(x) = f(\bar{x} + r_k x)$.

By Lemma 4.8, up to a subsequence, $u_k \to u_\infty$ in B_2 in the sense of (4.2), and then by Lemma 4.9, u_∞ minimizes $\mathcal{E}^\Phi(\cdot) + |B_{3/2}| f(\bar{x})^2$, and thus also $\mathcal{E}^\Phi$, in $B_{3/2}$. Since $\Phi \in I^*_{\Lambda,\alpha,\Upsilon}$, we know $\mathcal{H}^{n-2}(\mathrm{Sing}(u) \cap B_{3/2}) = 0$. In particular, $\mathcal{H}^{n-2}(\mathrm{Sing}(u_\infty) \cap \overline{B_1}) = 0$ and thus $\mathcal{H}^{n-2}_\infty(\mathrm{Sing}(u_\infty) \cap \overline{B_1}) = 0$ as well. This contradicts Corollary 4.14. □

We now prove Proposition 5.1. The idea is to use the fact that the $G = 1 + \Phi(\nabla u)$ is a subsolution to the linearized equation with right-hand side bounding $|D^2 u|^2$ (recall (5.3)), and relate this to the stability inequality (2.7).

Proof of Proposition 5.1. By scaling it suffices to assume that p is the origin and $r = 1$. Moreover, by Lemma 5.3 and since $\Phi \in I^*_{\Lambda,\alpha,\Upsilon}$, we know $\mathcal{H}^{n-2}(\mathrm{Sing}(u) \cap B_2) = 0$. Let $U = \{u > 0\} \cap B_{3/2}$, let $\Gamma = \partial U \cap B_{3/2}$ and let ν be the outer unit normal to U. Since u is a minimizer of $\mathcal{E}^\Phi_{f,\mu}$ on B_2, we have $\|\nabla u\|_{L^\infty(B_{3/2})} \leq L$ where L depends on n, Λ and an upper bound for $\|f\|_{L^\infty(B_2)}$ by Lemma 4.1. Throughout the proof, $C > 0$ denotes a constant that may change from line to line and depends only on the universal parameters (in fact, only n and Λ) and an upper bound for $\|f\|_{L^\infty(B_2)}$).

Step 1: In the first step we test the equation (5.3) for G against G multiplied by a cutoff function that vanishes on $\mathrm{Sing}(u)$. This step does not require smallness of the singular set. Let $\zeta \in C^1_c(B_{3/2} \setminus \mathrm{Sing}(u))$. We multiply $\mathrm{div}(D^2\Phi(\nabla u)\nabla G)$ by $\zeta^2(x) G$ and integrate by parts to obtain

$$\int_U \zeta^2 G \ \mathrm{div}(D^2\Phi(\nabla u)\nabla G) = \int_\Gamma \zeta^2 G D^2\Phi(\nabla u)[\nabla G, \nu]\, d\mathcal{H}^{n-1} - \int_U \langle D^2\Phi(\nabla u)\nabla G, \nabla(G\zeta^2)\rangle = I - II. \tag{5.6}$$

We bound the right-hand side above as follows. Using the free boundary condition (5.3) for G and regrouping terms, we see that I is coincides with the left-hand side of the stability inequality (2.7). Since ζ vanishes on $\mathrm{Sing}(u)$, the stability inequality (2.7) holds for ζ. Therefore,

$$\begin{aligned} I &= \int_\Gamma \zeta^2 G \, \langle D\Phi(\nabla u), \nu\rangle \langle D^2\Phi(\nabla u) D^2 u \nu, \nu\rangle = \int_\Gamma \frac{\langle D\Phi(\nabla u), \nu\rangle \langle D^2\Phi(\nabla u) D^2 u \nu, \nu\rangle}{1 + \Phi(\nabla u)} (G\zeta)^2 \\ &\leq \int_U D^2\Phi(\nabla u)[\nabla(G\zeta), \nabla(G\zeta)] + 2\mu^2 G^2 \zeta^2 = \int_U (G^2\, D^2\Phi(\nabla u)[\nabla\zeta, \nabla\zeta] + 2\mu^2 G^2 \zeta^2) + II. \end{aligned} \tag{5.7}$$

Combining this with (5.6) and applying the Lipschitz bound for u (which implies $G \leq C$), we find

$$\int_U \zeta^2 G \,\mathrm{div}(D^2\Phi(\nabla u)\nabla G) \leq C \int_U (|\nabla\zeta|^2 + \zeta^2)\,.$$

At the same time, we can bound the left-hand side of below using (5.3), which, after rearranging terms, yields

$$\Lambda^{-2} \int_U G\zeta^2 |D^2 u|^2 \leq C \int_U (|\nabla\zeta|^2 + \zeta^2)\, dx + 2\mu \int_U \zeta^2\, G D\Phi(\nabla u) \cdot \nabla f\, dx \tag{5.8}$$

Here we have once again used the Lipschitz bound for u. It remains bound the second term on the right-hand side of (5.8) from above.[2]. We integrate by parts to find

$$\begin{aligned} \int_U \zeta^2\, G D\Phi(\nabla u) \cdot \nabla f = &- \int_U \zeta^2 G f \ \mathrm{div}(D\Phi(\nabla u)) - \int_U (2\zeta f\, G D\Phi(\nabla u) \cdot \nabla\zeta + \zeta^2 f \nabla G \cdot D\Phi(\nabla u)) \\ &+ \int_\Gamma \zeta^2 G f\, D\Phi(\nabla u) \cdot \nu\, d\mathcal{H}^{n-1} \end{aligned} \tag{5.9}$$

The integrand of the boundary term is nonpositive a.e., since f is assumed to be nonnegative and for all $x \in \mathrm{Reg}(u)$ (in particular for $\mathcal{H}^{n-1}$-a.e. $x \in \Gamma$), we have $D\Phi(\nabla u) \cdot \nu = -\frac{1}{|\partial_\nu u|} D\Phi(\nabla u) \cdot \nabla u \leq 0$, as $D\Phi(p) \cdot p \geq 0$ for all p. Moreover, thanks to the equation for u and the nonnegativity of u, G, and f, we see that $fG\,\mathrm{div}(D\Phi(\nabla u)) = 2\mu^2 ufG - 2\mu G f^2 \geq -2\mu G f^2$. Therefore, also using the Lipschitz bound to esitmate the second term on the right-hand side of (5.9), we find

$$\begin{aligned} \int_U \zeta^2\, G D\Phi(\nabla u) \cdot \nabla f &\leq 2\mu \int_U \zeta^2 f^2 + C \int_U (\zeta f G |\nabla\zeta| + \zeta^2 f |D^2 u|) \\ &\leq C \int (|\nabla\zeta|^2 + \zeta^2) + \frac{C}{\delta} \int_U \zeta^2 + C\delta \int_U |D^2 u|^2 \zeta^2. \end{aligned} \tag{5.10}$$

[2]We could do this immediately to get a constant in (5.1) depending on $\|\nabla f\|_{L^\infty}$, but as shown here, this dependence is not necessary.

Here we choose $\delta > 0$ small enough so that $C\delta < \Lambda^{-2}/2$. Therefore, applying the bound (5.10) to the right hand side of (5.8) and keeping in mind that $G \geq 1$, we can absorb this term into the left-hand side of (5.8) to obtain

$$\int_U \zeta^2 |D^2 u|^2 \leq C \int_U (|\nabla \zeta|^2 + \zeta^2)\, dx \qquad \forall \zeta \in C_c^1(B_{3/2} \setminus \mathrm{Sing}(u))\,. \tag{5.11}$$

Step 2: In this step we shrink the cutoff function ζ to extend (5.11) to general $\varphi \in C_c^1(B_{5/4})$, using a capacity argument going back to Sternberg and Zumbrun, compare [80, Lemma 2.4].

Thanks to Lemma 5.3 and the assumption $\Phi \in I^*_{\Lambda,\alpha,\Upsilon}$, we know $\mathcal{H}^{n-2}(\mathrm{Sing}(u) \cap B_2) = 0$. Let $\Sigma = \mathrm{Sing}(u) \cap \overline{B}_{5/4}$. Since $\mathcal{H}^{n-2}(\Sigma) = 0$, for any $\varepsilon > 0$ we may find a countable collection of balls $\{B_{r_k}(z_k)\}$ covering Σ with $\sum_k r_k^{n-2} < \varepsilon$. By Theorem 4.12, Σ is a compact set, so we can choose a finite subcover, again denoted $\{B_{r_k}(z_k)\}$ after re-indexing. For each k, let $\psi_k \in C^\infty(\mathbb{R}^n)$ with $\psi_k = 0$ in $B_{r_k}(x_k)$, $\psi_k = 1$ on $\mathbb{R}^n \setminus B_{2r_k}(x_k)$, and $|\nabla \psi_k| \leq 2/r_k$. Let $\psi = \min_k \psi_k$, so that $\psi = 0$ on $U' = \cup_k B_{r_k}(z_k) \supset \Sigma$ and $\psi = 1$ on $\mathbb{R}^n \setminus U$ where $U = \cup_k B_{2r_k}(z_k)$. Note that U contains an δ-tubular neighborhood of Σ for some $\delta > 0$. Now, for any $\varphi \in C_c^\infty(B_2)$ with support contained in $\overline{B}_{5/4}$, we let $\zeta = \psi\varphi$, which satisfies $\zeta \in C_c^1(B_2 \setminus \mathrm{Sing}(u))$, and thus (5.11) holds for this choice of ζ.

By the chain rule and the fact that $0 \leq \psi \leq 1$ by construction, we have

$$|\nabla \zeta(x)|^2 \leq 2|\nabla \varphi(x)|^2 + 2|\varphi|^2 \sum_k r_k^{-2} \chi_{B_{r_k}(z_k)} \tag{5.12}$$

and thus using this to bound the the right-hand side of (5.11) with ζ from above yields

$$\int_U \zeta^2 |D^2 u|^2 \leq C \int_U |\nabla \varphi|^2\, dx + \|\varphi\|_{L^\infty}^2 \Big(\sum_k r_k^{n-2} + 1 + \|f\|_{L^\infty}\Big)$$

Here we again use $\|\nabla u\|_{L^\infty(B_{3/2})} \leq L$. Take a sequence $\varepsilon_k \to 0$ and the corresponding $\psi = \psi_{\varepsilon_k}$ and $\zeta_k = \psi_{\varepsilon_k}\varphi$. By Fatou's lemma, $\int_U \varphi^2 |D^2 u|^2 \leq C\|\varphi\|_{C^1}^2$. Choosing a cutoff φ with $\varphi \equiv 1$ in B_1 and $|\nabla \varphi| \leq C(n)$, we conclude. □

Remark 5.4. Observe that the proof of Proposition 5.1 only used that u is a *stable* - not minimizing - solution to the free boundary problem (2.6) whose free boundary was smooth outside a codimension-two null set. Curvature estimates like (5.1) for stable classical solutions (or stable solutions for which $\mathrm{Sing}(u)$ is known to be small) have various applications, see e.g. [10]. So, let us note that the proof also works for classical stable solutions, or stable solutions with $\mathcal{H}^{n-2}(\mathrm{Sing}(u)) = 0$, when $\Phi \in I_{\Lambda,\alpha,\Upsilon}$ is regular outside the origin. This in particular includes the important 2-homogeneous case. To deal with the singularity of Φ at the origin, the only modification to the proof of Proposition 5.1 is to test the equation for G with a function $\zeta^2\, \eta \circ G$ where η is a cutoff function on the level sets of G; this contributes an extra term in the integration-by-parts that has a favorable sign. This cutoff is defined and the proof is carried out in a degenerate elliptic setting in Proposition 7.1 in Section 7 below, and works with the obvious modifications when $\Phi \in I_{\Lambda,\alpha,\Upsilon}$ is $C^{2,\alpha}$ away from the origin.

5.2. $L^{2+\eta}$ **Hessian Estimates.** Next, we establish an improved L^q bound for the Hessian; compare (1.9). We give statement and proof for minimizers of $\mathcal{E}^\Phi$, not $\mathcal{E}^\Phi_{f,\mu}$, for clarity of presentation and since this is all that is needed in our application. The same proof yields analogous estimates for minimizers of $\mathcal{E}^\Phi_{f,\mu}$ with constants depending on $\|f\|_{C^1}$. Recall the universal constant $\bar{\delta} > 0$ was defined in Lemma 4.5 and that $r_\Lambda = \bar{\delta}/2$ from Definition 4.6. The main estimate of this section is the following.

Proposition 5.5. *Let $\Phi \in I_{\Lambda,\alpha,\Upsilon}$ and suppose u is a minimizer of $\mathcal{E}^\Phi(\cdot)$ in B_2 with $0 \in \partial\{u > 0\}$ satisfying the L^2 Hessian estimate (5.1) for all $p \in \Gamma(u)$ and $r \in (0,1]$ with $B_{2r}(p) \subset B_2$. There are universal constants $\eta > 0$ and $C > 0$ such that*

$$\int_{B_{3/4} \cap \{|\nabla u| \geq \bar{\delta}\}} |D^2 u|^{2+\eta} \leq C.$$

In the course of proving Proposition 5.5, we will make use of the following lemmas.

Lemma 5.6. *Let $U \subset \mathbb{R}^n$ be connected and open, and let $u \in C^1(U) \cap W^{2,1}_{\mathrm{loc}}(U)$. Fix $\delta \geq 0$ and suppose that $D^2 u = 0$ a.e. in $E_\delta := U \cap \{|\nabla u| > \delta\}$. If $E_\delta \neq \emptyset$, then $u(x) = e \cdot x + c$ in U for some constant vector e with $|e| > \delta$.*

Proof. Fix a connected component V of E_δ. Since the weak derivatives of ∇u vanish on V, by connectedness and continuity we have $\nabla u \equiv e$ on V, for a constant vector $e \in \mathbb{R}^n$ with $|e| > \delta$.

We claim that V is relatively closed in U, so that since it is also relatively open in U, $V \neq \emptyset$, and U is connected, we deduce $V = U$.

If $x \in \overline{V} \cap U$, continuity gives $\nabla u(x) = e$. Since $|e| > \delta$, there is a ball $B_r(x)$ contained in E_δ. This ball intersects V, so connectedness of $B_r(x)$ and V being connected component implies $B_r(x) \subset V$. Hence $x \in V$, as desired. □

The proof only needs $W^{2,1}_{\mathrm{loc}}$ regularity on $U \cap \{|\nabla u| > \delta\}$ and continuity of ∇u on U. This localized version will be used in Section 7 when establishing the p-Laplacian analogues of Lemma 6.1, in the verification of Ingredient (2), and Lemma 5.8, in the verification of Ingredient (3).

Lemma 5.7. *Let u minimize $\mathcal{E}^\Phi$ in B_1, with $0 \in \Gamma(u)$. Let $0 \leq \delta \leq \bar{\delta}$, and suppose that $D^2 u = 0$ a.e. in $B_1 \cap \{u > 0, |\nabla u| > \delta\}$. There exists $\rho_0 = \rho_0(n, \Lambda) \in (0, 1]$ and $e \in \mathbb{R}^n$ such that*

$$u(x) = (e \cdot x)_+ \quad \text{in } B_{\rho_0}, \qquad \mathcal{G}(e) = 1.$$

Proof. By Lemma 4.5, every connected component V of $B_1 \cap \{u > 0\}$, intersecting $B_{1/8}$ nontrivially, contains a ball $B_{\bar{\delta}}(p_V) \subset B_{1/2}$ on which $|\nabla u| \geq 8\bar{\delta} > \delta$. Lemma 5.6 thus guarantees that u coincides with an nonzero affine function on each connected component of $\{u > 0\} \cap B_1$. Since the balls $B_{\bar{\delta}}(p_V)$ are disjoint and contained in $B_{1/2}$, there are only finitely many such components. By Remark 4.4, we conclude the proof. □

The main tool to prove Proposition 5.5 is the following doubling estimate.

Lemma 5.8. *Let $\Phi \in I_{\Lambda,\alpha,\Upsilon}$ and suppose u is a minimizer of $\mathcal{E}^\Phi(\cdot)$ in B_4 satisfying the L^2 Hessian estimate (5.1) for all $p \in \Gamma(u)$ and $r > 0$ such that $B_{2r}(p) \subset B_4$. Suppose additionally that $|\nabla u| \leq L$ in B_4 for some $L \in (0, R_\Lambda)$. There is a constant $A \gg 1$ depending on L and the universal parameters such that, if $u(0) > 0$, $|\nabla u(0)| \geq \bar{\delta}$, and $|D^2 u|(0) \geq A$, then*

$$\int_{B_1 \cap \{u>0\}} |D^2 u|^2 \leq A \int_{B_{1/5} \cap \mathcal{B}_A(u)} |D^2 u|^2, \tag{5.13}$$

where $\mathcal{B}_A(u) = \{u > 0\} \cap \{|\nabla u| \geq \bar{\delta}\} \cap \{\frac{1}{A} \leq |D^2 u| \leq A\}$. Here $\bar{\delta}$ is the universal constant from Lemma 4.5 and R_Λ is defined in Definition 4.6.

Proof. Step 0: Initial proximity to the free boundary. Let $\rho_0 = \rho_0(n, \Lambda) \in (0, 1]$ be as in Lemma 5.7. To begin, we claim that for A sufficiently large, the assumptions of Lemma 5.8 guarantee that

$$\Gamma(u) \cap B_{\rho_0/100} \neq \varnothing. \tag{5.14}$$

Suppose not; since $u(0) > 0$ this means $B_{\rho_0/100} \subset \{u > 0\}$. In this case, applying the interior estimates (2.16) of Lemma 2.5 to $u - u(0)$ in $B_{\rho_0/100}$ shows that $\|\nabla u\|_{C^{0,\beta}(B_{\rho_0/200})} \leq CL$ for a universal $C > 0$. Since $|\nabla u(0)| \geq \bar{\delta}$, if we set $s = \min\{(\frac{\bar{\delta}}{4CL})^{1/\beta}, \frac{\rho_0}{200}\}$, the Hölder estimate gives $3\bar{\delta}/4 \leq |\nabla u| \leq L$ on B_s. Since $3\bar{\delta}/4 > r_\Lambda$ and $L < R_\Lambda$, this is a fixed compact subannulus of $\mathcal{A}(\Lambda)$, and the interior Hessian estimate applies. In turn, applying the Hessian bound (2.17) of Lemma 2.5 in this smaller ball yields

$$|D^2 u(0)| \leq \frac{C}{s^2}$$

for C depending on L and the universal parameters. This is a contradiction for $A > \frac{C}{s^2}$. Thus (5.14) holds.

Step 1: Set up and compactness. Suppose the conclusion of the lemma fails. We find a sequence of integrands $\{\Phi_k\} \subset I_{\Lambda,\alpha,\Upsilon}$ and minimizers u_k of $\mathcal{E}^{\Phi_k}$ in B_4 with $u_k(0) > 0, |\nabla u_k(0)| \geq \bar{\delta}$, $|D^2 u_k(0)| \geq k$, and $|\nabla u_k| \leq L$ such that

$$\int_{B_1 \cap \{u_k>0\}} |D^2 u_k|^2 \geq k \int_{B_{1/5} \cap \mathcal{B}_k(u_k)} |D^2 u_k|^2. \tag{5.15}$$

For k large, Step 0 guarantees that (5.14) holds with u_k in place of u. In particular, there is a point in $B_{\rho_0/100}$ for which (5.1) holds with $r = 1 + 1/100$. In particular, by set containment, (5.1) implies $\int_{B_1 \cap \{u_k>0\}} |D^2 u_k|^2 \leq C$. Pairing this with (5.15) and the trivial bound $\int_{B_{1/5} \cap \{|D^2 u_k| \leq 1/k\}} |D^2 u_k|^2 \leq C/k^2$ gives

$$\int_{B_{1/5} \cap \mathcal{D}_k} |D^2 u_k|^2 \leq \frac{C}{k}, \qquad \text{where} \qquad \mathcal{D}_k = \{u_k > 0\} \cap \{|\nabla u_k| \geq \bar{\delta}\} \cap \{|D^2 u_k| \leq k\} \tag{5.16}$$

for all k large and for a universal constant $C > 0$.

Up to a subsequence, $\Phi_k \to \Phi \in I_{\Lambda,\alpha,\Upsilon}$. Moreover, since (5.14) holds for each u_k, Lemma 4.8 and Lemma 4.9 guarantee that, up to a further subsequence, u_k converges locally uniformly and locally in H^1 in B_4 to a function u

that minimizes $\mathcal{E}^{\Phi}(\cdot)$ on compact subsets of B_4. Moreover, let $\bar{x}_k$ be a closest point of $\Gamma(u_k)$ to the origin. We claim that $|\bar{x}_k| \to 0$. Otherwise, after passing to a subsequence, there would be a fixed $\sigma > 0$ such that $B_\sigma \subset \{u_k > 0\}$ for every k. Repeating the interior argument of Step 0 on B_σ, using $|\nabla u_k(0)| \geq \bar{\delta}$ and $|\nabla u_k| \leq L$, would give a bound for $|D^2 u_k(0)|$ independent of k, contradicting $|D^2 u_k(0)| \geq k$. Thus $\bar{x}_k \to 0$, and convergence of the free boundaries gives $0 \in \Gamma(u)$.

Step 2: Identifying u. Let $E = B_{1/5} \cap \{u > 0\} \cap \{|\nabla u| > \bar{\delta}\}$. We claim $D^2 u = 0$ a.e. on E. Indeed, take open sets $D \Subset D' \Subset E$. The local uniform convergence of u_k and ∇u_k (the latter thanks to estimate (2.16) of Lemma 2.5) guarantee that $D' \subset \{u_k > 0\} \cap \{|\nabla u_k| > \bar{\delta}\}$ for all sufficiently large k. Hence, $\nabla u_k(x) \in \mathcal{A}(\Lambda)$ for all such k and for all $x \in \overline{D'}$. Therefore, $|D^2 u_k| \leq \bar{C}$ on D by estimate (2.17) of Lemma 2.5. Here $\bar{C}$ depends on D, D' as well as on L and the universal parameters. This means, by definition, $D \subset \mathcal{D}_k$ for all k large, with $\mathcal{D}_k$ as defined in (5.16). So, (5.16) implies that $D^2 u_k \to 0$ in $L^2(D)$. Distributional convergence of the second derivatives then gives $D^2 u = 0$ in D. Exhausting E, we obtain $D^2 u = 0$ a.e. in E.

Applying Lemma 5.7 to $v(y) = 10u(y/10)$ in B_1, we obtain, up to a rotation, that $u(x) = a(x \cdot e_1)_+$ in $B_{\rho_0/10}$, where $a > 0$ satisfies $D\Phi(ae_1) \cdot (ae_1) - \Phi(ae_1) = 1$. As observed in Remark 4.7, thus guarantees that ae_1 lies in a compact subset of $\mathcal{A}(\Lambda)$.

Step 3: $C^{2,\alpha}$ estimates on $\{u_k > 0\}$ and conclusion. By Step 2 and $\bar{x} = 0$, $u(x) = a(x \cdot e_1)_+$ in $B_{\rho_0/10}$, with $\mathcal{G}(ae_1) = 1$. For each k, let $a_k > 0$ be the unique number satisfying

$$D\Phi_k(a_k e_1) \cdot (a_k e_1) - \Phi_k(a_k e_1) = 1.$$

The convergence of the integrands imply $a_k \to a$. Since $\bar{x}_k \to 0$, then $B_{\rho_0/20}(\bar{x}_k) \subset B_{\rho_0/10}$ for k large. Local uniform convergence and nondegeneracy imply that, for all such k,

$$(a_k(x - \bar{x}_k) \cdot e_1 - \bar{\varepsilon}\rho_0/20)_+ \leq u_k(x) \leq (a_k(x - \bar{x}_k) \cdot e_1 + \bar{\varepsilon}\rho_0/20)_+ \qquad \text{in } B_{\rho_0/20}(\bar{x}_k),$$

where ε_0 is the flatness threshold in Theorem 4.12. Theorem 4.12 and the quantitative observation in the proof of Corollary 4.13 give $C^{2,\alpha}$ boundary graphs with uniformly bounded rescaled norms on a fixed smaller neighborhood of $\bar{x}_k$. Consequently, the constants in Lemma 2.5 are independent of k. The boundary gradient estimate and the Bernoulli condition also place ∇u_k in a fixed compact subannulus of $\mathcal{A}(\Lambda)$ on a further fixed smaller neighborhood. Applying (2.19) with $w = u_k$ and $\Omega = \{u_k > 0\}$, we therefore obtain

$$\|D^2 u_k\|_{L^\infty(B_{c\rho_0/20}(\bar{x}_k) \cap \{u_k > 0\})} \leq \frac{C}{\rho_0/20},$$

where $c, C > 0$ are independent of k. Since $\bar{x}_k \to 0$ and $u_k(0) > 0$, this estimate applies at the origin for all sufficiently large k, contradicting $|D^2 u_k(0)| \geq k$. □

We now see how Lemma 5.8 is applied to get improved integrability in the curvature estimate.

Proof of Proposition 5.5. We proceed in several steps.

Step 0: Set up. Since $u(0) = 0$, we may apply Lemma 4.1 in B_2 to find $|\nabla u| \leq L = L(\Lambda, n)$ in $B_{15/8}$. Keeping in mind Definition 4.6, $L < R_\Lambda$. Let A be the constant from Lemma 5.8 corresponding to this choice of L. Let $M \geq 4A$ be a constant depending on the universal parameters to be specified below. For each $k \in \mathbb{N}$, set

$$D_k := B_{\frac{3}{4} + (\frac{1}{4})^k} \cap \{u > 0\} \cap \{|\nabla u| > \bar{\delta}\} \cap \{|D^2 u| > M^k\}. \tag{5.17}$$

Step 1: Applying the doubling estimate to the rescaled problem. For each $x_0 \in D_k$, take $r = \frac{A}{M^k}$ and $u_r(x) := \frac{1}{r} u(x_0 + rx)$. Since $x_0 \in B_1$ and $M \geq 4A$, we see that u_r is a minimizer of $\mathcal{E}^{\Phi}(\cdot)$ in B_4 satisfying

$$u_r(0) = \frac{1}{r} u(x_0) > 0, \qquad |\nabla u_r|(0) = |\nabla u(x_0)| > \delta, \qquad |D^2 u_r(0)| = r|D^2 u(x_0)| > \frac{A}{M^k} M^k = A.$$

Moreover, $|\nabla u_r| = |\nabla u(x_0 + r\,\cdot)| \leq L$ in B_4. Therefore, we can apply the doubling estimate of Lemma 5.8 to conclude

$$\int_{B_1 \cap \{u_r > 0\}} |D^2 u_r|^2 \leq A \int_{B_{1/5} \cap \{u_r > 0\} \cap \{|\nabla u_r| \geq \delta\} \cap \{\frac{1}{A} \leq |D^2 u_r| \leq A\}} |D^2 u_r|^2.$$

In terms of the original minimizer u, this gives

$$\int_{B_r(x_0)\cap\{u>0\}} |D^2u|^2 \le A \int_{B_{r/5}(x_0)\cap\{u>0\}\cap\{|\nabla u|\ge\delta\}\cap\{\frac{M^k}{A^2}\le|D^2u|\le M^k\}} |D^2u|^2.$$

As a result, if we pick $M = A^2$ and let $\mathcal{K}_k = \{u > 0\} \cap \{|\nabla u| \ge \delta\} \cap \{M^{k-1} \le |D^2u| \le M^k\}$, this becomes

$$\int_{B_r(x_0)\cap\{u>0\}} |D^2u|^2 \le A \int_{B_{r/5}(x_0)\cap\mathcal{K}_k} |D^2u|^2. \tag{5.18}$$

Step 2: Inductive step. For each $x \in D_k$, we choose r as in the previous step. Thus $\{B_r(x)\}_{x\in D_k}$ forms a covering of D_k. We take a Vitali subcovering $\{B_r(x_j)\}$. Since $x \in B_{\frac{3}{4}+(\frac{1}{4})^k}$ and $r = \frac{A}{M^k} = \frac{A}{A^{2k}}$, after possibly increasing A, we have

$$B_r(x) \subset B_{\frac{3}{4}+(\frac{1}{4})^{k-1}} \qquad \text{for every } x \in D_k. \tag{5.19}$$

Now, apply (5.18) in each ball $\{B_r(x_j)\}$ in the covering. Using (5.19) followed by (5.18) and the fact that the balls $B_{r/5}(x_k)$ are pairwise disjoint, we find

$$\int_{D_k} |D^2u|^2 \le \sum_j \int_{B_r(x_j)\cap\{u>0\}} |D^2u|^2 \le A\sum_j \int_{B_{r/5}(x_j)\cap\mathcal{K}_k} |D^2u|^2 \le A \int_{B_{\frac{3}{4}+(\frac{1}{4})^{k-1}}\cap\mathcal{K}_k} |D^2u|^2\,. \tag{5.20}$$

The domain of integration in the last integral satisfies the containment

$$B_{\frac{3}{4}+(\frac{1}{4})^{k-1}} \cap \mathcal{K}_k \subset D_{k-1} \setminus \left(B_{\frac{3}{4}+(\frac{1}{4})^{k-1}} \cap \{u > 0\} \cap \{|\nabla u| \ge \delta\} \cap \{|D^2u| > M^k\}\right) \subset D_{k-1} \setminus D_k.$$

Thus, we conclude from (5.20) that $\int_{D_k} |D^2u|^2 \le A \int_{D_{k-1}} |D^2u|^2 - A \int_{D_k} |D^2u|^2$, or, in other words,

$$\int_{D_k} |D^2u|^2 \le \frac{A}{1+A} \int_{D_{k-1}} |D^2u|^2. \tag{5.21}$$

Step 3: Conclusion. An iteration of the previous step gives

$$\int_{D_k} |D^2u|^2 \le \left(\frac{A}{1+A}\right)^{k-1} \int_{D_1} |D^2u|^2 \le \left(\frac{A}{1+A}\right)^{k-1} \int_{B_1\cap\{u>0\}} |D^2u|^2.$$

Since $0 \in \Gamma(u)$, we may applying the curvature estimate of Proposition 5.1 to the right-hand side to find

$$\int_{D_k} |D^2u|^2 \le C \left(\frac{A}{1+A}\right)^{k-1}. \tag{5.22}$$

Recalling the definition of D_k in (5.17) and applying Chebychev inequality to (5.22), we find

$$\left|\{|D^2u| \ge M^k\} \cap \{|\nabla u| \ge \delta\} \cap \{u > 0\} \cap B_{3/4}\right| \le |D_k| \le C \left(\frac{A}{1+A}\right)^{k-1} M^{-2k}$$

for all k. The conclusion follows by the layer-cake formula. □

6. Proofs of the main theorems

In this section, we prove Theorems 1.3 and 1.7. As we described in the introduction, to prove Theorem 1.3, we first show that $I^*_{\Lambda,\alpha,\Upsilon} = I_{\Lambda,\alpha,\Upsilon}$ using a topological argument to show that $I^*_{\Lambda,\alpha,\Upsilon}$ is both open and closed in the connected nonempty set $I_{\Lambda,\alpha,\Upsilon}$. Then, we apply the improved $L^{2+\eta}$ curvature estimates to complete the proof. Theorem 1.7 will then follow from Theorem 1.3, the corresponding estimates established in the previous section, and an approximation argument.

6.1. Preparatory lemmas.

We need two additional ingredients before proving Theorem 1.3. The first is an ε-regularity statement with flatness measured in terms of the norm of the Hessian. It will be applied twice: once with $\eta = \delta = 0$ and then with $\eta > 0$ and $\delta = \bar\delta$. Recall $\bar\delta > 0$ is the constant determined in Lemma 4.5.

Lemma 6.1. *For any $\eta \in [0, n-2)$ and $\delta \in [0, \bar\delta]$, there exists $\varepsilon > 0$ depending on δ, η, and the universal parameters such that if $\Phi \in I_{\Lambda,\alpha,\Upsilon}$ and u is a minimizer of $\mathcal{E}^\Phi$ in B_1 with $0 \in \Gamma(u)$ and*

$$\int_{B_1\cap\{u>0\}\cap\{|\nabla u|\ge\delta\}} |D^2u|^{2+\eta} < \varepsilon,$$

then $0 \in \mathrm{Reg}(u)$.

Proof. Suppose for a contradiction that we have $\eta \in [0, n-2)$, $\delta \in [0, \bar{\delta}]$, and a sequence of integrands $\Phi_k \in I_{\Lambda,\alpha,\Upsilon}$ and minimizers u_k of $\mathcal{E}^{\Phi_k}(\cdot)$ in B_1 with $0 \in \Gamma_{u_k}$ such that $\int_{B_1 \cap \{u_k > 0\} \cap \{|\nabla u_k| \geq \delta\}} |D^2 u_k|^{2+\eta} \to 0$ but $0 \in \mathrm{Sing}(u_k)$ for each k. By compactness for integrands and Lemmas 4.8 and 4.9, up to passing to a subsequence, $\Phi_k \to \Phi \in I_{\Lambda,\alpha,\Upsilon}$ and $u_k \to u$ such that $0 \in \Gamma(u)$ and u is a minimizer of $\mathcal{E}^{\Phi}$ on compact subsets of B_1.

In the case $\delta = 0$, Lemma 4.9 guarantees that

$$\int_{\{u>0\} \cap B_1} |D^2 u|^{2+\eta} \, dx \leq \liminf_k \int_{\{u_k > 0\} \cap B_1} |D^2 u_k|^{2+\eta} = 0.$$

So, by Lemma 5.7, in $\{u > 0\} \cap B_{\rho_0}$, $u = (x \cdot e)_+$ for some $e \in \mathbb{R}^n$ satisfying the free boundary condition $D\Phi(e) \cdot e - \Phi(e) = 1$. In particular, $\Gamma(u)$ is a hyperplane and $0 \in \mathrm{Reg}(u)$. This contradicts Corollary 4.14, completing the proof in this case.

For $\delta > 0$, we argue as in Step 2 of the proof of Lemma 5.8 above. Set $E_\delta = B_1 \cap \{u > 0\} \cap \{|\nabla u| > \delta/2\}$ and take an open set $D \subset\subset E_\delta$. By local uniform convergence of u_k and ∇u_k, for all sufficiently large k we have $D \subset \{u_k > 0\} \cap \{|\nabla u_k| \geq \delta\}$. Hence $D^2 u_k \to 0$ in $L^p(D)$. Distributional convergence of the second derivatives gives $D^2 u = 0$ in D. Exhausting E_δ, we obtain $D^2 u = 0$ a.e. in E_δ.

So, once more, Lemma 5.7 guarantees that in $\{u > 0\} \cap B_{\rho_0}$, we have $u(x) = (e \cdot x)_+$ for some $e \neq 0$ with $D\Phi(e) \cdot e - \Phi(e) = 1$. In particular, $0 \in \mathrm{Reg}(u)$, again contradicting propagation of singularities of Corollary 4.14 from the sequence u_k. □

The second preparatory ingredient, which is a consequence of Lemma 6.1, shows how curvature estimates like Proposition 5.1 and Proposition 5.5 give a measure bound on the singular set. Once again, this statement will be applied twice: first with $\eta = \delta = 0$, and then with $\delta, \eta > 0$.

Corollary 6.2. *Let $\eta \in [0, n-2)$ and $\delta \in [0, \bar{\delta}]$. Suppose $\Phi \in I_{\Lambda,\alpha,\Upsilon}$ and let u be a minimizer of $\mathcal{E}^{\Phi}$ in B_1 with*

$$\int_{B_1 \cap \{u>0\} \cap \{|\nabla u| \geq \delta\}} |D^2 u|^{2+\eta} < \infty \,. \tag{6.1}$$

Then $\mathcal{H}^{n-2-\eta}(\mathrm{Sing}(u) \cap B_1) = 0$.

Proof. Let $x \in \mathrm{Sing}(u) \cap B_1$ and $r \in (0, \mathrm{dist}(x, \partial B_1))$. Applying the contrapositive of Lemma 6.1 to the rescaled function $u_r(y) = u(x + ry)/r$, which leaves the gradient lower bound invariant, we find that

$$\int_{B_r(x) \cap \{u>0\} \cap \{|\nabla u| \geq \delta\}} |D^2 u|^{2+\eta} \, dx \geq \varepsilon r^{n-2-\eta} \tag{6.2}$$

Here $\varepsilon > 0$ is the constant from Lemma 6.1. For $s \in (0,1)$, let $\Sigma_s = \mathrm{Sing}(u) \cap B_s$, and let $N_r(E) = \{x : \mathrm{dist}(x, E) < r\}$ denote the radius-r tubular neighborhood of a set E. We claim that for each $s \in (0,1)$,

$$\lim_{r \to 0} \frac{|N_r(\Sigma_s)|}{r^{2+\eta}} = 0 \,. \tag{6.3}$$

To see this, fix $r < 1 - s$ and take a maximal r-dense subset $\{x_j\}_{j=1}^{J_r}$ of Σ_s. This way, the balls $B_r(x_j)$ are pairwise disjoint and their union is contained in $N_r(\Sigma_s) \subset B_1$, while $\{B_{2r}(x_j)\}$ covers Σ_s and $\{B_{3r}(x_j)\}$ covers $N_r(\Sigma_s)$. Applying (6.2) on each $B_r(x_j)$ and summing up, we find

$$\begin{aligned} |N_r(\Sigma_s)| \leq 3^n J_r \omega_n r^n &\leq \frac{3^n \omega_n r^{2+\eta}}{\varepsilon} \sum_{i=1}^{J_r} \int_{\{u>0\} \cap \{|\nabla u| \geq \delta\} \cap B_r(x_j)} |D^2 u|^{2+\eta} \, dx \\ &\leq \frac{3^n \omega_n r^{2+\eta}}{\varepsilon} \int_{\{u>0\} \cap \{|\nabla u| \geq \delta\} \cap N_r(\Sigma_s)} |D^2 u|^{2+\eta} \, dx \,. \end{aligned} \tag{6.4}$$

By (6.1) and the fact that $|\Sigma_s| = 0$ (in fact $\mathcal{H}^{n-1}(\Sigma_s) < \infty$) by Corollary 4.3, the integral on the right-hand side of (6.4) tends to zero as $r \to 0$. Thus, dividing both sides of (6.4) by $r^{2+\eta}$ and sending $r \to 0$, we obtain (6.3). By a standard argument, this in particular shows that $\mathcal{H}^{n-2-\eta}(\Sigma_s) = 0$. Taking $s \to 1$ completes the proof. □

6.2. **Proofs of the main theorems.** We now prove Theorem 1.3.

Proof of Theorem 1.3. We proceed in several steps. Recall the space of good integrands $I^*_{\Lambda,\alpha,\Upsilon}$ defined in (1.10).

Step 1: $I^*_{\Lambda,\alpha,\Upsilon}$ is open in $I_{\Lambda,\alpha,\Upsilon}$. To establish the openness of $I^*_{\Lambda,\alpha,\Upsilon}$ in $I_{\Lambda,\alpha,\Upsilon}$, we show that the complement is closed. Consider a sequence $\{\Phi_k\} \subset I_{\Lambda,\alpha,\Upsilon} \setminus I^*_{\Lambda,\alpha,\Upsilon}$ with $\Phi_k \to \Phi \in I_{\Lambda,\alpha,\Upsilon}$. We claim that $\Phi \notin I^*_{\Lambda,\alpha,\Upsilon}$. Since $\Phi_k \notin I^*_{\Lambda,\alpha,\Upsilon}$, for each k we may find a minimizer u_k in B_1 such that $0 \in \Gamma_{u_k}$ and

$$\mathcal{H}^{n-2}(\mathrm{Sing}(u_k) \cap B_1) > 0.$$

Applying the same measure theory fact as in the proof of Lemma 5.3, see [78, Theorem 3.26(2)], for $\mathcal{H}^{n-2}$-a.e. $x \in \mathrm{Sing}(u_k) \cap B_1$ we have

$$\limsup_{r\to 0} \frac{\mathcal{H}^{n-2}_\infty(\mathrm{Sing}(u_k) \cap B_1 \cap B_r(x))}{\omega_{n-2} r^{n-2}} \geq \frac{1}{2^{n-2}}\,.$$

In particular, for each k we may find $x_k \in B_1$ and $r_k \in (0, \mathrm{dist}(x_k, \partial B_1))$ such that the rescaled function $v_k(y) = u_k(x_k + r_k y)/r_k$, which again is a minimizer of $\mathcal{E}^{\Phi_k}$ in B_1, has

$$\mathcal{H}^{n-2}_\infty(\mathrm{Sing}(v_k) \cap \overline{B}_{1/2}) \geq 2^{-2n}.$$

By Lemmas 4.8 and 4.9, up to a subsequence, $v_k \to v$ (in the various senses of this lemmas) where v is a minimizer of $\mathcal{E}^\Phi$ on compact subsets of B_1. Then by Corollary 4.14, we find $\mathcal{H}^{n-2}_\infty(\mathrm{Sing}(v) \cap \overline{B}_{1/2}) \geq 2^{-2n}$. Since $\mathcal{H}^{n-2}_\infty(E) \leq \mathcal{H}^{n-2}(E)$ for any set E, this means that $\mathcal{H}^{n-2}(\mathrm{Sing}(v) \cap B_1) \geq 2^{-2n} > 0$, and therefore $\Phi \notin I^*_{\Lambda,\alpha,\Upsilon}$.

Step 2: $I^*_{\Lambda,\alpha,\Upsilon}$ is closed in $I_{\Lambda,\alpha,\Upsilon}$. Take a sequence $\{\Phi_k\} \subset I^*_{\Lambda,\alpha,\Upsilon}$ with $\Phi_k \to \Phi \in I_{\Lambda,\alpha,\Upsilon}$. As observed in Remark 2.1, smooth integrands $\Phi \in C^\infty(\mathbb{R}^n)$ are dense in $I_{\Lambda,\alpha,\Upsilon}$. Therefore, thanks to the openness shown in Step 1 above, we may assume without loss of generality that $\Phi_k \in C^\infty(\mathbb{R}^n) \cap I^*_{\Lambda,\alpha,\Upsilon}$.

Let u be a minimizer of $\mathcal{E}^\Phi(\cdot, \Omega)$. To prove that $I^*_{\Lambda,\alpha,\Upsilon}$ is closed, we must show that $\mathcal{H}^{n-2}(\mathrm{Sing}(u) \cap \Omega) = 0$ so that $\Phi \in I^*_{\Lambda,\alpha,\Upsilon}$. Taking a countable exhaustion of Ω and using compactness, it suffices to show that for any $x \in \Gamma(u) \cap \Omega$, we have $\mathcal{H}^{n-2}(\mathrm{Sing}(u) \cap B_{d/8}(x)) = 0$ where $d = \mathrm{dist}(x, \partial\Omega) > 0$. After rescaling, this claim becomes the following:

Claim: If $u \in H^1(B_4)$ minimizes $\mathcal{E}^\Phi$ in B_4 and $0 \in \Gamma(u)$, then $\mathcal{H}^{n-2}(\mathrm{Sing}(u) \cap B_{1/2}) = 0$.

Applying Corollary 6.2 with $\eta = \delta = 0$, to prove the claim it suffices to prove that

$$\int_{\{u>0\} \cap B_{1/2}} |D^2 u|^2 \, dx < \infty\,. \tag{6.5}$$

By Lemma 4.1, we have $\sup_{B_2} u + \sup_{B_2} |\nabla u| \leq L$ for $L = L(n, \Lambda)$. So, letting $f_k = u * \xi_{1/k}$ where ξ is a standard mollifier and $\xi_\varepsilon = \frac{1}{\varepsilon^n}\xi(x/\varepsilon)$, we have the same bound $\sup_{B_2} f_k + \sup_{B_2} |\nabla f_k| \leq L$ for each f_k. In particular, up to a subsequence, $f_k \to u$ uniformly in B_1 and in $L^2(B_1)$.

Now, for each $k \in \mathbb{N}$, let u_k be a minimizer of the energy functional $\mathcal{E}^{\Phi_k}_{f_k,1}$ on B_1 with boundary data $u_k = u$ on ∂B_1. Taking u as a competitor for each u_k, we find that

$$\begin{aligned}\mathcal{E}^{\Phi_k}_{f_k,1}(u_k) \leq \mathcal{E}^{\Phi_k}_{f_k,1}(u) &= \int_{B_1} \Phi_k(\nabla u) + |u - f_k|^2 \, dx \leq \int_{B_1} \Phi(\nabla u) + \int_{B_1} |\Phi_k(\nabla u) - \Phi(\nabla u)| dx + \int_{B_1} |u - f_k|^2 \, dx \\ &\leq \int_{B_1} \Phi(\nabla u) + \|\Phi_k - \Phi\|_{C^0(B_L)} |B_1| + \int_{B_1} |u - f_k|^2 \, dx \to \int_{B_1} \Phi(\nabla u)\end{aligned}$$

In the final inequality, we use local C^1 convergence of Φ_k to Φ and the Lipschitz bound for u. In particular, $\sup_k \mathcal{E}^{\Phi_k}_{f_k,1}(u_k) < \infty$. In particular, the sequence $\{u_k\}$ is uniformly bounded in $H^1(B_1)$, so up to a subsequence, $u_k \rightharpoonup v$ in $H^1(B_1)$ for some $v \in H^1(B_1)$. A slight modification of the proof of Lemma 4.1 shows that $\sup_{B_\rho}(|\nabla u_k| + |u_k|) < \infty$ for each $\rho < 1$ (here the functions are anchored by their boundary data rather than their zero set). Hence up to a diagonal subsequence as in the proof of Lemma 4.8, $u_k \to v$ locally uniformly in B_1. Since $f_k \to u$ uniformly, Lemma 4.9(2) guarantees that v is a minimizer of $\mathcal{E}^\Phi_{u,1}$ in B_1. Using the minimality of v and u for $\mathcal{E}^\Phi_{u,1}$ and $\mathcal{E}^\Phi$ respectively, we find that

$$\mathcal{E}^\Phi(v) + \int_{B_1} |u - v|^2 = \mathcal{E}^\Phi_{u,1}(v) \leq \mathcal{E}^\Phi_{u,1}(u) = \mathcal{E}^\Phi(u) \leq \mathcal{E}^\Phi(v).$$

The ordering of the left-most and right-most expressions imply that $\int_{B_1} |v - u|^2 = 0$ and thus $u = v$ in B_1. In summary, $u_k \to u$ in B_1 locally uniformly and weakly in $H^1(B_1)$, and all conclusions of Lemma 4.9 hold.

Since $0 \in \Gamma(u)$ and from the Hausdorff convergence of the free boundaries from Lemma 4.9(4), we may find a sequence $\{x_k\} \subset B_1$ with $x_k \in \Gamma(u_k)$ for each $k \in \mathbb{N}$ and $x_k \to 0$. For any $\varepsilon > 0$, we have $\operatorname{dist}(x_k, \partial B_1) \geq 1 - \varepsilon$ for k large enough, so we may apply Proposition 5.1 with $x = x_k$ and $r = (1-\varepsilon)/2$. Thus u_k satisfies the curvature estimate

$$\int_{\{u_k>0\}\cap B_{1/2-2\varepsilon}} |D^2 u_k|^2 \leq \int_{\{u_k>0\}\cap B_{(1-\varepsilon)/2}(x_k)} |D^2 u_k|^2 \leq C(1-\varepsilon)^{n-2}.$$

In the final inequality, $C > 0$ is the constant from Proposition 5.1, which depends on the universal parameters only thanks to the upper bound $\|f_k\|_{L^\infty} \leq L$ above. By Lemma 4.9(5), the same bound holds for u. Letting $\varepsilon \to 0$, we obtain (6.5), completing the proof of the claim. Thus $I^*_{\Lambda,\alpha,\Upsilon}$ is closed.

Step 3: Conclusion. The previous two steps show that $I^*_{\Lambda,\alpha,\Upsilon}$ is an open and closed subset of $I_{\Lambda,\alpha,\Upsilon}$. Since $I_{\Lambda,\alpha,\Upsilon}$ is connected, recall Remark 2.1, and nonempty by Theorem 2.4, we have $I^*_{\Lambda,\alpha,\Upsilon} = I_{\Lambda,\alpha,\Upsilon}$. In other words, for every $\Phi \in I_{\Lambda,\alpha,\Upsilon}$ and for every minimizer u of $\mathcal{E}^\Phi(\cdot, \Omega)$,

$$\mathcal{H}^{n-2}(\operatorname{Sing}(u) \cap \Omega) = 0.$$

If $n = 2$, this already gives $\operatorname{Sing}(u) \cap \Omega = \emptyset$, so we may assume that $n \geq 3$.

We first observe that the approximation argument in Step 2, Proposition 5.1, and Lemma 4.9(5) give

$$\int_{B_{1/4}\cap\{w>0\}} |D^2 w|^2 \, dx \leq C$$

for every minimizer w of $\mathcal{E}^\Phi$ in B_4 with $0 \in \Gamma(w)$, where C is universal. Indeed, by Step 3 the smooth approximating integrands can all be chosen in $I^*_{\Lambda,\alpha,\Upsilon}$, and the penalization data used in Step 2 have a uniform L^∞ bound. Applying this estimate to $w(y) = u(p + 4ry)/(4r)$, we obtain

$$\int_{B_r(p)\cap\{u>0\}} |D^2 u|^2 \, dx \leq C r^{n-2} \qquad \text{whenever } p \in \Gamma(u) \text{ and } B_{16r}(p) \subset \Omega. \tag{6.6}$$

Thus this estimate holds for the original minimizer, without any additional smoothness assumption on Φ.

Fix $x_0 \in \Gamma(u) \cap \Omega$ and $R > 0$ such that $B_{8R}(x_0) \subset \Omega$, and set $v(y) := u(x_0 + Ry)/R$ for $y \in B_8$. Then v minimizes $\mathcal{E}^\Phi$ in B_8 and $0 \in \Gamma(v)$. If $q \in \Gamma(v)$ and $B_{4\rho}(q) \subset B_2$, then $B_{16\rho}(q) \subset B_8$. Consequently, (6.6), applied to v, verifies the L^2 hypothesis of Proposition 5.5 in B_2. This proposition therefore gives a universal exponent $\eta > 0$ such that

$$\int_{B_{3/4}\cap\{v>0\}\cap\{|\nabla v|\geq\bar\delta\}} |D^2 v|^{2+\eta} \, dy \leq C.$$

Decreasing η if necessary, using Hölder's inequality, we may assume that $0 < \eta < n-2$. Since $\nabla v(y) = \nabla u(x_0 + Ry)$, rescaling back gives

$$\int_{B_{3R/4}(x_0)\cap\{u>0\}\cap\{|\nabla u|\geq\bar\delta\}} |D^2 u|^{2+\eta} \, dx \leq C R^{n-2-\eta}. \tag{6.7}$$

Applying Corollary 6.2 on $B_{3R/4}(x_0)$, after translation and rescaling, with $\delta = \bar\delta$ and this choice of η, yields $\mathcal{H}^{n-2-\eta}\big(\operatorname{Sing}(u) \cap B_{3R/4}(x_0)\big) = 0$. The exponent η depends only on the universal parameters, and not on Φ, u, x_0, or R. Finally, a countable collection of such balls covers $\Gamma(u) \cap \Omega$. By countable subadditivity, we deduce (1.7), which completes the proof of Theorem 1.3. □

Remark 6.3. The proof of Theorem 1.3 shows that

$$\mathcal{H}^{n-2}(\operatorname{Sing}(u)) = 0 \qquad \text{for any minimizer } u \text{ of } \mathcal{E}^\Phi(\cdot) \tag{6.8}$$

for Φ in a more general class of integrands $\tilde I_\Lambda$. We say $\Phi \in \tilde I_\Lambda$ whenever Φ satisfies properties (1) and (2) in the definition of $I_{\Lambda,\alpha,\Upsilon}$ in (1.10), i.e. Φ is nonnegative with $\Phi(0) = 0$ and satisfies the ellipticity bounds (1.4), and additionally

$$\Phi \in C^2(\overline{B_{4\sqrt\Lambda}} \setminus B_{1/4\sqrt\Lambda}).$$

Compared to $I_{\Lambda,\alpha,\Upsilon}$, this class of integrands has no uniform modulus of continuity for the Hessian on an annulus. This class in particular contains (a) integrands satisfying (1) and (2) that are globally C^2, and (b) nonnegative 2-homogeneous integrands satisfying the ellipticity assumption (1.4) whose restriction to the sphere is C^2.

To see the claimed statement, first note that for any such integrand, there is some modulus of continuity $\omega : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$\big|D^2 u(x) - D^2 u(y)\big| \leq \omega(|x-y|) \qquad \text{for all } x, y \in \overline{B_\Lambda} \setminus B_{1/\Lambda}. \tag{6.9}$$

Since the conclusion (6.8) does not depend on ω, it thus suffices to prove (6.8) hold for the class of integrands $\tilde{I}^{\omega}_{\Lambda}$ in the class above that additionally satisfy (6.9). Observe that $\tilde{I}^{\omega}_{\Lambda}$ is compact, again with respect to the topology induced by local uniform convergence. Moreover, thanks to (2.2), for any $p \in \mathbb{R}^n$ satisfying the equation $D\Phi(p) \cdot p - \Phi(p) = 1$ corresponding to the free boundary condition (2.6), p lives in the annulus where Φ is C^2. These two properties allow the proof of the ε-regularity theorem, Theorem 4.12, to go through. Since additionally $\tilde{I}^{\omega}_{\Lambda}$ is connected and nonempty and smooth functions are dense in $\tilde{I}^{\omega}_{\Lambda}$, the remainder of the proof of (6.8) goes through verbatim.

It may be possible to prove Theorem 4.12 for the class of integrands satisifying (1) and (2), which would then imply that (6.8) for every such integrand as well. We leave this as an interesting open problem.

Finally, we note that in the proof of Theorem 1.3, we use the quantitative Hölder modulus of continuity of Φ in $\mathcal{A}(\Lambda)$ to prove the η-gain in the improved curvature estimates of Proposition 5.5. Here what is needed is some higher regularity of the free boundary as in Corollary 4.13, and an interior and boundary C^0 estimate for the Hessian, (2.17) and (2.19), where the latter assumes the same regularity of the boundary as what holds at flat points of the free boundary. Thus, any other assumptions on Φ giving rise to such uniform estimate would suffice.

Finally, the proof of Theorem 1.7 is essentially contained in the proof of Theorem 1.3 above:

Proof of Theorem 1.7. The first estimate (1.8) follows from (6.6), with $p = 0$ and $r = 1$. For the second estimate, we cover $\Gamma(u) \cap \overline{B}_{5/4}$ by a bounded number (depending only on n) of balls $B_{1/8}(x_j)$, with centers $x_j \in \Gamma(u) \cap \overline{B}_{5/4}$. Since $B_{1/2}(x_j) \subset B_2$, by (6.6), we can apply Proposition 5.5 to $4u(x_j + y/4)$ in B_2. By rescaling, we get

$$\int_{B_{3/16}(x_j) \cap \{u>0, |\nabla u| \geq \bar{\delta}\}} |D^2 u|^{2+\eta} dx \leq C.$$

We observe that $B_{3/16}(x_j)$ cover every point of B_1 whose distance from $\Gamma(u)$ is less than $1/16$. At the remaining points of $B_1 \cap \{u > 0, |\nabla u| \geq \bar{\delta}\}$, then the interior $C^{1,\beta}$ estimate provides a ball of universal radius on which $|\nabla u| \geq 3\bar{\delta}/4 > r_\Lambda$. By the uniform upper bound on the gradient, ∇u belongs to a fixed compact subannulus of $\mathcal{A}(\Lambda)$. The interior Hessian estimate therefore gives $|D^2 u| \leq C$ on the remaining set. We can just sum the two contributions, we deduce (1.9), since $\bar{\delta} = 2r_\Lambda$. □

7. Extensions of the method and the p-Laplacian case

The arguments used to prove Theorem 1.3 are robust. In this section we show how they can be applied to obtain singular set estimates for the free boundary for minimizers of the Alt-Caffarelli energies of p-Laplace type. Our goal is to demonstrate the method and to show how minor adaptations of the arguments above already yield the best known results on the size of the singular set in this context.

7.1. **A Meta Theorem.** To begin, let us recall in broad strokes what ingredients are needed to run the proof of Theorem 1.3. For a compact and connected collection of integrands I containing $|\cdot|^2$, let I^* denote the (nonempty) subset of those integrands such that any minimizer has $\mathcal{H}^{n-2}(\mathrm{Sing}(u)) = 0$. To show that $I = I^*$, we need:

(1) Geometric estimates, compactness and convergence, and ε-regularity that are uniform for all integrands in I, for minimizers of the base functional $\mathcal{E}^{\Phi}$ as well as the perturbed functional (2.4).

(2) A uniform curvature estimate along the lines of (1.8) that holds for minimizers when $\Phi \in I^*$, which has suitable lower semicontinuity properties and gives rise to an ε-regularity theorem like Lemma 6.1. Again, this should also be shown for the perturbed functional (2.4).

To then upgrade the estimate to $\mathcal{H}^{n-2-\eta}(\mathrm{Sing}(u)) = 0$, we additionally need:

(3) A curvature estimate along the lines of (1.9), which again gives rise to an ε-regularity theorem like Lemma 6.1. This upgraded estimate follows from a doubling estimate like Lemma 5.8, which basically needs uniform C^0 estimates for the Hessian of the interior equation, both in the interior and up to a "nice enough" boundary, and a higher regularity result like Corollary 4.13 giving the corresponding "nice enough" regularity at flat free boundary points.

Indeed, given ingredient (1), and in particular uniform ε-regularity and its standard corollary of propagation of singularities (c.f. Corollary 4.14), the argument of Step 1 of the proof of Theorem 1.3 shows I^* is open in I. To show closedness, let $\Phi_k \in I^*$ with $\Phi_k \to \Phi$ and a minimizer u of $\mathcal{E}^{\Phi}$. Ingredient (1) will yield compactness and convergence lemmas akin to Lemma 4.8 and 4.9. This means that, for the perturbed functionals chosen as in Step 2 of the proof of Theorem 1.3, minimizers u_k of the perturbed functionals will converge to u. Ingredient (2)

guarantees that the u_k satisfy a uniform curvature estimate and that this curvature estimate passes to the limit. Then, the corresponding ε-regularity theorem and the argument of Step 2 of the proof of Theorem 1.3 shows that I^* is closed in I. Finally, Ingredient (3) and the argument of Step 3 of the proof of Theorem 1.3 give the upgraded estimate.

We have already seen that it is possible to deal with cases where Φ degenerates or is singular outside a good set. To do so, it is important to additionally have a "good ball lemma" like Lemma 4.5 showing that ∇u lies in the good set for a definite set of points. It is also important for Φ to be well-behaved in a neighborhood of the slopes satisfying the free boundary condition.

7.2. **The p-Laplacian.** Let us run through this framework in the context of the p-Laplacian counterpart of the Alt-Caffarelli energy, establishing Theorem 1.6. For $p \in (1,\infty)$, consider

$$\mathcal{E}_p(u,\Omega) = \int_\Omega (|\nabla u|^p + \chi_{\{u>0\}})\,dx$$

and its corresponding penalized functional $\mathcal{E}_{p,\mu,f} = \mathcal{E}_p(\cdot) + \int |\mu u - f|^2$ for $\mu \in [0,1]$ and $f \in C^\infty(\Omega)\cap L^\infty(\Omega)$; compare (2.4). A minimizer of $\mathcal{E}_{p,\mu,f}(\cdot)$ with suitably regular free boundary $\Gamma(u)$ satsifies the free boundary problem

$$\begin{cases} \operatorname{div}(|\nabla u|^{p-2}\nabla u) = \frac{2}{p}\mu(\mu u - f) & \text{on } U \\ |\nabla u|^p = \frac{1}{p-1} & \text{on } \Gamma(u). \end{cases} \tag{7.1}$$

Here we let $U = \Omega\cap\{u>0\}$. For bounded right-hand sides, interior $C^{1,\beta}$ estimates for the p-Laplacian follow from [30], while estimates up to the boundary on controlled smooth domains are provided by [65], with estimates just like (2.16) and (2.18). Their constants can be chosen uniformly for p in a fixed compact interval in $(1,\infty)$. On a region where $0<\delta\le|\nabla u|\le L$, the equation is uniformly elliptic. For Hölder right-hand sides, Schauder estimates then give $C^{2,\beta}$ bounds akin to (2.17) and (2.19).

Let $I\subset(1,\infty)$ be a connected compact interval containing 2, and let I^* be the set of those $p\in I$ such that $\mathcal{H}^{n-2}(\operatorname{Sing}(u)) = 0$ for any minimizer u of $\mathcal{E}_p(\cdot)$. Note I^* is nonempty, as $2\in I^*$ by Theorem 2.4. Let us verify that each of the ingredients of the previous section hold in order to prove Theorem 1.6.

Ingredient (1) above has already been established in this setting. Geometric estimates were shown by Petrosyan-Valinoci [71] for the main functional $\mathcal{E}_p(\cdot)$, and as in Section 4, minor adaptations of their proofs extend their results to minimizers of the perturbed functional $\mathcal{E}_{p,\mu,f}(\cdot)$. Minimizers can be shown to be viscosity solutions by arguing as in Proposition 4.11 or as in [32]. For viscosity solutions, ε-regularity was shown by Danielli-Petrosyan [11] with zero right-hand side and with nonzero right-hand side [44]. An inspection of the proofs and a simple contradiction argument shows that all estimates are uniform for p in the compact set I and for $\|f\|_{L^\infty}\le C$; in fact, this is already made explicit in [44]. The geometric estimates also give rise to compactness and convergence lemmas by repeating the proofs of Lemma 4.8 and 4.9.

We need to establish Ingredient (2). The key curvature estimate is the following, compare to Proposition 5.1:

Proposition 7.1. *For $p\in I^*$, $\mu\in[0,1]$, and $f\in C^\infty(\Omega)\cap L^\infty(\Omega)$ nonnegative, there exist $C>0$ depending on n, I, and an upper bound for $\|f\|_{L^\infty(\Omega)}$ such that if u is a minimizer of $\mathcal{E}_{p,\mu,f}$ in Ω, then for all $x_0\in\Gamma(u)$ and $r\in(0,1]$ with $B_{2r}(x_0)\subset\Omega$, we have*

$$\int_{B(x_0,r)\cap\{u>0\}\cap\{|\nabla u|>0\}} |D^2u|^2|\nabla u|^{p-2+(p-2)_+} \le Cr^{n-2}. \tag{7.2}$$

We prove Proposition 7.1 in Section 7.3 below. The weight appearing on the left-hand side of (7.2) is natural in the p-Laplacian setting. When $p\in(1,2)$, this weight is bounded below thanks to the Lipschitz bound, but can be genuinely degenerate in the case $p>2$.

Nevertheless, the estimate (7.2) is lower semicontinuous with respect to local H^1 and local uniform convergence, and gives rise to an ε-regularity lemma like Lemma 6.1, arguing along the same lines as we have above.

Indeed, suppose $u_k\to u$ in B_2 the sense of Lemma 4.8 and 4.9, suppose $0\in\Gamma$, and suppose each u_k satisfies the conclusion of Proposition 7.1 in B_2. We claim that (7.2) holds for u with $x_0=0$ and $r=1$.

To see this, we argue as in Step 2 of the proof of Lemma 5.8 and the proof of Lemma 6.2. For $\delta>0$, let $E_\delta = B_1\cap\{|\nabla u|>\delta\}\cap\{u>0\}$, and fix open sets $D\Subset D'\Subset E_\delta$. Since $u_k\to u$ locally uniformly, interior

$C^{1,\beta}$ estimates for the p-Laplacian also guarantee that $\nabla u_k \to \nabla u$ locally uniformly. Hence for k large enough, $|\nabla u_k| > \delta$ and $u_k > 0$ on D'. In turn, uniform $C^{2,\beta}$ estimates away from the critical set show that

$$\int_D |\nabla u|^{p-2+(p-2)_+} |D^2 u|^2 = \lim_{k\to\infty} \int_D |\nabla u_k|^{p-2+(p-2)_+} |D^2 u_k|^2 \le C. \tag{7.3}$$

In the second inequality we apply (7.2) to u_k with $x_0 = x_k$ and $r = 1 - o_k(1)$. Taking an exhaustion of E_δ extends the domain of integration to E_δ on the left-hand side of (7.3). Since this holds for all $\delta > 0$, we obtain the claim.

The ε-regularity lemma akin to Lemma 6.1 is deduced from the ε-regularity of [11, 44] by repeating this argument, as we did in the proof of Lemma 6.1. In identifying the limiting minimizer, we use the localized version of Lemma 5.6: on each positivity component, ∇u is continuous, while $W^{2,1}_{\mathrm{loc}}$ regularity is needed only where $|\nabla u| > \delta$. Lemma 7.2 below ensures that this set is nonempty. Therefore, Ingredient (2) holds. Thus $I = I^*$.

This shows that $\mathcal{H}^{n-2}(\mathrm{Sing}(u)) = 0$ for a minimizer of $\mathcal{E}_p(\cdot)$ for any $p \in I$. To improve the estimate to $\mathcal{H}^{n-2-\eta}(\mathrm{Sing}(u)) = 0$, where η depends on n and I, we need Ingredient (3). The same challenge arises here as arose in our main theorem: the interior equation only admits uniform C^2 estimates outside a neighborhood of the critical set. (Thanks to the Lipschitz estimate [71], we do not have to worry about degeneracy when the gradient blows up.) This challenge can be dealt with in the same way, with a "good ball lemma" like Lemma 4.5:

Lemma 7.2. *Suppose that u minimizes $\mathcal{E}_p$ in B_1 and $0 \in \Gamma(u)$. There is a $\bar\delta = \bar\delta(n, I)$ such that every connected component V of $B_1 \cap \{u > 0\}$ that intersects $B_{1/8}$ contains a ball $B_{\bar\delta}(y_V) \subset V \cap B_{1/2} \cap \{|\nabla u| \ge 8\bar\delta\}$.*

This lemma is proven using the uniform Lipschitz, nondegeneracy, and interior $C^{1,\beta}$ estimates via exactly the same argument as the proof of Lemma 4.5. In view of the uniform interior and boundary $C^{2,\beta}$ estimates that hold away from the critical set, Ingredient (3) is proven exactly as Proposition 5.5. We thus see that Theorem 1.6 holds.

7.3. **Proof of the curvature estimate.** As we have seen above, the new estimate needed for Theorem 1.6 is Proposition 7.1. We now prove the proposition. Like Proposition 5.1, its proof is based on the stability inequality and a positive supersolution to the linearized equation. Similar computations to those in Appendix A show that a minimizer of $\mathcal{E}_{p,\mu,f}(\cdot)$ in Ω satisfies the following stability inequality:

$$\int_{\{u>0\}\cap\{|\nabla u|>0\}} A(\nabla u)[\nabla\xi, \nabla\xi] + \frac{2}{p}\mu^2\xi^2\, dx \ge -(p-1)^{3/p} \int_{\Gamma(u)} \partial_{\nu\nu} u\, \xi^2\, d\mathcal{H}^{n-1} \qquad \text{for all } \xi \in C^1_c(\Omega\setminus \mathrm{Sing}(u)). \tag{7.4}$$

Here and below, ν is the outer unit normal to $\{u > 0\}$ and

$$A(\nabla u) = |\nabla u|^{p-2}\Big(\mathrm{Id} + (p-2)\frac{\nabla u}{|\nabla u|} \otimes \frac{\nabla u}{|\nabla u|}\Big). \tag{7.5}$$

Proof of Proposition 7.1. The basic scheme of the proof is the same as the in Proposition 5.1: a power of $|\nabla u|$ is a subsolution to the linearized equation and its right-hand side controls the norm of the Hessian. We test this equation with a cutoff function and relate it to the stability inequality. The case $p \ge 2$ is essentially the same as Proposition 5.1, while $p < 2$ requires a more refined analysis using the Kato-type inequality (7.14) below. In what follows, C and c denote positive constants that may change from line to line and depends on n, I, and $\|f\|_{L^\infty(B_2)}$. We will also write A as shorthand for $A(\nabla u)$.

Step 0: Set up. By scaling, we may assume $x_0 = 0$ and $r = 1$, so u minimizes $\mathcal{E}_{p,\mu,f}(\cdot)$ on B_2. Thanks to the Lipschitz estimates, we may assume $|\nabla u| \le L$ in $B_{3/2}$. Let $U = \{u > 0\} \cap B_{3/2}$, $\Gamma = \partial U \cap B_{3/2}$, and let ν denote the outer unit normal to U. We will argue separately for $p \ge 2$ and $p < 2$, starting with the simpler case $p \ge 2$.

Step 1 for $p \ge 2$: Cutoff function vanishing on $\mathrm{Sing}(u)$. In this step, let $p \ge 2$ and let $G = |\nabla u|^p$. For $\varepsilon > 0$, let $U_\varepsilon = U \cap \{G > \varepsilon\}$ be the superlevel sets of G. Direct computation a la Lemma 5.2 shows that for each $\varepsilon > 0$,

$$\begin{cases} \mathrm{div}(A(\nabla u)\nabla G) \ge C|\nabla u|^{2(p-2)}|D^2 u|^2 - 2\mu|\nabla u|^{p-2}\nabla u\cdot\nabla f & \text{on } U_\varepsilon \\ \langle A\nabla G, \nu\rangle = -p(p-1)^{(3-p)/p}\partial_{\nu\nu} u & \text{on } \Gamma(u), \end{cases} \tag{7.6}$$

Let $\varphi \in C^1_c(B_{3/2}\setminus \mathrm{Sing}(u))$. Let $\varepsilon \ll 1$ be a regular value of G so that $\partial U_\varepsilon \cap U$ is C^1 and U_ε is a set of finite perimeter. Let $\eta : \mathbb{R}_+ \to \mathbb{R}$ be an increasing function such that $\eta(t) = 0$ for $t \le \varepsilon$ and $\eta \equiv 1$ on $\{t > \frac{1}{2(p-1)}\}$. We multiply $\mathrm{div}(A(\nabla u)\nabla G)$ by

$$\varphi^2\, \eta\circ G$$

on U_ε and integrate by parts. The portion of boundary term on $\partial U_\varepsilon \cap U$ vanishes since $\eta(G(x)) = \eta(\varepsilon)$ is zero there. Moreover, for all $x \in \Gamma \cap \mathrm{spt}\varphi$ we have $x \in U_\varepsilon$ and $\eta(G(x)) = 1$, thanks to the assumption that φ vanishes on $\mathrm{Sing}(u)$ and the free boundary condition $G(x) = 1/(p-1)$ on $\mathrm{Reg}(u)$. Thus, writing $A = A(\nabla u)$, we obtain

$$\begin{aligned}\int_{U_\varepsilon} \varphi^2 \eta(G)\,\mathrm{div}(A\nabla G) &= -\int_{U_\varepsilon} \varphi^2\,\eta'(G)\,\langle A\nabla G, \nabla G\rangle - \int_{U_\varepsilon} 2\eta(G)\varphi\,\langle A\nabla G, \nabla\varphi\rangle + \int_\Gamma \varphi^2 A[\nabla G, \nu]\,d\mathcal{H}^{n-1} \\ &= I + II + III.\end{aligned} \tag{7.7}$$

We bound the right-hand side above. Term I has a sign that allows us to discard it: since $\eta' \geq 0$ and $\langle Aq, q\rangle \geq 0$, we see that $I \leq 0$. For term II, keeping in mind (7.5), we have $|A\nabla G| \leq C|\nabla u|^{2p-3}|D^2 u|$. So, for $\delta > 0$ small to be specified later and using $\eta \leq 1$ and the Lipschitz bound for u,

$$|II| \leq C\delta \int_{U_\varepsilon} \eta(G)\varphi^2 |\nabla u|^{2(p-2)}|D^2 u|^2 + \frac{C}{\delta}\int_U |\nabla\varphi|^2. \tag{7.8}$$

The boundary condition for G in (7.6) shows that III is the left-hand side of the stability inequality (7.4), so

$$III = -C\int_\Gamma \partial_{\nu\nu} u\,\varphi^2 \leq C\int_U A[\nabla\varphi, \nabla\varphi] \leq C\int_U |\nabla\varphi|^2\,.$$

In the final inequality, we used $p - 2 \geq 0$ and the Lipchitz bound for u to say $A[\nabla\varphi, \nabla\varphi] \leq C|\nabla\varphi|^2$. Next, we bound the left-hand side of (7.7) below using (7.6). Choosing δ to absorb the first term on the right-hand side of (7.8) into this lower bound, exactly as in the proof of Proposition 5.1, we obtain

$$\int_{U_\varepsilon} \eta(G)\varphi^2 |\nabla u|^{2(p-2)}|D^2 u|^2 \leq C\int_{U_\varepsilon} \eta(G)\varphi^2\,|\nabla u|^{p-2}\nabla u\cdot\nabla f\,dx + C\int_U |\nabla\varphi|^2\,dx$$

Now, take an increasing sequence η_k tending to $\chi_{(\varepsilon,\infty)}$ and then send $\varepsilon \to 0$. Using dominated convergence theorem for the right-hand side and monotone convergence theorem on the left-hand side, we find

$$\int_U \varphi^2 |D^2 u|^2 |\nabla u|^{2(p-2)} \leq C\int_U \varphi^2\,|\nabla u|^{p-2}\nabla u\cdot\nabla f\,dx + C\int_U |\nabla\varphi|^2\,dx\,. \tag{7.9}$$

Finally, integrating by parts in the first term on the right-hand side and bounding this term precisely as in (5.9) in the proof of Proposition 5.1, using $f \geq 0$ and the interior equation for u, we find

$$\int_U \varphi^2\,|\nabla u|^{p-2}\nabla u\cdot\nabla f \leq 2\mu\int_U \varphi^2 f^2 - 2\int_U \varphi f\,|\nabla u|^{p-2}\nabla u\cdot\nabla\varphi \leq C\int_U (\varphi^2 f^2 + |\nabla u|^{2(p-1)}|\nabla\varphi|^2)\,dx. \tag{7.10}$$

Combined with (7.9), this gives that, for all $\varphi \in C^1_c(B_{3/2}\setminus \mathrm{Sing}(u))$,

$$\int_U \varphi^2 |D^2 u|^2 |\nabla u|^{2(p-2)} \leq C\int_U \varphi^2 + |\nabla\varphi|^2\,dx\,. \tag{7.11}$$

Step 1 for $1 < p < 2$: Cancellation with a linear truncation. Let $\varphi \in C^1_c(B_{3/2}\setminus\mathrm{Sing}(u))$. We denote

$$Q = \sum_{i=1}^n A[\nabla u_i, \nabla u_i] - A[\nabla|\nabla u|, \nabla|\nabla u|] \geq 0.$$

Differentiating (7.1) we get

$$\begin{aligned}\mathrm{div}(A\nabla|\nabla u|) &= \frac{Q}{|\nabla u|} + \frac{2\mu^2}{p}|\nabla u| - \frac{2\mu}{p}\frac{\nabla u}{|\nabla u|}\cdot\nabla f \quad \text{in } U\setminus\{|\nabla u|(x) = 0\}, \\ A[\nabla|\nabla u|, \nu] &= -(p-1)^{3/p} u_{\nu\nu}|\nabla u| \quad \text{on } \Gamma.\end{aligned} \tag{7.12}$$

We choose regular values $\varepsilon \downarrow 0$ with $\varepsilon < (p-1)^{-1/p}/2$, and set $U_\varepsilon = U \cap \{|\nabla u|(x) > \varepsilon\}$, and $h_\varepsilon = (|\nabla u| - \varepsilon)_+$. We multiply (7.12) by $\varphi^2 h_\varepsilon$, integrate over U_ε, and apply (7.4) with $\xi = \varphi h_\varepsilon$. The stability test can be extended across the regular free boundary; its support in the positive phase stays away from $\{|\nabla u|(x) = 0\}$. The boundary term on $\{|\nabla u| = \varepsilon\}$ vanishes. Since $\nabla h_\varepsilon = \nabla|\nabla u|$ in U_ε, the terms containing $A[\nabla|\nabla u|, \nabla|\nabla u|]$ and the mixed gradient terms cancel exactly. Using $|\nabla u| = h_\varepsilon + \varepsilon$, we obtain

$$\int_{U_\varepsilon} \varphi^2 \frac{h_\varepsilon}{|\nabla u|}Q + \frac{2\mu^2\varepsilon}{p}\int_{U_\varepsilon}\varphi^2 h_\varepsilon \leq \int_{U_\varepsilon} h_\varepsilon^2 A[\nabla\varphi, \nabla\varphi] + \frac{2\mu}{p}\int_{U_\varepsilon}\varphi^2 h_\varepsilon\frac{\nabla u}{|\nabla u|}\cdot\nabla f + \varepsilon\int_\Gamma -(p-1)^{3/p}u_{\nu\nu}h_\varepsilon\varphi^2 d\mathcal{H}^{n-1}.$$

The last term tends to zero, since on Γ we have that $-(p-1)^{3/p}u_{\nu\nu}$ is integrable and $h_\varepsilon = (p-1)^{-1/p} - \varepsilon$. Moreover,

$$h_\varepsilon^2 A[\nabla\varphi, \nabla\varphi] \leq |\nabla u|^p |\nabla\varphi|^2.$$

Discarding the nonnegative zero-order term on the left-hand side, and passing to the limit by monotone and dominated convergence we deduce

$$\int_{U\setminus\{|\nabla u|(x)=0\}} \varphi^2 Q \le \int_{U\setminus\{|\nabla u|(x)=0\}} |\nabla u|^2 A[\nabla\varphi,\nabla\varphi] + \frac{2\mu}{p}\int_U \varphi^2 \nabla u\cdot\nabla f. \tag{7.13}$$

A direct computation, which we postpone until the end of the proof, yields the Kato-type inequality since $\Delta_p u = g := \frac{2}{p}\mu(\mu u - f)$:

$$Q \ge c_p |\nabla u|^{p-2}|D^2 u|^2 - |\nabla u|^{2-p} g^2, \qquad c_p = \frac{(p-1)^2}{2n-1}. \tag{7.14}$$

In particular, since u, $|\nabla u|$, and f are bounded and $2-p>0$, (7.13) and (7.14) imply that

$$\int_{U\setminus\{|\nabla u|(x)=0\}} \varphi^2 |\nabla u|^{p-2}|D^2 u|^2 < \infty. \tag{7.15}$$

We now remove the dependence on ∇f in (7.13). Integrating by parts, we know that

$$\int_{U_\varepsilon} \varphi^2 h_\varepsilon \frac{\nabla u}{|\nabla u|}\cdot\nabla f = -\int_{U_\varepsilon} f\varphi^2 \operatorname{div}(h_\varepsilon\frac{\nabla u}{|\nabla u|}) - 2\int_{U_\varepsilon} f\varphi h_\varepsilon \frac{\nabla u}{|\nabla u|}\cdot\nabla\varphi - ((p-1)^{-1/p}-\varepsilon)\int_\Gamma f\varphi^2\, d\mathcal{H}^{n-1},$$

as $h_\varepsilon = 0$ on $\{|\nabla u| = \varepsilon\}$ and the outward normal component of $h_\varepsilon \nabla u/|\nabla u|$ on Γ is $-((p-1)^{-1/p}-\varepsilon)$. Using that

$$\operatorname{div}(h_\varepsilon\frac{\nabla u}{|\nabla u|}) = \left(1-\frac{\varepsilon}{|\nabla u|}\right)\Delta u + \frac{\varepsilon}{|\nabla u|} D^2 u[\frac{\nabla u}{|\nabla u|},\frac{\nabla u}{|\nabla u|}] \quad \text{in } U_\varepsilon,$$

and recalling that $f\ge 0$, we obtain

$$\int_{U_\varepsilon} \varphi^2 h_\varepsilon \frac{\nabla u}{|\nabla u|}\cdot\nabla f \le \sqrt{n}\,\|f\|_\infty \int_{U_\varepsilon}\varphi^2|D^2u| + 2\|f\|_\infty\int_{U_\varepsilon}|\varphi||\nabla\varphi||\nabla u|.$$

Thanks to (7.15), we can pass the latter inequality to the limit as $\varepsilon\to 0$ and deduce

$$\int_U \varphi^2\nabla u\cdot\nabla f \le \sqrt{n}\,\|f\|_\infty \int_{U\setminus\{|\nabla u|(x)=0\}}\varphi^2|D^2u| + 2\|f\|_\infty\int_U|\varphi||\nabla u||\nabla\varphi|,$$

which in turn, by Young's inequality and the bound $|\nabla u|\le L$, provides

$$\frac{2\mu}{p}\int_U \varphi^2\nabla u\cdot\nabla f \le \frac{c_p}{2}\int_{U\setminus\{|\nabla u|(x)=0\}}\varphi^2|\nabla u|^{p-2}|D^2u|^2 + C\int_U(\varphi^2+|\nabla\varphi|^2).$$

Combining this with (7.13) and (7.14) proves

$$\int_{U\setminus\{|\nabla u|(x)=0\}}\varphi^2|\nabla u|^{p-2}|D^2u|^2 \le C\int_U(\varphi^2+|\nabla\varphi|^2), \tag{7.16}$$

where the constant C depends only on n, I, and $\|f\|_\infty$.

Step 3: Capacity argument. We now unify the cases for $p\ge 2$ and $p<2$. Arguing as in Step 2 of the proof of Proposition 5.1, we can localize a cutoff function around the singular set to extend (7.11) and (7.16) to general $\varphi\in C^1_c(B_{5/4})$. Indeed, by the same blowup argument as Lemma 5.3 and the assumption $\Phi\in I^*$, we know $\mathcal{H}^{n-2}(\operatorname{Sing}(u)\cap B_2)=0$. The set $\Sigma = \operatorname{Sing}(u)\cap\overline{B}_{5/4}$ is compact thanks to epsilon regularity [11, 44]. So, since $\mathcal{H}^{n-2}(\Sigma)=0$, we can repeat the construction of the proof of Proposition 5.1: for any $\varepsilon>0$, we obtain a cutoff function ψ vanishing on $\operatorname{Sing}(u)$ so that for any $\varphi\in C^\infty_c(B_2)$ with support contained in $B_{5/4}$, we have

$$|\nabla(\psi\varphi)(x)|^2 \le 2|\nabla\varphi(x)|^2 + 2|\varphi|^2\sum_k r_k^{-2}\chi_{B_{r_k}(z_k)}, \tag{7.17}$$

and $\psi\varphi\in C^1_c(B_{3/2}\setminus\operatorname{Sing}(u))$, and $\psi\varphi\to\varphi$ a.e. as $\varepsilon\to 0$. This penultimate point means that (7.16) holds for $\zeta=\psi\varphi$, so

$$\int_U(\psi\varphi)^2|D^2u|^2|\nabla u|^{p-2+(p-2)_+} \le C\int_U|\nabla\varphi|^2\,dx + C\|\varphi\|^2_{L^\infty}(\sum_k r_k^{n-2}+1)$$

Take a sequence $\varepsilon_k\to 0$ and the corresponding $\psi=\psi_{\varepsilon_k}$ and $\zeta_k=\psi_{\varepsilon_k}\varphi$, applying Fatou's lemma to pass to the limit on the left-hand side. Choosing a cutoff φ with $\varphi\equiv 1$ in B_1 and $|\nabla\varphi|\le C(n)$, we conclude.

Step 4: Proof of the Kato-type inequality. It remains to show (7.14). At a point $x \in U \setminus \{|\nabla u| = 0\}$, choose an orthonormal basis such that $\nu = -\nabla u/|\nabla u| = e_1$. This way, we have $\sum |\nabla u_k \cdot \nu|^2 = \sum(\partial_k|\nabla u|)^2 = |\nabla|\nabla u||^2$ and $|\nabla|\nabla u| \cdot \nu|^2 = u_{11}^2$. Hence,

$$\begin{aligned}\sum_k \langle A\nabla u_k, \nabla u_k\rangle - \langle A\nabla|\nabla u|, \nabla|\nabla u|\rangle &= |\nabla u|^{p-2}\Big\{ \sum_{j,k\geq 1} u_{jk}^2 - \sum_{k\geq 1} u_{1k}^2 + (p-2)\Big(\sum_{k\geq 1} u_{1k}^2 - u_{11}^2\Big)\Big\} \\ &= |\nabla u|^{p-2}\Big\{ \sum_{j,k>1} u_{jk}^2 + (p-1)\sum_{k>1} u_{1k}^2\Big\}.\end{aligned} \tag{7.18}$$

The interior equation in (7.1) in this basis becomes $g = \operatorname{div}(|\nabla u|^{p-2}\nabla u) = |\nabla u|^{p-2}(\Delta u + (p-2)u_{11})$, or, after rearranging,

$$(p-1)\, u_{11} = -\sum_{k>1} u_{kk} + g\,|\nabla u|^{2-p}.$$

Squaring this expression and applying Cauchy–Schwarz inequality then yields

$$u_{11}^2 \leq \frac{2(n-1)}{(p-1)^2}\sum_{j,k>1} u_{jk}^2 + \frac{2}{(p-1)^2}\, g^2|\nabla u|^{4-2p}.$$

In turn, we find that

$$|D^2u|^2 = \sum_{j,k>1} u_{jk}^2 + 2\sum_{k>1} u_{1k}^2 + u_{11}^2 \leq \Big(1 + \frac{2(n-1)}{(p-1)^2}\Big)\sum_{j,k>1} u_{jk}^2 + 2\sum_{k>1} u_{1k}^2 + \frac{2}{(p-1)^2}\, g^2|\nabla u|^{4-2p}. \tag{7.19}$$

As above, we let $c_p := \frac{(p-1)^2}{2n-1}$. Since $1 < p \leq 2$ and $n \geq 2$, we have $c_p(1 + \frac{2(n-1)}{(p-1)^2}) = \frac{(p-1)^2+2(n-1)}{2n-1} \leq 1$. Moreover, $2c_p \leq p-1$ and $\frac{2c_p}{(p-1)^2} = \frac{2}{2n-1} \leq 1$. Therefore, multiplying (7.19) by $c_p|\nabla u|^{p-2}$ gives us

$$c_p|\nabla u|^{p-2}|D^2u|^2 \leq |\nabla u|^{p-2}\Big\{ \sum_{j,k>1} u_{jk}^2 + (p-1)\sum_{k>1} u_{1k}^2\Big\} + g^2|\nabla u|^{2-p}.$$

Together with (7.18), this gives (7.14). □

7.4. **Further Applications.** The framework above suggests extensions beyond the standard p-Laplacian. Indeed, there are several classes of integrands for which a singular set bound like Theorem 1.3 is an open problem, and for which we believe the meta theorem above may be applicable. Proving such a theorem via this method in these settings would require establisxhing Ingredients (1)-(3) uniformly along a compact, connected family containing a quadratic integrand, together with an appropriate good-ball lemma.

A first class consists of anisotropic Orlicz energies

$$\Phi(\xi) = G(F(\xi)), \qquad 0 < m \leq \frac{tG''(t)}{G'(t)} \leq M < \infty \quad (t > 0),$$

where F is a sufficiently smooth norm away from the origin, $D^2(F^2/2)$ is uniformly elliptic there, and $G(0) = G'(0) = 0$, with $G' > 0$ and sufficient regularity on $(0, \infty)$. This includes smooth anisotropic p-energies $F(\xi)^p$, as well as the nonhomogeneous examples $|\xi|^p + a|\xi|^q$, with $a \geq 0$ and $1 < p \leq q < \infty$. Suitable logarithmic modifications of power growth provide further candidates. The geometric estimates of [71] and the radial Orlicz free-boundary theory of [67] provide relevant starting points. Uniform estimates for the penalized problems and the required weighted curvature bounds would still have to be established.

A related direction concerns spatially dependent energies, for example $\int_\Omega (|\nabla u|^{p(x)}/p(x) + \lambda(x)\chi_{\{u>0\}})\, dx$, with sufficiently regular $p(x)$ bounded away from 1 and infinity, and positive $\lambda(x)$ bounded away from zero. Blow-ups freeze these coefficients, hence the Hausdorff dimension estimate transfers the corresponding constant-exponent singular-set estimates, provided compactness to minimizers of the frozen functionals and uniform propagation of singularities hold. This approach would not require a new curvature estimate for the variable-coefficient equation; see [44] for the variable-exponent improvement-of-flatness theory.

The orthotropic, or pseudo-p-Laplacian, associated with $\Phi(\xi) = \sum_{i=1}^n |\xi_i|^p$, presents a different difficulty. For $p \neq 2$, its Hessian is degenerate or singular on coordinate hyperplanes. These hyperplanes contain nonzero gradients satisfying the Bernoulli condition $D\Phi(\xi)\cdot\xi - \Phi(\xi) = (p-1)\sum_{i=1}^n |\xi_i|^p = 1$; for instance, $\xi = (p-1)^{-1/p}e_1$. Thus, a lower bound for $|\nabla u|$ alone does not provide the ellipticity required in Ingredient (3). An extension to this setting would require curvature and regularity estimates adapted to the coordinatewise degeneracy, rather than a direct adaptation of the argument above.

Appendix A. First and second variation

Let $f \in C^\infty(\Omega)$ be a nonnegative function and $\mu \in [0,1]$. We compute the first and second variations of the functional $\mathcal{E}^\Phi_{f,\mu}$ defined in (2.4) at a minimizer of the functional. To make it apparent where derivatives fall, we will use the notation

$$A(p) = D\Phi(p) \qquad \text{and} \qquad Q(p) = D^2\Phi(p).$$

Taking outer variations $u_\varepsilon = u + \varepsilon\varphi$ for $\varphi \in C^\infty_c(U)$ shows that a minimizer u of $\mathcal{E}^\Phi_{f,\mu}$ satisfies

$$-\operatorname{div}(A(\nabla u)) + 2\mu(\mu u - f) = 0 \quad \text{in } \{u > 0\} \cap \Omega\,. \tag{A.1}$$

Throughout the section, we assume $u \in W^{1,2}(\Omega)$ is continuous and nonnegative. Letting $U = \{u > 0\} \cap \Omega$, we assume that $\Gamma = \partial U \cap \Omega$ is of class C^2 and that u solves (A.1). We let ν denote the outer unit normal to U.

Let $\{\Psi_t\}_{t\in(-\varepsilon,\varepsilon)}$ be a smooth one-parameter family of diffeomorphisms on $\mathbb{R}^n$ with $\{\Psi_t \neq Id\} \subset\subset \Omega$ for all $t \in (-\varepsilon,\varepsilon)$ and $\Psi_0 = Id$. Let $X = \frac{d}{dt}|_{t=0}\Psi_t$ denote the initial velocity of this flow, and let $U_t := \Psi_t(U)$ and $\Gamma_t = \Psi_t(\Gamma)$ so that $\partial U_t \cap \Omega = \Gamma_t$. Let u_t be the unique weak solution to

$$-\operatorname{div}(A(\nabla u_t)) + 2\mu(\mu u_t - f) = 0 \qquad \text{in } U_t$$

with $u_t = 0$ on Γ and $u_t = u$ on $\partial\Omega$.

Lemma A.1. *In the setup above,*

$$\frac{d}{dt}\Big|_{t=0} \mathcal{E}^\Phi_{f,\mu}(u_t) = \int_\Gamma \big(1 + \Phi(\nabla u) - A(\nabla u)\cdot\nabla u\big)\, X\cdot\nu. \tag{A.2}$$

In particular, if u is a minimizer of $\mathcal{E}^\Phi_{f,\mu}$, then $A(\nabla u)\cdot\nabla u - \Phi(\nabla u) = 1$ on Γ.

Before showing Lemma A.1, we collect a few facts. To write the first variation as an integral over Γ rather than U, we will use an anisotropic version of the Rellich identity: for $u \in C^2(U)$ and for a C^1 vector field X,

$$\operatorname{div} X\, \Phi(\nabla u) - DX^T\nabla u\cdot A(\nabla u) = \operatorname{div}\big(\Phi(\nabla u)X - (X\cdot\nabla u)A(\nabla u)\big) + (X\cdot\nabla u)\operatorname{div}(A(\nabla u)). \tag{A.3}$$

The proof of (A.3) is a direct computation:

$$\begin{aligned}&\operatorname{div}\big(\Phi(\nabla u)X - (X\cdot\nabla u)A(\nabla u)\big)\\ &\quad= \Phi(\nabla u)\operatorname{div} X + D^2u(A(\nabla u), X) - D^2u(X, A(\nabla u)) - DX^T\nabla u\cdot A(\nabla u) - (X\cdot\nabla u)\operatorname{div}(A(\nabla u)).\end{aligned}$$

We also observe that

$$D\Psi_t = Id + tDX + o(t) \qquad \text{and} \qquad (D\Psi_t)^{-1} = Id - tDX + o(t), \tag{A.4}$$

and recall that the Jacobian $J\Psi_t = |\det D\Psi_t|$ has

$$\frac{d}{dt}\Big|_{t=0} J\Psi_t = \operatorname{div} X, \tag{A.5}$$

see, e.g. [66, Lemma 17.4]. With these tools in hand, we can prove the lemma:

Proof of Lemma A.1. We compute derivatives of the three terms in the energy separately. First, by the area formula, we have $\int_\Omega \chi_{\{u_t>0\}}\,dx = \int_U J\Phi_t(y)\,dy$, and so (A.5) gives

$$\frac{d}{dt}\big|_{t=0}\int_\Omega \chi_{\{u_t>0\}}\,dx = \int_U \operatorname{div} X dy = \int_\Gamma X\cdot\nu. \tag{A.6}$$

Now, let $\tilde u_t(x) = u_t\circ\Psi_t$ be the pullback of u_t, so that the positivity set of $\tilde u_t$ is U. By the chain rule, we have

$$\nabla u_t|_{\Psi_t(x)} = D\Psi_t^{-T}(x)\,\nabla\tilde u_t(x) \qquad \text{and} \qquad \frac{d}{dt}\Big|_{t=0}\tilde u_t = \dot u + \nabla u\cdot X \tag{A.7}$$

where we let $\dot u = \frac{d}{dt}|_{t=0}u_t$. Therefore, applying the area formula followed by the first identity in (A.7), we write the first term in the energy as

$$\int_{U_t}\Phi(\nabla u_t(y))\,dy = \int_{U_t}\Phi\big(\nabla u_t|_{\Psi_t\circ\Psi_t(y)}\big)\,dy = \int_U \Phi\big(\nabla u_t|_{\Psi_t(x)}\big)J\Psi_t(x)\,dx = \int\Phi\big(D\Psi_t^{-T}(x)\nabla\tilde u_t(x)\big)\,J\Psi_t(x)\,dx.$$

From the second term in (A.4), we have $\frac{d}{dt}|_{t=0}\,D\Psi_t^{-T} = -DX^T$. Therefore, letting $w = \frac{d}{dt}\big|_{t=0}\tilde u_t$, we have

$$\frac{d}{dt}\Big|_{t=0} D\Psi_t^{-T}(x)\nabla\tilde u_t(x) = -DX^T\nabla u + \nabla w\,. \tag{A.8}$$

Hence, using this and (A.5) followed by the Rellich identity (A.3), we have

$$\begin{aligned}\frac{d}{dt}\Big|_{t=0}\int_{U_t}\Phi(\nabla u_t)\,dx &= \int_U \operatorname{div} X\,\Phi(\nabla u) - A(\nabla u)\cdot DX^T\nabla u + A(\nabla u)\cdot\nabla w\\ &= \int_U \operatorname{div}\big(\Phi(\nabla u)X - (X\cdot\nabla u)A(\nabla u)\big) + (X\cdot\nabla u)\operatorname{div}(A(\nabla u)) + A(\nabla u)\cdot\nabla w\\ &= \int_\Gamma \big(\Phi(\nabla u)X - (X\cdot\nabla u)A(\nabla u)\big)\cdot\nu + \int_U (X\cdot\nabla u - w)\operatorname{div}(A(\nabla u))\,.\end{aligned} \tag{A.9}$$

In the final identity, we applied the divergence theorem and integrated by parts in the final bulk term. Note that the integration by parts produces no boundary term, since $\tilde{u}_t(x) = u_t\circ\Phi_t(x) = 0$ on Γ for all t, so differentiating shows $w(x) = 0$ for all $x\in\Gamma$.

We may simplify the boundary term in (A.9). Since u is constant on Γ, we have $\nabla u = \partial_\nu u\,\nu$, thus

$$(\nabla u\cdot X)(A(\nabla u)\cdot\nu) = (\partial_\nu u)(X\cdot\nu)(A(\nabla u)\cdot\nu) = (X\cdot\nu)(A(\nabla u)\cdot\nabla u) \tag{A.10}$$

Combining this with (A.9) and using $\operatorname{div}(A(\nabla u)) = 2\mu(\mu u - f)$ and the second identity in (A.7), we have

$$\frac{d}{dt}\Big|_{t=0}\int_{U_t}\Phi(\nabla u_t)\,dx = \int_\Gamma (\Phi(\nabla u) - A(\nabla u)\cdot\nabla u)\,X\cdot\nu - 2\mu\int_U(\mu u - f)\frac{d}{dt}\Big|_{t=0}u_t\,. \tag{A.11}$$

Finally, we now differentiate the last term in the energy:

$$\frac{d}{dt}\Big|_{t=0}\int_\Omega(\mu u_t - f)^2\,dx = 2\mu\int_\Omega(\mu u_t - f)\frac{d}{dt}\Big|_{t=0}u_t(x)\,dx = 2\mu\int_U(\mu u_t - f)\frac{d}{dt}\Big|_{t=0}u_t(x)\,dx \tag{A.12}$$

In the last identity we use that $\frac{d}{dt}|_{t=0}u_t = 0$ on $\Omega\setminus U$. Combining (A.6), (A.11), and (A.12) completes the proof. □

Now we compute the second variation of $\mathcal{E}^\Phi_{f,\mu}$ at a critical point.

Lemma A.2. *Let u, u_t be as in the setup above, and assume further that $A(\nabla u)\cdot\nabla u - \Phi(\nabla u) = 1$ on Γ. Then*

$$\frac{d^2}{dt^2}\Big|_{t=0}\mathcal{E}^\Phi_{f,\mu}(u_t) = \int_U Q(\nabla u)[\nabla\dot{u},\nabla\dot{u}] + 2\mu^2\dot{u}^2\,dx - \int_\Gamma\big\langle Q(\nabla u)D^2u[\nu],\nu\big\rangle\frac{1+\Phi(\nabla u)}{(A(\nabla u)\cdot\nu)}\,(X\cdot\nu)^2. \tag{A.13}$$

where as above $\dot{u} = \frac{d}{dt}|_{t=0}u_t$. If u is a minimizer of $\mathcal{E}^\Phi_{f,\mu}$, then for all $\psi\in C^\infty(\Omega)$,

$$\int_U Q(\nabla u)[\nabla\psi,\nabla\psi] + 2\mu^2\psi^2\,dx \geq \int_\Gamma\big\langle D^2\Phi(\nabla u)D^2u[\nu],\nu\big\rangle\frac{(D\Phi(\nabla u)\cdot\nu)}{1+\Phi(\nabla u)}\psi^2\,\mathcal{H}^{n-1}. \tag{A.14}$$

Proof. Let ν_t be the outer unit normal to U_t, $G_t(x) = 1+\Phi(\nabla u) - A(\nabla u)\cdot\nabla u$ and $V_t = X(x)\cdot\nu_t(x)$. We can then express (A.2) as

$$\frac{d}{dt}\,\mathcal{E}^\Phi_{f,\mu}(u_t) = \int_{\Gamma_t}G_t(y)V_t(y)\,d\mathcal{H}^{n-1}(y) = \int_\Gamma G_t(\Psi_t(x))\cdot V_t(\Psi_t(x))\,J^\Gamma\Psi_t(x)\,d\mathcal{H}^{n-1} \tag{A.15}$$

for all $t\in(-\varepsilon,\varepsilon)$. In the second identity we use the area formula on hypersurfaces (see [66, Theorem 11.3]). Keeping in mind that $G_0(\Psi_0(x)) = G_0(x) = 0$ on Γ, the chain rule gives

$$\frac{d^2}{dt^2}\Big|_{t=0}\mathcal{E}^\Phi_{f,\mu}(u_t) = \int_\Gamma\frac{d}{dt}\Big|_{t=0}G_t(\Psi_t(x))\,V_0. \tag{A.16}$$

To compute $\frac{d}{dt}|_{t=0}G_t(\Psi_t)$, recall from (A.7) and (A.8) that $\frac{d}{dt}|_{t=0}\nabla u_t|_{\Psi_t(x)} = -DX^T\nabla u + \nabla w$. Just as before, here we let $\tilde{u}_t(x) = u_t\circ\Psi_t$ and $w = \frac{d}{dt}|_{t=0}\tilde{u}_t$. So, applying the chain rule, we find

$$\begin{aligned}\frac{d}{dt}\Big|_{t=0}G_t(\Psi_t(x)) &= \frac{d}{dt}\Big|_{t=0}\Phi(\nabla u_t|_{\Psi_t(x)}) - \big\langle A(\nabla u),\frac{d}{dt}\Big|_{t=0}\nabla u_t|_{\Psi_t(x)}\big\rangle - \Big\langle Q(\nabla u)\Big[\frac{d}{dt}\Big|_{t=0}\nabla u_t|_{\Psi_t(x)}\Big],\nabla u\Big\rangle\\ &= -\Big\langle Q(\nabla u)\Big[\frac{d}{dt}\Big|_{t=0}\nabla u_t|_{\Psi_t(x)}\Big],\nabla u\Big\rangle = -\big\langle Q(\nabla u)[-DX^T\nabla u + \nabla w],\nabla u\big\rangle.\end{aligned} \tag{A.17}$$

Differentiating both sides of the second identity of (A.7) with respect to x, we find

$$\nabla w = \nabla\dot{u} + \nabla(\nabla u\cdot X) = \nabla\dot{u}(x) + D^2uX(x) + DX^T\nabla u.$$

Thus, since $X = V_0\nu$ and $\nabla u(x) = (\partial_\nu u)\nu$ for $x\in\Gamma$, (A.17) becomes

$$\frac{d}{dt}\Big|_{t=0}G_t(\Psi_t(x)) = -\big\langle Q(\nabla u)[\nabla\dot{u} + D^2u[X]],\nabla u\big\rangle = -\big\langle Q(\nabla u)[\nabla\dot{u}],\nabla u\big\rangle - \big\langle Q(\nabla u)D^2u[\nu],\nabla u\big\rangle V_0$$

for $x \in \Gamma$. Thus by (A.16),

$$\frac{d^2}{dt^2}\Big|_{t=0} \mathcal{E}^{\Phi}_{f,\mu}(u_t) = -\int_\Gamma \langle Q(\nabla u)[\nabla \dot u], \nabla u\rangle V_0 - \int_\Gamma \langle Q(\nabla u) D^2 u[\nu], \nabla u\rangle V_0^2. \tag{A.18}$$

Next, we use the fact that $\dot u$ solves the equation

$$\begin{cases} -\operatorname{div}(Q(\nabla u)\nabla \dot u) + 2\mu^2 \dot u = 0 & \text{on } U, \\ \dot u = -\partial_\nu u\, V_0 & \text{on } \Gamma, \\ \dot u = 0 & \text{on } \partial\Omega \cap \partial U. \end{cases} \tag{A.19}$$

Indeed, the interior equation holds by differentiating the equation $-\operatorname{div}(A(\nabla u_t)) + 2\mu(\mu u_t - f) = 0$ so

$$0 = \frac{d}{dt}\Big|_{t=0} \big(-\operatorname{div}(A(\nabla u_t)) + 2\mu(\mu u_t - f)\big) = -\operatorname{div}\big(Q(\nabla u)\nabla \dot u\big) + 2\mu^2 \dot u$$

with respect to t. To see the boundary condition on Γ in (A.19), we again use that $w = 0$ on Γ and apply the second identity in (A.7). Multiplying the equation by $\dot u$ and integrating by parts in U, we find

$$\begin{aligned} 0 &= \int_U Q(\nabla u)[\nabla \dot u, \nabla \dot u] + 2\mu^2 (\dot u)^2\, dx - \int_\Gamma \dot u Q(\nabla u)[\nabla \dot u, \nu]\, d\mathcal{H}^{n-1} \\ &= \int_U Q(\nabla u)[\nabla \dot u, \nabla \dot u] + 2\mu^2 (\dot u)^2\, dx + \int_\Gamma V_0\, Q(\nabla u)[\nabla \dot u, \nabla u]\, d\mathcal{H}^{n-1}. \end{aligned} \tag{A.20}$$

So putting (A.20) into (A.18) yields

$$\frac{d^2}{dt^2}\Big|_{t=0} \mathcal{E}^{\Phi}_{f,\mu}(u_t) = \int_U Q(\nabla u)[\nabla \dot u, \nabla \dot u] + 2\mu^2 \dot u^2\, dx - \int_\Gamma \langle Q(\nabla u) D^2 u[\nu], \nabla u\rangle V_0^2.$$

This shows (A.13). Next, if u is a minimizer, we have $\frac{d^2}{dt^2}|_{t=0} \mathcal{E}^{\Phi}_{f,\mu}(u_t) \geq 0$. Since $\dot u$ has the least energy of all extensions of $-\partial_\nu u\, V_0$ into U, if we let $\psi \in C^\infty_c(\Omega)$ be any extension of $-\partial_\nu u\, V_0 \in C^\infty_c(\Gamma)$, then the non-negativity of $\frac{d^2}{dt^2}|_{t=0} J(u_t)$ guarantees that

$$\int_U Q(\nabla u)[\nabla \psi, \nabla \psi] + 2\mu^2 \psi^2\, dx \geq \int_\Gamma \langle Q(\nabla u) D^2 u[\nu], \nabla u\rangle \frac{\psi^2}{(\partial_\nu u)^2} = \int_\Gamma \langle Q(\nabla u) D^2 u[\nu], \nu\rangle \frac{(A(\nabla u)\cdot \nu)}{1 + \Phi(\nabla u)} \psi^2 \tag{A.21}$$

where we used that $\partial_\nu u (A(\nabla u)\cdot \nu) = A(\nabla u)\cdot \nabla u = 1 + \Phi(\nabla u)$. □

Remark A.3. In the computations above, we assumed that Φ was smooth, which we are free to do in our proof via a mollification argument. However, if u is a minimizer of $\mathcal{E}^\Phi(\cdot, \Omega)$ with free boundary of class $C^{2,\alpha}$, where $\Phi \in I_{\Lambda,\alpha,\Upsilon}$ is an even and two-homogeneous integrand (hence not twice differentiable at the origin), the computations above go through and the variations take a cleaner form.

Indeed, by Lemma A.1, which does not require C^2 regularity of Φ, and using Euler's identity $\nabla\Phi(p)\cdot p = 2\Phi(p)$, the Euler Lagrange equation for u becomes

$$\begin{cases} \operatorname{div}(D\Phi(\nabla u)) = 0 & \text{in } \{u > 0\} \cap \Omega \\ \Phi(\nabla u) = 1 & \text{on } \partial\{u > 0\} \cap \Omega. \end{cases} \tag{A.22}$$

Making further use of Euler's identity, the stability inequality (A.14) takes the form

$$\int_{\partial\{u>0\}} D^2 u[D\Phi(\nabla u), \nu]\, \varphi^2\, \Phi(\nu)\, \mathcal{H}^{n-1} \leq \int_{\{u>0\}} D^2\Phi(\nabla u)[\nabla\varphi, \nabla\varphi]\, dx \qquad \text{for all } \varphi \in C^\infty_c(\Omega). \tag{A.23}$$

Here the integrand on the right-hand side is defined a.e, since the critical set $\mathcal{C} = \{|\nabla u| = 0\} \cap \{u > 0\}$ has Lebesgue measure zero, see e.g. [29, Prop. 3.2]. The proof of Lemma A.2 goes through in this setting because the free boundary condition $\Phi(\nabla u) = 1$ together with the $C^{2,\alpha}$ regularity of the free boundary allow us to localize the variation to a neighborhood of the free boundary, excluding degeneracy of the gradient.

Remark A.4. Continuing to assume that $\Phi \in I_{\Lambda,\alpha,\Upsilon}$ is even and 2-homogeneous and u is a minimizer of $\mathcal{E}^\Phi(\cdot, \Omega)$ with free boundary of class $C^{2,\alpha}$, we can further rewrite the stability inequality (A.23) to take an analogous form to the usual form of the stability inequality for the classical Alt-Caffarelli energy; see for instance [52, Eqn (2.4)].

Indeed, let $F = \sqrt{\Phi}$, and let $H^F_\Sigma(x)$ denote the anisotropic F-mean curvature of Γ arising as the first variation of the surface energy $\int_{\partial U} F(\nu_U)\, \partial\mathcal{H}^{n-1}$, oriented with respect to the inner unit normal. Then, making use of the

level set identity for the anisotropic mean curvature [29, Eqn (3.11)] and Euler's identity once again, the left-hand side of (A.23) can be rewritten so that (A.23) takes the form

$$\int_{\partial\{u>0\}} H^F_\Gamma \varphi^2 F(\nu)\, d\mathcal{H}^{n-1} \le \int_{\{u>0\}} D^2(F^2/2)(\nabla u)[\nabla\varphi, \nabla\varphi]\, dx \qquad \text{for all } \varphi \in C^\infty_c(\Omega)\,. \tag{A.24}$$

Appendix B. Proof of the convergence lemma

In this Appendix, we prove the compactness properties of minimizers stated in Lemma 4.8. For the classical Alt-Caffarelli energy, this result is rather standard. Here the main differences are the varying integrands and the lower-order perturbations. We sketch the main ingredients of the proof.

Proof. To simplify our notation, we set $F := \sup_k \|f_k\|_{L^\infty(B_1)} < \infty$, $\mathcal{E}_k := \mathcal{E}^{\Phi_k}_{f_k,\mu_k}$, $\mathcal{E} := \mathcal{E}^{\Phi}_{f,\mu}$ and

$$\omega_k(M) := \sup_{|p|\le M} |\Phi_k(p) - \Phi(p)| \longrightarrow 0 \qquad \text{for each fixed } M < \infty.$$

As a result, for each $s \in (0,1)$ and $w_k \in W^{1,\infty}(B_s)$ with $\|\nabla w_k\|_{L^\infty(B_s)} \le M$, we have

$$\left| \int_{B_s} \big(\Phi_k(\nabla w_k) - \Phi(\nabla w_k)\big)\, dx \right| \le |B_s|\, \omega_k(M) \longrightarrow 0. \tag{B.1}$$

For $s \in (0,1)$, locally uniform convergence of the sequence $\{u_k\}$ and Lemma 4.1 give

$$\sup_k \big(\|u_k\|_{L^\infty(B_s)} + \mathrm{Lip}(u_k; B_s)\big) \le L, \qquad \|u\|_{L^\infty(B_s)} + \mathrm{Lip}(u; B_s) \le L. \tag{B.2}$$

Moreover, uniform convergence of $\{f_k\}$ gives a modulus of continuity ω_f such that for $y \in B_s$ and $D \in (0,1)$,

$$|f_k((1-D)y) - f_k(y)| \le 2\|f_k - f\|_{L^\infty(B_1)} + \omega_f(D) = o_k(1) + \omega_f(D). \tag{B.3}$$

Step 1: The dilation competitor. In this step we show that for all $0 < \rho < s < 1$, if $v - u = 0$ on ∂B_ρ, then

$$\limsup_{k\to\infty} \mathcal{E}_k(u_k, B_s) \le \mathcal{E}(v, B_s). \tag{B.4}$$

Fix $0 < D < \min\{\frac{1}{2}, 1 - \frac{\rho}{s}\}$. Let $\ell : [(1-D)s, s] \to [0,1]$ be the affine function with $\ell((1-D)s) = 1$, $\ell(s) = 0$, so $|\ell'| = 1/(Ds)$. Writing $\pi(x) := s\,x/|x|$ for the radial projection onto ∂B_s, define

$$w_k(x) := \begin{cases} v\big(\frac{x}{1-D}\big), & |x| \le (1-D)s, \\ \ell(|x|)\, u(\pi(x)) + \big(1 - \ell(|x|)\big) u_k(\pi(x)), & (1-D)s < |x| < s, \\ u_k(x), & |x| \ge s. \end{cases}$$

Since w_k is an admissible competitor for u_k, we have

$$\mathcal{E}_k(u_k, B_s) \le \mathcal{E}_k(w_k, B_s). \tag{B.5}$$

We estimate $\mathcal{E}_k(w_k, B_s)$. Set $t = 1 - D$. For every $p \in \mathbb{R}^n$, Hessian bound on Φ_k gives

$$|\Phi_k(t^{-1}p) - \Phi_k(p)| \le \int_1^{t^{-1}} |D\Phi_k(\tau p)|\, |p|\, d\tau \le \frac{\Lambda}{2} t^{-2}|p|^2 \le CD|p|^2.$$

Therefore, with (B.3), we find

$$\int_{B_{ts}} \Phi_k(\nabla w_k)\, dx = t^n \int_{B_s} \Phi_k(t^{-1}\nabla v)\, dy \le \int_{B_s} \Phi(\nabla v)\, dy + CD \int_{B_s} |\nabla v|^2\, dy + o_k(1),$$

$$|\{w_k > 0\} \cap B_{ts}| = t^n |\{v > 0\} \cap B_s| \le |\{v > 0\} \cap B_s|,$$

and

$$\int_{B_{ts}} |\mu_k w_k - f_k|^2\, dx = t^n \int_{B_s} |\mu_k v(y) - f_k(ty)|^2\, dy \le \int_{B_s} |\mu v - f|^2\, dy + o_k(1) + C_v \omega_f(D).$$

In the annulus $A_D := B_s \setminus B_{(1-D)s}$, whose volume is at most $C(n)D$, we compute

$$\nabla w_k = \ell'(|x|)\, \frac{x}{|x|} \big(u(\pi(x)) - u_k(\pi(x))\big) + \ell\, \nabla\big(u \circ \pi\big) + (1-\ell)\, \nabla\big(u_k \circ \pi\big).$$

By (B.2), we have

$$|\nabla w_k| \le \frac{\eta_k}{Ds} + CL, \qquad \eta_k := \|u - u_k\|_{L^\infty(\overline{B_s})} \longrightarrow 0.$$

Hence, bounding $\Phi_k(p) \le C(\Lambda)|p|^2$, the characteristic function by 1, and the fidelity term by $C(1+F^2)$, we have $\mathcal{E}_k(w_k, A_D) \le C(\frac{\eta_k^2}{D} + D)$.Combining these estimates with (B.5) gives

$$\mathcal{E}_k(u_k, B_s) \le \mathcal{E}(v, B_s) + C_v\big(D + \omega_f(D)\big) + C\frac{\eta_k^2}{D} + o_k(1).$$

Taking first $\limsup_{k\to\infty}$ at fixed D, and then letting $D \downarrow 0$, we obtain (B.4).

Step 2: Lower semicontinuity. For every $s \in (0,1)$, we show that

$$\mathcal{E}(u, B_s) \le \liminf_{k\to\infty} \mathcal{E}_k(u_k, B_s). \tag{B.6}$$

Indeed, (B.2) and (B.1) give $|\int_{B_s}\big(\Phi_k(\nabla u_k) - \Phi(\nabla u_k)\big)\,dx| \le |B_s|\omega_k(L) \longrightarrow 0$. Convexity and weak $H^1(B_s)$ convergence then imply

$$\liminf_k \int_{B_s} \Phi_k(\nabla u_k)\,dx \ge \int_{B_s} \Phi(\nabla u)\,dx.$$

For the measure term, Fatou's lemma gives $\liminf_k |\{u_k > 0\} \cap B_s| \ge |\{u > 0\} \cap B_s|$. Finally, the fidelity term converges, since $u_k \to u$ in $L^2(B_s)$, $f_k \to f$ uniformly, and $\mu_k \to \mu$.

Step 3: proof of (2). Let $0 < \rho < s < 1$ and let $v - u \in H_0^1(B_\rho)$. Combining (B.6) and (B.4), we have

$$\mathcal{E}(u, B_s) \le \liminf_{k\to\infty} \mathcal{E}_k(u_k, B_s) \le \limsup_{k\to\infty} \mathcal{E}_k(u_k, B_s) \le \mathcal{E}(v, B_s).$$

Since $v = u$ on $B_s \setminus B_\rho$, subtracting the common finite contribution on that annulus gives

$$\mathcal{E}(u, B_\rho) \le \mathcal{E}(v, B_\rho) \qquad \text{whenever } v - u \in H_0^1(B_\rho), \quad \rho < 1. \tag{B.7}$$

Thus u is a local minimizer in B_1.

Step 4: Proof of (1). Fix $s \in (0,1)$. Taking $v = u$ in (B.4) and combining with (B.6),

$$\mathcal{E}_k(u_k, B_s) \longrightarrow \mathcal{E}(u, B_s).$$

In particular, using (B.1) and (B.2), $\int_{B_s} \Phi(\nabla u_k) \to \int_{B_s} \Phi(\nabla u)$. The uniform convexity of Φ gives

$$\int_{B_s} |\nabla u_k - \nabla u|^2 dx \le C \int_{B_s} \Big[\Phi(\nabla u_k) - \Phi(\nabla u) - D\Phi(\nabla u)\cdot(\nabla u_k - \nabla u)\Big] dx \longrightarrow 0.$$

Step 5: proof of (3) and (4). Now that the limit u is an energy minimizer, it satisfies the Lipschitz bound as well as non-degeneracy properties. The convergence of the positive sets as well as the free boundaries follow from standard arguments; see, e.g., [85].

Step 6: proof of (5). For notational simplicity we prove the claim with $r = 1$. We may assume that the left-hand side of (5) is finite and, passing to a subsequence realizing the lim inf, that

$$\sup_k \|D^2 u_k\|_{L^p(\{u_k>0\}\cap B_1)} \le C_0 < \infty.$$

Set $\Omega^+ := B_1 \cap \{u > 0\}$. For $0 < \varepsilon < 1$, let

$$V_\varepsilon := \big\{x \in \Omega^+ : \operatorname{dist}(x, \mathbb{R}^n \setminus \Omega^+) > \varepsilon\big\}.$$

Then $\overline{V_\varepsilon} \Subset \Omega^+$, and $V_\varepsilon \uparrow \Omega^+$ as $\varepsilon \downarrow 0$. If $V_\varepsilon = \emptyset$, the desired inequality on V_ε is trivial. Otherwise $m_\varepsilon := \min_{\overline{V_\varepsilon}} u > 0$, and locally uniform convergence gives $u_k \ge m_\varepsilon/2$ on V_ε for all sufficiently large k. In particular, $V_\varepsilon \subset \{u_k > 0\}$ and $\|D^2 u_k\|_{L^p(V_\varepsilon)} \le C_0$. Let $q = p/(p-1)$. For $\varphi \in C_c^\infty(V_\varepsilon)$ and any i, j, the weak H^1 convergence and integration by parts give

$$\int_{V_\varepsilon} \partial_i u\, \partial_j \varphi\, dx = \lim_{k\to\infty} \int_{V_\varepsilon} \partial_i u_k\, \partial_j \varphi\, dx = -\lim_{k\to\infty} \int_{V_\varepsilon} \partial_{ij} u_k\, \varphi\, dx,$$

and the last integrals are bounded by $C_0\|\varphi\|_{L^q(V_\varepsilon)}$. By duality, u admits weak second derivatives $\partial_{ij}u \in L^p(V_\varepsilon)$. Moreover, since $(\partial_{ij}u_k)_k$ is bounded in $L^p(V_\varepsilon)$ and its distributional limit is $\partial_{ij}u$, we have $D^2u_k \rightharpoonup D^2u$ componentwise, weakly in $L^p(V_\varepsilon)$. By weak lower semicontinuity of the convex functional $w \mapsto \int |w|^p$,

$$\int_{V_\varepsilon} |D^2u|^p\,dx \le \liminf_{k\to\infty} \int_{V_\varepsilon} |D^2u_k|^p\,dx \le \liminf_{k\to\infty} \int_{\{u_k>0\}\cap B_1} |D^2u_k|^p\,dx.$$

Since $V_\varepsilon \uparrow \{u>0\}\cap B_1$ as $\varepsilon \downarrow 0$, the monotone convergence theorem concludes the proof of (5). □

Appendix C. Proof of ε-Regularity

We sketch the proof of Theorem 4.12, emphasizing the differences from [26]. The first step is the following partial Harnack inequality; c.f. [26, Theorem 4.1] [25, Theorem 3.1] or [85, Theorem 7.7].

Lemma C.1 (Partial Harnack)**.** *There exist $\bar\varepsilon = \bar\varepsilon(n,\Lambda,\alpha) > 0$ and $c = c(n,\Lambda) \in (0,1)$ such that the following holds. Suppose u is a viscosity solution to (2.10) in B_1 for some $g \in C(B_1)\cap L^\infty(B_1)$ and suppose $0 \in \overline{\{u>0\}}$. If there exist $e \in \mathbb{R}^n$ with $\mathcal{G}(e) = 1$ and $a_0 < b_0$ with $|a_0 - b_0| \le \bar\varepsilon$ such that*

$$(x\cdot e + a_0)_+ \le u(x) \le (x\cdot e + b_0)_+ \qquad \text{in } B_1 \tag{C.1}$$

and

$$\|g\|_{L^\infty(B_1)} \le |a_0 - b_0|^2, \tag{C.2}$$

then there are a_1, b_1 with $a_0 \le a_1 < b_1 \le b_0$ and $|b_1 - a_1| \le (1-c)|a_0 - b_0|$ such that

$$(x\cdot e + a_1)_+ \le u(x) \le (x\cdot e + b_1)_+ \qquad \text{in } B_{1/64\Lambda}.$$

Proof. The proof of this lemma only uses that $\mathcal{M}^-_{1/\Lambda,\Lambda}(D^2u) \le g \le \mathcal{M}^+_{1/\Lambda,\Lambda}(D^2u)$ in the viscosity sense in $\{u>0\}$ and that $\mathcal{G}(\nabla u) = 1$ on $\partial\{u>0\}$ in the viscosity sense. The proof essentially differs from [26, Theorem 4.1] only in the construction of the barrier w below, since here the free boundary condition does not depend on $|\nabla u|$ only.

Let $\bar\varepsilon \le 1/(24\Lambda^2)$ and $\tau = |e|/(24\Lambda)$. Note that $|e| \in [1/\Lambda, \Lambda]$ (as $\mathcal{G}(e) = 1$) and thus $\bar\varepsilon \le \tau$. Moreover, since $0 \in \overline{\{u>0\}}$, we know $b_0 \ge 0$ and thus $a_0 \ge -\tau$. We consider two cases: $a_0 \ge \tau$ and $|a_0| \le \tau$.

Case 1: First suppose $a_0 \ge \tau$. Then u satisfies $\mathcal{M}^-_{1/\Lambda,\Lambda}(D^2u) \le g \le \mathcal{M}^+_{1/\Lambda,\Lambda}(D^2u)$ in the viscosity sense in $B_{1/(24\Lambda)} \subset B_1 \cap \{x\cdot e > -\tau\}$. We let

$$\ell(x) = x\cdot e + a_0 \qquad \text{and} \qquad \varepsilon = b_0 - a_0, \tag{C.3}$$

so that the assumption (C.1) implies $\ell(x) \le u(x) \le \ell(x) + \varepsilon$ in $B_{1/(24\Lambda)}$. If $u(0) \ge \ell(0) + \varepsilon/2$, let $v = u - \ell$. In $B_{1/(24\Lambda)}$, the function v is nonnegative and again satisfies $\mathcal{M}^-(D^2v, 1/\Lambda, \Lambda) \le g \le \mathcal{M}^+(D^2v, 1/\Lambda, \Lambda)$ in the viscosity sense. So, by Harnack inequality (see [7, Theorem 4.3]) there is a constant $C = C(n,\Lambda) > 0$ such that

$$\frac{\varepsilon}{2} \le v(0) \le C\Big(\inf_{B_{1/(48\Lambda)}} v + \frac{1}{(24\Lambda)^2}\|g\|_{L^\infty}\Big) \le C\Big(\inf_{B_{1/(48\Lambda)}} v + \frac{1}{(24\Lambda)^2}\varepsilon^2\Big)$$

Choosing $\bar\varepsilon = \bar\varepsilon(n,\Lambda)$ small enough, $C/(24\Lambda)^2\varepsilon^2 \le \varepsilon/4$, so absorbing this and rearranging terms, we get

$$\ell(x) + \frac{\varepsilon}{4C} \le u(x) \qquad \text{in } B_{1/(48\Lambda)},$$

which yields the conclusion with $a_1 = a_0 + \varepsilon/(4C)$ and $b_1 = b_0$ in this case. If instead $u(0) \le \ell(0) + \varepsilon/2$, we argue analogously with the function $w = \ell + \varepsilon - u$ to find $u(x) \le \ell(x) + (1 - \frac{1}{4C})\varepsilon$ in $B_{1/(48\Lambda)}$. This yields the conclusion with $b_1 = b_0 - \varepsilon/(4C)$ and $a_1 = a_0$.

Case 2: Now suppose $|a_0| \le \tau$. Let $\bar x = \frac{e}{3\Lambda|e|}$. For the linear function ℓ and number ε defined in (C.3) above, first consider the case when $u(\bar x) \ge \ell(\bar x) + \varepsilon/2$. In this case, we claim there exists $c = c(n,\Lambda) > 0$ such that

$$(x\cdot e + a_0 + c\varepsilon)_+ \le u(x) \qquad \text{for all } x \text{ in } B_{1/(64\Lambda)}\,. \tag{C.4}$$

This will prove the claim with $a_1 = a_0 + c\varepsilon$ and $b_1 = b_0$. To this end, first note that $x\cdot e > \tau$ for all $B_{1/(4\Lambda)}(\bar x)$, so this ball is contained in $\{u>0\}$. Applying Harnack inequality to the function $v = u - \ell$ just as in Case 1, we find

$$\frac{\varepsilon}{2} \le v(\bar x) \le C\Big(\inf_{B_{1/(8\Lambda)}(\bar x)} v + \frac{1}{16\Lambda^2}\|g\|_{L^\infty}\Big) \le C\inf_{B_{1/(8\Lambda)}(\bar x)} v + \frac{\varepsilon}{4}$$

provided $\bar\varepsilon = \bar\varepsilon(n,\Lambda)$ is chosen small enough. Rearranging, this means

$$u \ge \ell + c_*\varepsilon \qquad \text{in } B_{1/(8\Lambda)}(\bar x) \tag{C.5}$$

where $c_* = 1/(4C)$. Now, fix the matrix $M_e = (D^2\Phi(e))^{-1}$. Then letting $q(x) = M_e[x-\bar{x}, x-\bar{x}]$, and $\gamma = 64n\Lambda^7$, consider the barrier w defined piecewise by

$$w(x) = \begin{cases} 1 & \text{if } q(x) < 1/(64\Lambda^3) \\ 0 & \text{if } q > 1/(4\Lambda) \\ \bar{c}(e^{-\gamma q(x)} - e^{-\gamma/(4\Lambda)}) & \text{if } 1/(64\Lambda^3) < q < 1/(4\Lambda)\,. \end{cases}$$

Here $\bar{c} = \bar{c}(n,\Lambda)$ is chosen to make w continuous. Let $A(\bar{x})$ denote the annular region $A(\bar{x}) := \{1/(64\Lambda^3) < q(x) < 1/(4\Lambda)\}$ in which w is nonconstant. Using that $|x-\bar{x}|^2/\Lambda \le q(x) \le \Lambda|x-\bar{x}|^2$, we see that $\{q < 1/(4\Lambda)\} \subset B_1$ and $\{q < 1/(64\Lambda^3)\} \subset B_{1/(8\Lambda)}(\bar{x})$. Moreover, we have

$$\nabla w(x) = -2\bar{c}\gamma e^{-\gamma q(x)} M_e(x-\bar{x}) \qquad \text{and} \qquad D^2 w(x) = 2\bar{c}\gamma e^{-\gamma q(x)}\big(2\gamma M_e(x-\bar{x}) \otimes M_e(x-\bar{x}) - M_e\big) \tag{C.6}$$

In particular, for $x \in A(\bar{x}) \cap \{x \cdot e \le \tau\}$, we have

$$\nabla w(x) \cdot D\mathcal{G}(e) = -2\bar{c}\gamma e^{-\gamma q(x)}(x-\bar{x}) \cdot e \ge c(n,\Lambda)(-\tau + |e|/(3\Lambda)) \ge c(n,\Lambda). \tag{C.7}$$

Here we have used that $D\mathcal{G}(p) = D^2\Phi(p)[p]$ for any p for which $D^2\Phi(p)$ exists. Now, fix $t \in (0,1)$ and let

$$v_t(x) = \ell(x) + c_*\varepsilon w(x) - tc_*\varepsilon.$$

Noting that $D^2 v_t = c_*\varepsilon D^2 w$, we have

$$\mathcal{M}^-_{1/\Lambda,\Lambda}(D^2 v_t(x)) \ge c_*\varepsilon \mathcal{M}^-_{1/\Lambda,\Lambda}(D^2 w(x)) \ge 2c_*\varepsilon\bar{c}\gamma e^{-\gamma q(x)}\Big(\frac{2\gamma q(x)}{\Lambda^2} - n\Lambda^2\Big) \ge c(n,\Lambda)\varepsilon \tag{C.8}$$

for $x \in A(\overline{x})$, thanks to our choice of γ. Choosing $\bar{\varepsilon}$ small enough, (C.8) and (C.2) guarantee $\mathcal{M}^-_{1/\Lambda,\Lambda}(D^2 v_t(x)) > g$ in $A(\bar{x})$. We claim that $u > v_t$ in $A(\bar{x})$. Suppose not. Noting that $u > v_t$ on $B_{1/(8\Lambda)}(\bar{x})$ by (C.5) and on $B_1 \setminus \{q < 1/(4\Lambda)\}$ by construction, this means that $u - v_t$ has a local minimum at some x in $A(\bar{x})$, so a vertical translation of v_t touches u from below at this x. We note that x cannot lie in $A(\bar{x}) \cap \{u > 0\}$ since $\mathcal{M}^-_{1/\Lambda,\Lambda}(D^2 u(x)) \le g$ and $\mathcal{M}^-_{1/\Lambda,\Lambda}(D^2 v_t(x)) > g(x)$ in the viscosity sense in this set. Next, x cannot lie in $A(\bar{x}) \cap \Gamma(u)$, since

$$\mathcal{G}(\nabla v_t) = \mathcal{G}(e) + D\mathcal{G}(e) \cdot (\nabla v_t - e) + o(\varepsilon) = 1 + c_*\varepsilon \nabla w(x) \cdot D^2\Phi(e)[e] + o(\varepsilon) > 1$$

for ε small enough, thanks to (C.7), again violating u being a viscosity solution. Finally, we cannot have $x \in A(\bar{x}) \cap \{u = 0\}$, since at such a point we would have $\nabla(u - v_t)(x) = -e - c_*\varepsilon\nabla w(x) = 0$, contradicting (C.7) again.

So, $u > v_t$ in $A(\overline{x})$ for all $t \in (0,1)$. Hence, taking $t \to 0$, we have $u \ge \ell + c_*\varepsilon w$ in $A(\overline{x})$. In particular, since w is bounded below in $B_{1/(64\Lambda)}$, this establishes the claim in this case. If instead $u(\bar{x}) \le \ell(\bar{x}) + \varepsilon/2$, an analogous proof shows $(x \cdot e + b_0 - c\varepsilon)_+ \ge u(x)$ for all x in $B_{1/(64\Lambda)}$. This completes the proof. □

By iterating Lemma C.1, we find the following:

Corollary C.2. *Let $\Phi_k \in I_{\Lambda,\alpha,\Upsilon}$ with $\Phi_k \to \Phi$, $g_k \in C(B_1) \cap L^\infty(B_1)$, and $\varepsilon_k \to 0$. For each k, suppose u_k satisfies (2.10) with g_k in place of g and with $\mathcal{L} = \mathcal{L}_{\Phi_k}$, and for e_k with $\mathcal{G}_k(e_k) = 1$ we have*

$$(x \cdot e_k - \varepsilon_k)_+ \le u_k(x) \le (x \cdot e_k + \varepsilon_k)_+ \qquad \text{in } B_1$$

and

$$\|g_k\|_{L^\infty(B_1)} \le \varepsilon_k^2. \tag{C.9}$$

Then let $\tilde{u}_k(x) := (u_k(x) - x \cdot e_k)/\varepsilon_k$ and graph$(\tilde{u}_k) := \{(x, \tilde{u}_k(x)) : x \in \overline{\Omega}_{u_k} \cap B_{1/2}\} \subset \mathbb{R}^{n+1}$. Then there exist $e \in \mathbb{R}^n$ with $\mathcal{G}(e) = 1$ and a Hölder continuous function $\tilde{u} : B_{1/2} \cap \{e \cdot x \ge 0\} \to \mathbb{R}$ such that, up to a subsequence, $e_k \to e$, $\tilde{u}_k \to \tilde{u}$ locally uniformly on $B_{1/2} \cap \{e \cdot x > 0\}$ and graph$(\tilde{u}_k) \to$ graph$(\tilde{u})$ in the Hausdorff distance. Here graph$(\tilde{u}) = \{(x, \tilde{u}(x) : x \in B_{1/2} \cap \{x \cdot e \ge 0\}\}$.

The proof is the same as those of [85, Lemmas 7.13 and 7.14] and is thus omitted. Following [25], the next step toward proving Lemma C.4 is the following linearization; c.f. [85, Lemma 7.15].

Lemma C.3. *With the same assumptions and notation as Corollary C.2, $\tilde{u}$ has $|\tilde{u}| \le 1$ in $B_{1/2} \cap \{x \cdot e \ge 0\}$ and solves*

$$D^2\Phi(e) : D^2\tilde{u} = 0 \text{ in } \{x \cdot e > 0\} \cap B_{1/2}, \qquad D^2\Phi(e)[e] \cdot \nabla\tilde{u} = 0 \text{ on } \{x \cdot e = 0\} \cap B_{1/2}. \tag{C.10}$$

Proof. First we show $\tilde{u}$ solves the equation in the interior. Suppose $\phi \in C^\infty(B_1)$ touches $\tilde{u}$ from below at $x_0 \in \{x \cdot e > 0\} \cap B_{1/2}$. We can find $x_k \to x_0$ with $x_k \in \{u_k > 0\} \cap B_{1/2}$ and $c_k \to 0$ such that $\phi + c_k$ touches $\tilde{u}_k$ from below locally at x_k. At such a point, $Q_k(x) = x \cdot e_k + \varepsilon_k(\phi(x) + c_k)$ touches u_k from below at x_k. Since $\nabla Q_k(x_k) \in \mathcal{A}(\Lambda)$ for k large, the assumption that u_k is a viscosity solution guarantees that

$$D^2\Phi_k(e_k + \varepsilon_k \nabla\phi(x_k)) : D^2\phi(x_k) \leq g_k(x_k)/\varepsilon_k.$$

The right hand side tends to zero thanks to (C.9), while the left-hand side converges to $D^2\Phi(e) : D^2\phi(x_0)$ thanks to the uniform convergence of $D^2\Phi_k$ on $\mathcal{A}(\Lambda)$ and the smoothness of ϕ. Thus $D^2\Phi(e) : D^2\phi(x_0) \leq 0$. An analogous argument holds for a smooth function ϕ touching $\tilde{u}$ from above at $x_0 \in \{x \cdot e > 0\}$. Thus $\tilde{u}$ satisfies $D^2\Phi(e) : D^2\tilde{u} = 0$ in $\{x \cdot e > 0\}$ in the viscosity sense; this is a constant coefficient linear equation, so standard regularity theory shows that $\tilde{u}$ is analytic in $\{x \cdot e > 0\} \cap B_{1/2}$ and solves the equation classically.

Next, suppose P is a quadratic polynomial that touches $\tilde{u}$ locally from below at $x_0 \in \{x \cdot e = 0\} \cap B_{1/2}$. Without loss of generality, we may assume $x_0 = 0$, and by replacing $P(x)$ with $P(x) - \eta|x|^2 - \eta(e \cdot x) + C(\eta)(x \cdot e)^2$ for η small, we may assume that P locally touches $\tilde{u}$ strictly from below at 0 and that $\mathcal{M}^-(D^2P, 1/\Lambda, \Lambda) \geq 1$ in a small neighborhood of the origin. So, as before, we can find $x_k \in \overline{\{u_k > 0\}}$ with $x_k \to 0$ and $c_k \to 0$ such that $P + c_k$ touches $\tilde{u}_k$ from below at x_k.

In terms of u_k, this means the polynomial $Q_k = \varepsilon_k(P + c_k) + e_k \cdot x$ touches u_k from below at x_k and satisfies $\mathcal{M}^-(D^2Q_k, 1/\Lambda, \Lambda) \geq \varepsilon_k$. In view of (C.9) and the fact that $\mathcal{L}_{\Phi_k} u_k = g_k$ in $\{u_k > 0\}$ in the viscosity sense, we cannot have $x_k \in \{u_k > 0\}$ and hence $x_k \in \Gamma_{u_k}$ for k large enough. At such a free-boundary contact point, Q_k^+ touches u_k from below. Hence, as u_k satisfies the free boundary condition in the viscosity sense and using the uniform C^1 modulus of continuity, we have

$$1 \geq \mathcal{G}_k(\nabla Q_k(u_k)) = \mathcal{G}_k(e_k) + \varepsilon_k D^2\Phi_k(e_k)[e_k] \cdot \nabla P(x_k) + o(\varepsilon_k) = 1 + \varepsilon_k D^2\Phi_k(e_k)[e_k] \cdot \nabla P(x_k) + o(\varepsilon_k).$$

Here we again used that $D\mathcal{G}(p) = D^2\Phi(p)[p]$ whenever $D^2\Phi(p)$ exists (and in particular for any $p \in \mathcal{A}(\Lambda)$. Therefore, $D^2\Phi_k(e_k)[e_k] \cdot \nabla P(x_k) \leq 0$, and so passing to the limit, $D^2\Phi_k(e_k)[e_k] \cdot \nabla P(x_0) \leq 0$ We argue analogously if P touches $\tilde{u}$ from above at 0 to find $D^2\Phi_k(e_k)[e_k] \cdot \nabla P(x_0) \geq 0$. Thus the oblique boundary condition in (C.10) in the viscosity sense, and by boundary regularity for viscosity solutions with oblique boundary conditions, see e.g. [62, Theorem 1.2], $\tilde{u}$ is regular up to the boundary and this holds classically. □

From here, another standard arguement yields the following improvement of flatness statement; c.f [25, Lemma 4.1], [85, Theorem 7.4].

Lemma C.4. *There exist $\varepsilon_0 = \varepsilon_0(\Lambda, n) \in (0,1)$, $C = C(\Lambda, n) > 0$, $\sigma(\Lambda, n) \in (0,1)$, and $r_0 = r_0(\Lambda, n) \in (0, \sigma^2/2)$, and such that the following holds. Given $g \in C(B_1) \cap L^\infty(B_1)$, and $e \in \mathbb{R}^n$ with $\mathcal{G}(e) = 1$, suppose u is a viscosity solution to (2.10) in B_1 with $0 \in \Gamma(u)$ such that*

$$(x \cdot e - \varepsilon)_+ \leq u(x) \leq (x \cdot e + \varepsilon)_+ \qquad \text{in } B_1 \tag{C.11}$$

and

$$\|g\|_{L^\infty(B_1)} \leq \varepsilon^2 \tag{C.12}$$

for some $\varepsilon \leq \varepsilon_0$. Then there is some $e' \in \mathbb{R}^n$ with $\mathcal{G}(e') = 1$ and $|e - e'| < C\varepsilon$ such that $u_{r_0}(x) = u(r_0 x)/r_0$ satisfies

$$(x \cdot e' - \sigma\varepsilon)_+ \leq u_{r_0}(x) \leq (x \cdot e' + \sigma\varepsilon)_+ \qquad \text{in } B_1\,. \tag{C.13}$$

Finally, Theorem 4.12 follows from Lemma C.4 via a standard argument, as in [25, 85].

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