SHELAH'S CATEGORICITY CONJECTURE FROM A SUCCESSOR FOR TAME ABSTRACT ELEMENTARY CLASSES

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Abstract. We prove a categoricity transfer theorem for tame abstract elementary classes.

Theorem 0.1. *Suppose that $\mathcal{K}$ is a χ-tame abstract elementary class and satisfies the amalgamation and joint embedding properties and has arbitrarily large models. Let $\lambda \geq \mathrm{Max}\{\chi, \mathrm{LS}(\mathcal{K})^+\}$. If $\mathcal{K}$ is categorical in λ and λ^+, then $\mathcal{K}$ is categorical in λ^{++}.*

Combining this theorem with some results from [Sh 394], we derive a form of Shelah's Categoricity Conjecture for tame abstract elementary classes:

Corollary 0.2. *Suppose $\mathcal{K}$ is a χ-tame abstract elementary class satisfying the amalgamation and joint embedding properties. Let $\mu_0 := \mathrm{Hanf}(\mathcal{K})$. If $\chi \leq \beth_{(2^{\mu_0})^+}$ and $\mathcal{K}$ is categorical in some $\lambda^+ > \beth_{(2^{\mu_0})^+}$, then $\mathcal{K}$ is categorical in μ for all $\mu > \beth_{(2^{\mu_0})^+}$.*

Introduction

Let $\mathcal{K}$ be a class of models all of the same language. $\mathcal{K}$ is said to be *categorical in a cardinal* μ if and only if all the models from $\mathcal{K}$ of cardinality μ are isomorphic. In 1954 Jerzy Łoś [Lo] conjectured that for a first-order theory T in a countable language its class of models $\mathrm{Mod}(T)$ behaves like the class of algebraically closed fields of fixed characteristic. Namely, if there exists an uncountable cardinal λ such that $\mathrm{Mod}(T)$ is categorical in λ then $\mathrm{Mod}(T)$ is categorical in every uncountable cardinal. Michael Morley [Mo] in 1965 published a proof of this conjecture and asked about its generalization for first-order theories in uncountable languages. Saharon Shelah [Sh 31] around 1970 managed to prove the relativized Łoś conjecture for theories in uncountable languages. The work of Morley and Shelah introduced a large number of new concepts and devices into model theory and established the field known as *stability theory* or by the name of *classification theory*.

Already in the forties and fifties Alfred Tarski and Andrzej Mostowski realized that first-order logic is too weak to deal with some of the most

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basic objects of mathematics and proposed logics with greater expressive power like $L_{\omega_1,\omega}$ and $L_{\omega_1,\omega}(\mathbf{Q})$. Examples due to Morley and Jack Silver in the sixties gave the impression that while these logics have good expressive power, there is very little structure there and almost any conjecture has a counterexample.

Some ground breaking work of Shelah from the mid seventies ([Sh 48], [Sh 87a] and [Sh 87b]) together with earlier important work by H. Jerome Keisler [Ke2] led Shelah to realize that machinery of stability theory could be developed for the context of non-first order theories. Around 1977 Shelah proposed a far reaching conjecture to serve as test problem to progress in the field.

Conjecture 0.3 (Shelah's Categoricity Conjecture)**.** *Given a countable language L and T a theory in $L_{\omega_1,\omega}$, if* $\mathrm{Mod}(T)$ *is categorical in* λ *for some* $\lambda > \beth_{\omega_1}$,[1] *then* $\mathrm{Mod}(T)$ *is categorical in* χ *for every* $\chi \geq \beth_{\omega_1}$.

While $L_{\omega_1,\omega}$ surfaced as a general context for developing a non-elementary model theory, it is not broad enough to capture some mathematically significant examples such as Boris Zilber's class of algebraically closed fields with pseudo-exponentiation which is connected with Schanuel's conjecture of transcendental number theory [Zi]. Furthermore, by focusing on a specific non-first-order logic such as $L_{\omega_1,\omega}$ one may be near-sighted, not envisioning the underlying stability theory. In [Sh 88] the notion of *Abstract Elementary Class* (see Definition 0.5) was introduced by Saharon Shelah. This is a semantic generalization of $L_{\omega_1,\omega}(\mathbf{Q})$-theories, according to [Sh 702] AECs are the most general context to have a reasonable model theory. In Shelah own words: "I have preferred this context, certainly the widest I think has any chance at all." The guiding conjecture in the development of a classification theory of non-elementary classes is now a strengthening of Conjecture 0.3 and appears in the list of open problems in [Shc]:

Conjecture 0.4 (Shelah's Categoricity Conjecture)**.** *Let $\mathcal{K}$ be an abstract elementary class. If $\mathcal{K}$ is categorical in some* $\lambda > \mathrm{Hanf}(\mathcal{K})$, [2] *then for every* $\mu \geq \mathrm{Hanf}(\mathcal{K})$*, $\mathcal{K}$ is categorical in μ.*

Similar to the solutions of Łoś conjecture, a categoricity transfer theorem for non-first-order logic is expected to provide the basic conceptual tools necessary for a stability theory for non-first-order logic which then may be applied to answer questions in other branches of mathematics. Already Shelah's proof of Conjecture 0.4 for excellent classes forshadowed tools developed by Zilber to study algebraically closed fields with psuedo-exponentiation.

In [GrVa1], we wanted to develop basic stability theory for AECs with amalgamation, in our attempt to prove certain technical statements necessary for establishing stability spectrum theorem and existence of Morley sequences we introduce a property we called tameness. Later we realized that

[1] $\beth_{\omega_1}$ is the Hanf number of this logic.

[2] For an explanation of $\mathrm{Hanf}(\mathcal{K})$ see Remark 0.6

a relativized version of tameness for saturated models appeared implicitly in Shelah's proof of his main theorem from [Sh 394].

In [GrVa1] we introduce the notion of tameness to abstract elementary classes as a context with enough generality to capture many mathematical examples, but surprisingly poignant enough to be accessible from a model theoretic point of view. Zilber's work on algebraically closed fields equipped with elliptic curves is tame. An interesting non-tame example is Shelah and Bradd Hart's example of an abstract elementary class which is categorical in $\aleph_k$ for every $k < n$, but fails to be categorical in $2^{\aleph_n}$.

While there are over a thousand published pages devoted to a partial solution of Conjecture 0.3, it remains wide open. Here we prove an approximation to Conjecture 0.4 for tame abstract elementary classes. Below we will review the history of work towards Conjecture 0.4 noting that our result is the most general approximation of Conjecture 0.4. Our proof is unprecedented in providing an upward categoricity transfer theorem without employing compactness machinery via manipulations of first-order or infinitary syntax.

Now let us make explicit some of the concepts we have mentioned:

Definition 0.5. Let $\mathcal{K}$ be a class of structures all in the same similarity type $\mathrm{L}(\mathcal{K})$, and let $\prec_{\mathcal{K}}$ be a partial order on $\mathcal{K}$. The ordered pair $\langle \mathcal{K}, \prec_{\mathcal{K}} \rangle$ is an *abstract elementary class, AEC* for short if and only if

- A0 (Closure under isomorphism)
  - (a) For every $M \in \mathcal{K}$ and every $\mathrm{L}(\mathcal{K})$-structure N if $M \cong N$ then $N \in \mathcal{K}$.
  - (b) Let $N_1, N_2 \in \mathcal{K}$ and $M_1, M_2 \in \mathcal{K}$ such that there exist $f_l : N_l \cong M_l$ (for $l = 1, 2$) satisfying $f_1 \subseteq f_2$ then $N_1 \prec_{\mathcal{K}} N_2$ implies that $M_1 \prec_{\mathcal{K}} M_2$.
- A1 For all $M, N \in \mathcal{K}$ if $M \prec_{\mathcal{K}} N$ then $M \subseteq N$.
- A2 Let M, N, M^* be $\mathrm{L}(\mathcal{K})$-structures. If $M \subseteq N$, $M \prec_{\mathcal{K}} M^*$ and $N \prec_{\mathcal{K}} M^*$ then $M \prec_{\mathcal{K}} N$.
- A3 (Downward Löwenheim-Skolem) There exists a cardinal $\mathrm{LS}(\mathcal{K}) \geq \aleph_0 + |\mathrm{L}(\mathcal{K})|$ such that for every $M \in \mathcal{K}$ and for every $A \subseteq |M|$ there exists $N \in \mathcal{K}$ such that $N \prec_{\mathcal{K}} M$, $|N| \supseteq A$ and $\|N\| \leq |A| + \mathrm{LS}(\mathcal{K})$.
- A4 (Tarski-Vaught Chain)
  - (a) For every regular cardinal μ and every $N \in \mathcal{K}$ if $\{M_i \prec_{\mathcal{K}} N \ : \ i < \mu\} \subseteq \mathcal{K}$ is $\prec_{\mathcal{K}}$-increasing (i.e. $i < j \Longrightarrow M_i \prec_{\mathcal{K}} M_j$) then $\bigcup_{i<\mu} M_i \in \mathcal{K}$ and $\bigcup_{i<\mu} M_i \prec_{\mathcal{K}} N$.
  - (b) For every regular μ, if $\{M_i \ : \ i < \mu\} \subseteq \mathcal{K}$ is $\prec_{\mathcal{K}}$-increasing then $\bigcup_{i<\mu} M_i \in \mathcal{K}$ and $M_0 \prec_{\mathcal{K}} \bigcup_{i<\mu} M_i$.

For M and $N \in \mathcal{K}$ a monomorphism $f : M \to N$ is called an $\mathcal{K}$-*embedding* if and only if $f[M] \prec_{\mathcal{K}} N$. Thus, $M \prec_{\mathcal{K}} N$ is equivalent to "id_M is a $\mathcal{K}$-embedding from M into N".

Remark 0.6. *Hanf*$(\mathcal{K})$ is widely accepted abuse of notation. In the case $\mathcal{K} = \mathrm{Mod}(\psi)$ for $\psi \in L_{\omega_1,\omega}$ in a countable language $\mathrm{Hanf}(\mathcal{K}) = \beth_{\omega_1}$, in the more general case $\mathrm{Hanf}(\mathcal{K}) = \beth_{(2^{2^{\mathrm{LS}(\mathcal{K})}})^+}$ where $\mathrm{LS}(\mathcal{K})$ is the Löwenheim-Skolem number of the class (the cardinality that appears in a Downward-Löwenheim-Skolem theorem for the class). See [Gr] for a formal definition.

In recent years there has been much activity in several concrete generalizations of first order model theory. They are

- model theory of Banach spaces (see [HeIo],[Io1],[Io2])
- homogeneous model theory, formerly known as finite diagrams stable in power (see [Be], [BeBu], [BuLe], [GrLe], [Hy], [HySh], [Le1], [Sh 3], [Sh 54]) and
- compact abstract theories (CATs) [BY].

In all of these contexts, a categoricity transfer theorem has been proved. However, these classes are very specialized. AECs and even $L_{\omega_1,\omega}$ have much more expressive power than these specializations. For instance, the classes of solvable groups and universal locally finite groups are AECs but cannot be captured by any of these specializations. Furthermore, the contexts itemized above are far too limited to handle Zilber's class of algebraically closed fields with pseudo-exponentiation and its variations. Each of these contexts turns to be tame.

We summarize the non-elementary categoricity transfer results known to date. Keisler in 1971 solved Conjecture 0.3 under the additional assumption of existence of a sequentially homogeneous model [Ke1]. Under these same assumptions Olivier Lessmann provides a Baldwin-Lachlan style proof of the categoricity transfer result in [Le1][3]. Unfortunately this is insufficient for Shelah's conjecture, since Marcus and Shelah found an example for an $L_{\omega_1,\omega}$-sentence which is categorical in all infinite cardinals but does not have a sequentially homogeneous model (see [Ma]).

Extending the work of Keisler, Shelah in 1984 proved Conjecture 0.3 for excellent classes [87a] and [87b]. The hardest part is to show under the assumption of $2^{\aleph_n} < 2^{\aleph_{n+1}}$ for all $n < \omega$ that $\mathrm{I}(\aleph_{n+1}, \psi) < \mu(n)$ (for all n) implies that a certain class of atomic models of a first order theory derived from ψ is excellent.

Other attempts to prove Shelah's Categoricity Conjecture involved making extra set theoretic assumptions. Michael Makkai and Shelah [MaSh] proved Conjecture 0.4 under the additional assumption that the categoricity cardinal λ is a successor and the class $\mathcal{K}$ is axiomatizable by a $L_{\kappa,\omega}$-theory where κ is above a strongly compact cardinal and $\lambda > \kappa$. Oren Kolman and Shelah [KoSh] began the generalization of [MaSh] replacing the hypothesis that κ is above a strongly compact cardinal with the assumption that κ is

[3]The referee pointed out to us that also Hyttinen in [Hy1] obtained a similar result. We feel that the reader will be interested to know that Lesmann's result is part of his 1998 PhD thesis and the relevant paper was submitted for publication and was widely circulated already in 1997.

above a measurable cardinal. Shelah completed this work in [Sh 472], but only managed to prove a partial downward categoricity transfer theorem.

Fact 0.7. *Let $\mathcal{K}$ be an AEC axiomatized by a $L_{\kappa,\omega}$-theory with κ measurable. If $\mathcal{K}$ is categorical in some $\lambda^+ > \mathrm{Hanf}(\mathcal{K})$, then $\mathcal{K}$ is categorical in every μ with $\mathrm{Hanf}(\mathcal{K}) < \mu \leq \lambda^+$.*

The most general context of AECs considered so far are AECs which satisfy the amalgamation property. In [Sh 394], Shelah proves a partial going down result for these classes:

Fact 0.8. [Sh 394] *Suppose that $\mathcal{K}$ satisfies the amalgamation and joint embedding properties. Let $\mu_0 := \mathrm{Hanf}(\mathcal{K})$. If $\mathcal{K}$ is categorical in some $\lambda^+ > \beth_{(2^{\mu_0})^+}$, then $\mathcal{K}$ is categorical in every μ such that $\beth_{(2^{\mu_0})^+} < \mu \leq \lambda^+$.*

One of the better approximations to Shelah's categoricity conjecture for AECs can be derived from a theorem due to Makkai and Shelah ([MaSh]):

Fact 0.9 (Makkai and Shelah 1990)**.** *Let $\mathcal{K}$ be an AEC, κ a strongly compact cardinal such that $\mathrm{LS}(\mathcal{K}) < \kappa$. Let $\mu_0 := \beth_{(2^\kappa)^+}$. If $\mathcal{K}$ is categorical in some $\lambda^+ > \mu_0$ then $\mathcal{K}$ is categorical in every $\mu \geq \mu_0$.*

It is easy to see (using the assumption that κ is strongly compact) that any AEC $\mathcal{K}$ as above has the AP (for models of cardinality $\geq \kappa$) and is also tame.

Our main result can be viewed as replacing the assumption of existence of a strongly compact cardinal in the Makkai and Shelah theorem by tameness and the amalgamation property. As a consequence of Corollary 4.4, we get:

Theorem 0.10. *Let $\mathcal{K}$ be an AEC, $\kappa := \beth_{(2^{\mathrm{LS}(\mathcal{K})})^+}$. Denote by $\mu_0 := \beth_{(2^\kappa)^+}$. Suppose that $\mathcal{K}_{>\kappa}$ has the amalgamation property and is tame. If $\mathcal{K}$ is categorical in some $\lambda^+ > \mu_0$ then $\mathcal{K}$ is categorical in every $\mu \geq \mu_0$.*

This is the first upward categoricity theorem we know in ZFC for AECs.

Shelah and Villaveces [ShVi] and [Va] begin to study AECs with no maximal models under GCH. The focus of this work is to initially prove that the amalgamation property follows from categoricity. After informing Shelah about the results presented in our paper, he sent an email indicating that using methods of [Sh 705] (good frames and $\mathcal{P}^-(n)$-diagrams) he has made progress towards a categoricity transfer theorem for AECs under some extra set theoretic assumptions.

All of the upward categoricity transfer results relied heavily on syntax, strong compactness or set theoretic assumptions. Until now, no upward categoricity result was known (or even suspected in light of the Hart and Shelah example) to hold.

In this paper we extend [Sh 87a], [Sh 87b] and [Sh 394] by presenting an upward categoricity transfer theorem for AECs that satisfy the amalgamation property with some level of tameness (see Definition 1.11).

We thank Andrés Villaveces for organizing the Bogotá Meeting in Model Theory 2003 where we initially presented this work. We are indebted to the participants of this meeting especially John Baldwin, Olivier Lessmann and Andrés Villaveces for removing several errors from a preliminary draft of this paper, for making suggestions and raising questions involving this paper.

1. Background

We assume that $\mathcal{K}$ is an abstract elementary class (AEC) and satisfies the amalgamation and joint embedding property. Ultimately, we will use a little less, by only assuming amalgamation and joint embedding for models of cardinality $\leq \lambda^+$ where λ^+ is a categoricity cardinal. But for readability we make the more global assumptions from the start.

We will be using the basic machinery of abstract elementary classes including Galois-types introduced by Shelah in [Sh 88] and [Sh 394]. For convenience we refer the reader to [Gr] for the definitions and essential results. This work both extends and generalizes some results from [Sh 394]. For the reader unfamiliar with [Sh 394] we have included statements and definitions of the material that we will use explicitly here.

Following the notation and terminology from [Gr], for a class $\mathcal{K}$ and a cardinal μ we let $\mathcal{K}_\mu := \{M \in \mathcal{K} \;:\; \|M\| = \mu\}$.

Let $\mu \geq \mathrm{LS}(\mathcal{K})$. We say that $\mathcal{K}$ has the *μ-amalgamation property* if and only if for any $M_\ell \in \mathcal{K}_\mu$ (for $\ell \in \{0,1,2\}$) such that $M_0 \prec_{\mathcal{K}} M_1$ and $M_0 \prec_{\mathcal{K}} M_2$ there are $N \in \mathcal{K}_\mu$ and $\mathcal{K}$-embeddings $f_\ell : M_\ell \to N$ such that $f_\ell \restriction M_0 = \mathrm{id}_{M_0}$ for $\ell = 1, 2$. $\mathcal{K}$ has the *amalgamation property* if and only if $\mathcal{K}$ has the μ-amalgamation property for all $\mu \geq \mathrm{LS}(\mathcal{K})$.

Definition 1.1. An AEC $\mathcal{K}$ is *Galois-stable in μ* if and only if for every $M \in \mathcal{K}_\mu$, the number of Galois-types over M is $\leq \mu$.

Fact 1.2 (Claim 1.7(a) of [Sh 394])**.** *If $\mathcal{K}$ is categorical in $\lambda \geq \mathrm{LS}(\mathcal{K})$, then $\mathcal{K}$ is Galois-stable in all μ with $\mathrm{LS}(\mathcal{K}) \leq \mu < \lambda$.*

A slight, but useful, improvement of Fact 1.2 can be derived from an upward stability transfer theorem for tame classes which appears in [BaKuVa].

Corollary 1.3. *If $\mathcal{K}$ is categorical in λ^+, then $\mathcal{K}$ is Galois-stable in μ for all $\mathrm{LS}(\mathcal{K}) \leq \mu \leq \lambda^+$.*

Remark 1.4. We can only guarantee stability as high as λ^+ since we don't know whether or not there is tameness for types over larger models (see Fact 1.14).

Working under the amalgamation property, Galois-stability implies the existence of Galois-saturated models in much the same way as stability implies the existence of saturated models in first order model theory. Here we review some facts about Galois-saturated models.

Definition 1.5. Let $\mu > \mathrm{LS}(\mathcal{K})$.

(1) $M \in \mathcal{K}$ is said to be *μ-Galois-saturated*[4] if and only if for every $N \prec_{\mathcal{K}} M$ with $N \in \mathcal{K}_{<\mu}$ and every Galois-type p over N, we have that p is realized in M. A model M is *Galois-saturated* if and only if it is $\|M\|$-Galois-saturated.
(2) $M \in \mathcal{K}$ is said to be *μ-model homogeneous* if and only if for every $N \prec_{\mathcal{K}} M$ with $N \in \mathcal{K}_{<\mu}$ and every $N' \in \mathcal{K}_{\|N\|}$ with $N \prec_{\mathcal{K}} N'$ there exists a $\mathcal{K}$-mapping $f : N' \to M$ with $f \restriction N = \mathrm{id}_N$. We write M is *model homogeneous* to mean that M is $\|M\|$-model homogeneous.

The following is a central property of classes with the amalgamation property:

Fact 1.6 (From [Sh 576], see also [Gr])**.** *Suppose that $\mathcal{K}$ satisfies the amalgamation property. Let $M \in \mathcal{K}_{>\mathrm{LS}(\mathcal{K})}$. The following are equivalent*
(1) *M is Galois-saturated.*
(2) *M is model homogeneous.*

The same proof gives a relativized version of Fact 1.6 which we will use later in this paper:

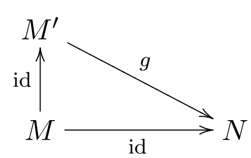
Fact 1.7. *Suppose that $\mathcal{K}$ satisfies the amalgamation property. Let $M \in \mathcal{K}_{>\mathrm{LS}(\mathcal{K})}$. The following are equivalent*
(1) *M is μ-Galois-saturated.*
(2) *M is μ-model homogeneous.*

The following is a well known result that has its origins in Bjarni Jónsson's [Jo] precursor of uniqueness of saturated models:

Fact 1.8. *Let $\mathcal{K}$ be an AEC. Suppose $\mu > \lambda \geq \mathrm{LS}(\mathcal{K})$. If $\mathcal{K}$ is categorical in λ then all model-homogeneous models of cardinality μ are isomorphic.*

A local relative to model homogeneity is that of being universal over.

Definition 1.9. (1) Let κ be a cardinal $\geq \mathrm{LS}(\mathcal{K})$. We say N is *κ-universal over M* if and only if for every $M' \in \mathcal{K}_\kappa$ with $M \prec_{\mathcal{K}} M'$ there exists a $\mathcal{K}$-embedding $g : M' \to N$ such that $g \restriction M = \mathrm{id}_M$:



(2) We say N is *universal over M* or N is *a universal extension of M* if and only if N is $\|M\|$-universal over M.
(3) For $M \in \mathcal{K}_\mu$, σ a limit ordinal with $\sigma \leq \mu^+$ and $M' \in \mathcal{K}_{\mu|\sigma|}$ we say that M' is a *(μ, σ)-limit over M* if and only if there exists a $\prec_{\mathcal{K}}$-increasing and continuous sequence of models $\langle M_i \in \mathcal{K}_\mu \mid i < \sigma \rangle$ such that

[4] We must make the distinction between saturated in first order logic (which is a property of M alone) and Galois-saturated models (which is depends on M and $\mathcal{K}$).

(a) $M = M_0$,
(b) $M' = \bigcup_{i<\sigma} M_i$ and
(c) M_{i+1} is universal over M_i.

Notice that in the case $\sigma < \mu^+$ then M' has cardinality μ and in the case $\sigma = \mu^+$ we have that $\|M'\| = \mu^+$.

Fact 1.10 guarantees the existence of limit models from stability and amalgamation assumptions:

Fact 1.10 ([Sh 600], see [GrVa1] for a complete proof)**.** *If $\mathcal{K}$ satisfies the amalgamation property and is Galois-stable in μ, then for every $M \in \mathcal{K}_\mu$, there exists $N \in \mathcal{K}_\mu$ such that $M \prec_{\mathcal{K}} N$ and N is universal over M. Thus for any $M \in \mathcal{K}_\mu$ and $\alpha \leq \mu^+$ there exists $N \precneqq_{\mathcal{K}} M$ which is (μ, α)-limit.*

Now we switch gears and recall the concept of tameness from [GrVa1].

Definition 1.11. Let χ be a cardinal number. We say the abstract elementary class $\mathcal{K}$ with the amalgamation property is χ-*tame* provided that for $M \in \mathcal{K}_{>\chi}$, $p \neq q \in$ ga-S(M) implies the existence of $N \prec_{\mathcal{K}} M$ of cardinality χ such that $p \restriction N \neq q \restriction N$.

A variant of χ-tameness involves limiting the scope of the models.

Definition 1.12. Assume $\chi < \mu$. We say that $\mathcal{K}$ is (μ, χ)-tame if and only if for all $M \in \mathcal{K}_\mu$ and all $p, q \in$ ga-S(M) whenever $p \neq q$, then there exists $N \prec_{\mathcal{K}} M$ of cardinality χ such that $p \restriction N \neq q \restriction N$.

Notation 1.13. When $\mathcal{K}$ is (μ, χ)-tame for all $\mu \leq \mu'$ we write $\mathcal{K}$ is $(\leq \mu', \chi)$-tame.

Fact 1.14 (From Main Claim 2.3 of part II on page 288 [Sh 394])**.** *If $\mathcal{K}$ is categorical in some $\lambda^+ > \beth_{(2^{\mathrm{Hanf}(\mathcal{K})})^+}$, then $\mathcal{K}$ is $(< \lambda^+, \chi)$-tame for all $\chi(\Phi^*) \leq \chi < \lambda^+$.*

Remark 1.15.
(1) We will be using tameness to prove the existence of rooted minimal types or monotonicity of minimal types (Proposition 2.2). Unfortunately, [Sh 394] only gives us tameness up to and not including the categoricity cardinal λ^+.
(2) Formally, Shelah proves that when two types over a saturated model of cardinality $\kappa < \lambda^+$ differ, then there is a submodel of cardinality $\chi(\Phi^*)$ over which they differ. However, if we assume categoricity in $\lambda^+ > \beth_{(2^{\mathrm{Hanf}(\mathcal{K})})^+}$, by Fact 0.8 all models of cardinality κ are saturated.
(3) $\chi(\Phi^*)$ has a formal definition in [Sh 394] related to Ehrenfeucht-Mostowski constructions. Since we will only use the fact that $\chi(\Phi^*)$ lies below Hanf$(\mathcal{K})$, we will not give its formal definition.

Question 1.16. *Does categoricity in $\lambda >$ Hanf$(\mathcal{K})$ imply (λ, χ)-tameness for some $\chi < \lambda$?*

2. Minimal Types

The main tool in our constructions will be a minimal type which is a variation of Definition $(*)_4$ of Theorem 9.7 of [Sh 394].

Definition 2.1. Let $M \in \mathcal{K}_\mu$ and $p \in \text{ga-S}(M)$ be given. We say p is *minimal* if and only if p is non-algebraic (no $c \in M$ realizes p) and for every $M' \in \mathcal{K}_\mu$ with $M \prec_{\mathcal{K}} M'$, there is exactly one non-algebraic extension of p to M'.

The proof of the following proposition draws on our tameness assumption. We will be interested in applying the monotonicity proposition to λ where λ is the categoricity cardinal. Recall that Shelah's work does not guarantee any level of tameness in the categoricity cardinal.

Proposition 2.2 (Monotonicity of Minimal Types)**.** *Suppose $\mathcal{K}$ is (λ, χ)-tame for some $\lambda \geq \mu \geq \chi$. If $p \in \text{ga-S}(M)$ is minimal with $M \in \mathcal{K}_\mu$, then for all $N \in \mathcal{K}_\lambda$ extending M and every $q \in \text{ga-S}(N)$ extending p, if q is non-algebraic then q is minimal.*

Proof. Suppose for the sake of contradiction that p and q are as in the statement of the proposition, with q non-algebraic but not minimal. Since q is not minimal, there exist distinct non-algebraic extensions of q, say $q', q'' \in \text{ga-S}(N')$ for some $N' \in \mathcal{K}_\lambda$ with $N \prec_{\mathcal{K}} N'$. By tameness, we can find $M' \in \mathcal{K}_\mu$ of cardinality μ such that $M \prec_{\mathcal{K}} M' \prec_{\mathcal{K}} N'$ and $q' \restriction M' \neq q'' \restriction M'$. Notice that $q' \restriction M'$ and $q'' \restriction M'$ are both non-algebraic extensions of p. This contradicts the minimality of p. $\dashv$

Fact 2.3 (Density of Minimal Types [Sh 394])**.** *If $\mathcal{K}$ is Galois-stable in μ, then for every $N \in \mathcal{K}_\mu$ and every $q \in \text{ga-S}(N)$, there are $M \in \mathcal{K}_\mu$ and $p \in \text{ga-S}(M)$ such that $N \preceq_{\mathcal{K}} M$, $q \leq p$ and p is minimal.*

To prove the extension property for minimal types, we need a few facts about non-splitting in AECs.

Definition 2.4. Let $\mu > \text{LS}(\mathcal{K})$ be a cardinal. For $M \in \mathcal{K}$ and $p \in \text{ga-S}(M)$, we say that p *μ-splits over* N if and only if $N \prec_{\mathcal{K}} M$ and there exist $N_1, N_2 \in \mathcal{K}_\mu$ and a $\prec_{\mathcal{K}}$-mapping $h : N_1 \cong N_2$ such that

(1) $N \prec_{\mathcal{K}} N_1, N_2 \prec_{\mathcal{K}} M$,
(2) $h(p \restriction N_1) \neq p \restriction N_2$ and
(3) $h \restriction N = \text{id}_N$.

Remark 2.5. Consider the $\mathcal{K}$-mapping h in the definition of μ-splitting. Notice that we do not require that there is an extension $h' \in \text{Aut}(M)$ of h.

The existence, uniqueness and extension properties for non-splitting types have been studied in [Sh 394], [ShVi] and [Va]. Here we state the formulations of these results which we will be using.

Existence of non-splitting types:

Fact 2.6 (Claim 3.3 of [Sh 394])**.** *Assume $\mathcal{K}$ is an abstract elementary class and is Galois-stable in μ. For every $M \in \mathcal{K}_{\geq\mu}$ and $p \in \text{ga-S}(M)$, there exists $N \in \mathcal{K}_\mu$ such that p does not μ-split over N.*

A consequence of the proof of the uniqueness result, Theorem I.4.15 of [Va], is the following:

Corollary 2.7. *Let $N, M, M' \in \mathcal{K}_\mu$ be such that M' is universal over M and M is a limit model over N. Suppose that $p \in \text{ga-S}(M)$ does not μ-split over N and p is non-algebraic. For every $M' \in \mathcal{K}$ extending M of cardinality μ, if $q \in \text{ga-S}(M')$ is an extension of p and does not μ-split over N, then q is non-algebraic.*

The version of this extension and existence result that we will use is the following which came about when John Baldwin removed the cofinality requirement in Lemma 6.3 of [Sh 394] with an argument using Ehrenfeucht-Mostowski models. This has allowed us to remove the assumption of $LS(\mathcal{K}) = \aleph_0$ from previous drafts of this paper.

Fact 2.8 (Corollary 2 of [Ba2])**.** *Suppose that $\mathcal{K}$ is categorical in some $\lambda > \text{LS}(\mathcal{K})$ and $\mathcal{K}$ has arbitrarily large models. Let μ be a cardinal such that $\text{LS}(\mathcal{K}) < \mu$ and let σ be a limit ordinal with $\text{LS}(\mathcal{K}) < \sigma < \mu^+$. Then, for every (μ, σ)-limit model M and every type $p \in \text{ga-S}(M)$, there exists $N \precneqq_{\mathcal{K}} M$ of cardinality μ such that for every $M' \in \mathcal{K}_{\leq\lambda}$ extending M, there exists $q \in \text{ga-S}(M')$ an extension of p such that q does not μ-split over N. In particular p does not μ-split over N.*

The last property of non-splitting that we will need is monotonicity:

Proposition 2.9. *If $M_0 \prec_{\mathcal{K}} N \prec_{\mathcal{K}} M$ and $p \in \text{ga-S}(M)$ does not μ-split over M_0, then $p \restriction N$ does not μ-split over M_0.*

Proof. Immediate from the definitions.

$\dashv$

Combining the machinery of non-splitting, we identify the following relative to Claim 4.3 of [Sh 576]:

Proposition 2.10 (Extension Property for Minimal Types)**.** *Suppose that $\mathcal{K}$ has arbitrarily large models. Let $\mathcal{K}$ be categorical in some $\lambda > \text{LS}(\mathcal{K})$ and (λ, χ)-tame for some $\chi < \lambda$. Let μ be such that $\text{LS}(\mathcal{K}) < \mu$. If $p \in \text{ga-S}(M)$ is minimal and M is a (μ, σ)-limit model for some limit ordinal $\text{LS}(\mathcal{K}) < \sigma < \mu^+$, then for every $M' \in \mathcal{K}_{\leq\lambda}$ extending M, there is a minimal $q \in \text{ga-S}(M')$ such that q extends p.*

Proof. Without loss of generality M' is universal over M. Let $p \in \text{ga-S}(M)$ be minimal. Since M is (μ, σ)-limit model, using Fact 2.8, we can find a proper submodel $N \prec_{\mathcal{K}} M$ of cardinality μ such that for every $M' \in \mathcal{K}_{\leq\lambda}$ there exists $q \in \text{ga-S}(M')$ extending p such that q does not μ-split over N. Suppose for the sake of contradiction that q is not minimal. Then tameness and Proposition 2.2 tells us that q must be algebraic. Let $a \in M'$ realize q and $M^a \in \mathcal{K}_\mu$ contain a with $M \prec_{\mathcal{K}} M^a \prec_{\mathcal{K}} M'$. Then $q \restriction M^a$ is also algebraic. However, since $q \restriction M^a$ does not μ-split over N and extends p, by Corollary 2.7 we see that $q \restriction M^a$ is not-algebraic. This gives us a contradiction. $\dashv$

We now introduce a strengthening of minimal types which allow us to transfer Vaughtian pairs from one cardinality to another in the subsequent section. The intuition is that there is a small part of a minimal types that controls its minimality.

Definition 2.11. Let $M \in \mathcal{K}_\mu$ be given. A type $p \in \text{ga-S}(M)$ is *rooted minimal* if and only if p is minimal and there is $N \prec_{\mathcal{K}} M$ of cardinality $< \mu$ such that $p \restriction N$ is minimal. We say that N is a *root* of p.

Proposition 2.12 (Existence of rooted minimal types)**.** *Let $\mathcal{K}$ be categorical in some $\lambda > \chi^+$ and (λ, χ)-tame with $\chi \geq \text{LS}(\mathcal{K})$. Then for every $M' \in \mathcal{K}_\lambda$, there exists a rooted minimal $q \in \text{ga-S}(M')$.*

Proof. Notice that categoricity in λ implies stability in μ with $\text{LS}(K) < \mu < \lambda$ by Fact 1.2. Choose $M \in \mathcal{K}_\mu$ be some $\mathcal{K}$-substructure of M' with $\mu \geq \chi$. Since $\mathcal{K}$ is stable in μ and categorical in λ, we may take M to be a (μ, σ)-limit model for some limit ordinal σ with $\text{LS}(\mathcal{K}) < \sigma < \mu$. Furthermore, by Fact 2.3 and monotonicity of minimal types, we can choose M such that there is a minimal type $p \in \text{ga-S}(M)$.

Then by Proposition 2.10, there exists a minimal $q \in \text{ga-S}(M')$ extending p. q is rooted. $\dashv$

Remark 2.13. In Section 4 we will prove the existence of rooted minimal types over models of cardinality λ when $\text{cf}(\lambda) = \omega$ under the assumption that $\text{LS}(\mathcal{K}) = \aleph_0$.

Proposition 2.14. *Suppose $\mathcal{K}$ is χ-tame. Let $N \in \mathcal{K}_{\geq \chi}$. If $p \in \text{ga-S}(M)$ is rooted minimal with $N \prec_{\mathcal{K}} M$ a submodel such that $p \restriction N$ is minimal, then for every N' with $N \prec_{\mathcal{K}} N' \prec_{\mathcal{K}} M$ we have that $p \restriction N'$ is minimal.*

Proof. Follows by tameness and monotonicity of minimal types. $\dashv$

3. Vaughtian Pairs

Next we prove a Vaughtian pair transfer theorem for rooted minimal types.

Definition 3.1. Let $\mu \leq \lambda$. Fix $M \in \mathcal{K}_\mu$ and $p \in \text{ga-S}(M)$ a minimal type. A (p, λ)-*Vaughtian pair* is a pair of models $N_0, N_1 \in \mathcal{K}_\lambda$ such that

(1) $M \prec_{\mathcal{K}} N_0 \precneqq_{\mathcal{K}} N_1$ and
(2) no $c \in N_1 \backslash N_0$ realizes p.

Fact 3.2 (Claim $(*)_8$ of Theorem 9.7 of [Sh 394])**.** *Assume that $\mathcal{K}$ is categorical in some successor cardinal λ^+. If $\lambda > \mathrm{LS}(\mathcal{K})$, then for every model $M \in \mathcal{K}_{\leq\lambda}$ and every minimal type $p \in$ ga-S(M), there are no (p,λ)-Vaughtian pairs.*

Theorem 3.3. *Fix $\mu > \mathrm{LS}(\mathcal{K})$. Let p be a rooted minimal type over a model M of cardinality μ. Fix a root $N \prec_{\mathcal{K}} M$ of cardinality κ, with $p \restriction N$ minimal. If $\mathcal{K}$ has a (p,μ)-Vaughtian pair, then there is a $(p\restriction N,\kappa)$-Vaughtian pair.*

Proof. Suppose that (N^0, N^1) form a (p,μ)-Vaughtian pair. Let C denote the set of all realizations of $p \restriction N$ inside N^1. Fix $a \in N^1\backslash N^0$.

We now construct $\langle N^0_i, N^1_i \in \mathcal{K}_\kappa \mid i < \kappa^+\rangle$ satisfying the following:

(1) $N^0_0 = N$
(2) $N^\ell_i \precneqq_{\mathcal{K}} N^\ell$ for $\ell = 0,1$
(3) the sequences $\langle N^0_i \mid i < \kappa^+\rangle$ and $\langle N^1_i \mid i < \kappa^+\rangle$ are both $\prec_{\mathcal{K}}$-increasing and continuous
(4) $a \in N^1_i\backslash N^0_i$ and
(5) $C_i := C \bigcap N^1_i \subseteq N^0_{i+1}$.

The construction follows from the following:

Claim 3.4. *If $d \in N^1$ realizes $p \restriction N^0_0$, then $d \in N^0$. Thus $C \subseteq N^0$.*

Proof of Claim 3.4. Suppose that $d \in N^1\backslash N^0$ realizes $p \restriction N^0_0$. Then ga-tp(d/N^0) is a non-algebraic extension of $p \restriction N^0_0$. Since $p \restriction N^0_0$ is minimal, we have that ga-tp$(d/M) = p$. Since (N^0, N^1) form a (p,μ)-Vaughtian pair, it must be the case that $d \in N^0$, contradicting our choice of d.

$\dashv$

The construction is enough: Pick a limit ordinal $\delta < \kappa^+$ satisfying: for all $i < \delta$ and $x \in N^1_i$, if there exists $j < \kappa^+$ such that $x \in C_j$, then there exists $j < \delta$, such that $x \in C_j$. In fact there is a club many such δ's.

Claim 3.5. *For every $c \in N^1_\delta \cap C$, we have $c \in N^0_\delta$.*

Proof of Claim 3.5. Since $\langle N^1_i \mid i < \kappa^+\rangle$ is continuous, there is $i < \delta$ such that $c \in N^1_i$. Thus by the definition of E, there is a $j < \delta$ with $c \in C_j$. By condition (5) of the construction, we would have put $c \in N^0_{j+1} \prec_{\mathcal{K}} N^0_\delta$.

$\dashv$

Notice that $N^1_\delta \neq N^0_\delta$ since $a \in N^1_\delta\backslash N^0_\delta$. Thus Claim 3.5 allows us to conclude that we have constructed a $(p\restriction N^0_0,\kappa)$-Vaughtian pair (N^0_δ, N^1_δ).

$\dashv$

Corollary 3.6. *Let $\lambda > \mathrm{LS}(K)$. If $\mathcal{K}$ is categorical in λ and λ^+ and p is rooted minimal type over a model of cardinality λ^+, then there are no (p,λ^+)-Vaughtian pairs.*

Proof. Suppose (N_0, N_1) is a (p,λ^+)-Vaughtian pair. Then by Theorem 3.3 and Proposition 2.14, there is a $(p\restriction N,\lambda)$-Vaughtian pair where $p \restriction N$ is minimal. Since $\mathcal{K}$ is categorical in λ, N is saturated. This contradicts Fact 3.2.

⊣

Corollary 3.7. *Let $\lambda > \mathrm{LS}(\mathcal{K})$. If $\mathcal{K}$ is categorical in λ and λ^+, then every rooted minimal type over a model N of cardinality λ^+ is realized λ^{++} times in every model of cardinality λ^{++} extending N.*

Proof. Suppose $M \in \mathcal{K}_{\lambda^{++}}$ realizes p only $\alpha < \lambda^+$ times.

Let $A := \{a_i \mid i < \alpha\}$ be an enumeration of the realizations of p in M. We can find $N_0 \in \mathcal{K}_{\lambda^+}$ such that $N \bigcup A \subseteq N_0 \prec_{\mathcal{K}} M$. Since M has cardinality λ^{++}, we can find $N_1 \in \mathcal{K}_{\lambda^+}$ such that $N_0 \precneqq_{\mathcal{K}} N_1 \prec_{\mathcal{K}} M$. Then (N_0, N_1) form a (p, λ^+)-Vaughtian pair contradicting Corollary 3.6.

⊣

4. Upward Categoricity Transfer Theorems

The following shows the strength of the assumption of no Vaughtian pairs. In order to prove that a model is saturated, it suffices to check that the model realizes one rooted minimal type many times. Furthermore, Theorem 4.1 provides a new sufficient condition for a model to be universal over a submodel.

Theorem 4.1. *Suppose $M_0 \in \mathcal{K}_\lambda$ and $r \in$ ga-S(M_0) is a minimal type such that $\mathcal{K}$ has no (r, λ)-Vaughtian pairs.*

Let α be an ordinal $< \lambda^+$ such that $\alpha = \lambda \cdot \alpha$. Suppose $M \in \mathcal{K}_\lambda$ has a resolution $\langle M_i \in \mathcal{K}_\lambda \mid i < \alpha\rangle$ such that for every $i < \alpha$, there is $c_i \in M_{i+1} \backslash M_i$ realizing r. Then M is saturated over M_0. Moreover if $\mathcal{K}$ is stable in λ, then M is a (λ, α)-limit model over M_0.

Notice that Proposition 2.12 and Corollary 3.6 guarantee that such r and M_0 exist when we assume that $\mathcal{K}$ is tame and categorical in λ with λ a successor cardinal. We use the letter r to represent this type since in our applications of this theorem, r will be rooted and we will want to distinguish this type from others.

Theorem 4.1 is similar to Claim 5.6 of [Sh 576]. It is also related to a result of [ShVi] and [Va] that the top of a relatively full tower of length $\alpha = \lambda \cdot \alpha$ is a (μ, α)-limit model.

Proof of Theorem 4.1. Let $\langle M_i \mid i < \alpha\rangle$ and r be given as in the statement of the theorem. Without loss of generality, we can assume that the resolution $\langle M_i \mid i < \alpha\rangle$ is continuous. Fix $q \in$ ga-S(M_0). We will prove that M realizes q by constructing a model M' which realizes q and a $\mathcal{K}$-mapping from M into this model M' simultaneously. Then we will show that this is in fact an isomorphism.

Since $\alpha = \lambda \cdot \alpha$, we can fix a collection of disjoint sets $\{S_i \mid i < \alpha\}$ such that $\alpha = \bigcup_{i<\alpha} S_i$ and each S_i is unbounded in α of cardinality λ and $\mathrm{Min}(S_i) \geq i$.

We define by induction on $i < \alpha$ sequences of models $\langle N'_i \mid i < \alpha\rangle$ and $\langle M'_i \mid i < \alpha\rangle$ and a sequence of $\mathcal{K}$-mappings $\langle h_i \mid i < \alpha\rangle$. Additionally, for $i < \alpha$ we fix a sequence $\langle a_\zeta \mid \zeta \in S_i\rangle$. We require:

1. M'_0 realizes both r and q,
2. $N'_i \prec_{\mathcal{K}} M'_i$,
3. $\langle N'_i \mid i < \alpha\rangle$ and $\langle M'_i \mid i < \alpha\rangle$ are $\prec_{\mathcal{K}}$-increasing and continuous sequences of models in $\mathcal{K}_\lambda$,
4. $N'_0 = M_0$,
5. $\langle a_\zeta \mid \zeta \in S_i\rangle$ is an enumeration of $\{a \in M'_i \mid a \models r\}$,
6. $a_i \in N'_{i+1}$,
7. $h_i : M_i \cong N'_i$,
8. $\langle h_i \mid i < \alpha\rangle$ is increasing and continuous with $h_0 = \mathrm{id}_{M_0}$ and
9. when $\mathcal{K}$ is stable in λ, we additionally require M'_{i+1} is universal over M'_i.

The construction is possible: For $i = 0$, we take $N'_0 = M_0$ and let M'_0 be a extension of M_0 of cardinality λ realizing r and q. If possible we choose M'_0 to be a universal extension of M_0 of cardinality λ. Set $h_0 = \mathrm{id}_{M_0}$. Let $\langle a_\zeta \mid \zeta \in S_0\rangle$ be some (possibly repeating) enumeration of $\{a \in M'_0 \mid a \models r\}$.

Suppose that we have defined for all $k \leq j$, N'_k, M'_k, h_k and $\langle a_\zeta \mid \zeta \in S_k\rangle$. Let a_j be given. Notice that a_j has been defined since $\min S_l > j$ for $l \geq i$. If a_j is already in N'_j, then we simply amalgamate the following diagram

$$\begin{array}{ccc} M_{j+1} & \xrightarrow{\ f\ } & M^* \\ {\scriptstyle\mathrm{id}}\uparrow & & \uparrow{\scriptstyle\mathrm{id}} \\ M_j & \xrightarrow[h_j]{} & M'_j \end{array}$$

setting $h_{j+1} := f$ and $N'_{j+1} = h_{j+1}(M_{j+1})$. Let M'_{j+1} be an of M'_j containing N'_{j+1} of cardinality λ. If possible, choose M'_{j+1} to be universal over M'_j. Fix $\langle a_\zeta \mid \zeta \in S_{j+1}\rangle$ some enumeration of $\{a \in M'_{j+1} \mid a \models r\}$.

In the event that $a_j \notin N'_j$, we need to be more careful with the amalgamation. Let f and M^* be as in the diagram above. Let us rewrite this diagram as

$$\begin{array}{ccccc} M_{j+1} & \xrightarrow{\ f\ } & M^* & & \\ {\scriptstyle\mathrm{id}}\uparrow & & \uparrow{\scriptstyle\mathrm{id}} & & \\ M_j & \xrightarrow[h_j]{} & N'_j & \xrightarrow[\mathrm{id}]{} & M'_j \end{array}$$

Since $a_j \notin N'_j$, we have that $a_j \in M'_j \backslash N'_j$. Thus $\text{ga-tp}(a_j/N'_j, M'_j)$ is non-algebraic. Now lets compare this to the realization c_j of r in $M_{j+1}\backslash M_j$.

Notice that $f(\text{ga-tp}(c_j/M_j, M_{j+1})) = \text{ga-tp}(f(c_j)/N'_j, M^*)$ is non-algebraic. Since $h_0 = \text{id}_{M_0}$, $\text{ga-tp}(f(c_j)/N'_j, M^*)$ is also an extension of r. By the minimality of r we can conclude that

$$\text{ga-tp}(a_j/N'_j, M'_j) = \text{ga-tp}(f(c_j)/N'_j, M^*).$$

So we can find an amalgam M^{**} such that the following diagram commutes

$$\begin{array}{ccccc} M_{j+1} & \xrightarrow{f} & M^* & \xrightarrow{g} & M^{**} \\ {\scriptstyle \text{id}}\uparrow & & \uparrow{\scriptstyle \text{id}} & & \uparrow{\scriptstyle \text{id}} \\ M_j & \xrightarrow[h_j]{} & N'_j & \xrightarrow[\text{id}]{} & M'_j \end{array}$$

and $g(f(c_j)) = a_j$. Let $h_{j+1} := g \circ f$ and $N'_{j+1} := h_{j+1}(M_{j+1})$. Let M'_{j+1} be an extension of M'_j cardinality λ containing N'_{j+1}. If possible, choose M'_{j+1} to be universal over M'_j. Fix $\langle a_\zeta \mid \zeta \in S_i\rangle$ some enumeration of $\{a \in M'_i \mid a \models r\}$. This completes the construction.

Let $N' := \bigcup_{i<\alpha} N'_i$, $M' := \bigcup_{i<\alpha} M'_i$ and $h := \bigcup_{i<\alpha} h_i$. Notice that M' realizes q and $h : M \cong N'$ with $h \restriction M_0 = \text{id}_{M_0}$. We will show that $N' = M'$ in order to conclude that M also realizes q. Suppose not. Then $N' \prec_{\mathcal{K}} M'$ and we can fix $a \in M' \backslash N'$. Since there are no (r, λ)-Vaughtian pairs, we can choose a such that $a \models r$.

By the definition of M', there is an $i < \alpha$ such that $a \in M'_i$. Then $a = a_\zeta$ for some $\zeta \in S_i$. At stage, $\zeta + 1$, we made sure that $a = a_\zeta \in N'_{\zeta+1} \subseteq N'$. This contradicts our choice of a. $\dashv$

We now restate Theorem 0.1

Theorem 4.2. *Suppose that $\mathcal{K}$ has arbitrarily large models, is χ-tame and satisfies the amalgamation property. If $\lambda \geq \chi \geq \text{LS}(\mathcal{K})$ and $\mathcal{K}$ is categorical in both λ and λ^+ then $\mathcal{K}$ is categorical in λ^{++}.*

Proof of Theorem 0.1. We will show that for every $N \in \mathcal{K}_{\lambda^{++}}$ and every $M \prec_{\mathcal{K}} N$ of cardinality λ^+, N realizes every type over M.

Let $M \prec_{\mathcal{K}} N$ have cardinality λ^+. First notice that Proposition 2.12 and categoricity in λ^+ guarantees that there exists a rooted minimal $r \in \text{ga-S}(M)$. By Corollary 3.7, we know that N realizes r λ^{++}-times.

Let $\alpha < \lambda^+$ be such that $\alpha = \lambda^+ \cdot \alpha$. By the Downward-Löwenheim Skolem Axiom of AECs, we can construct a $\prec_{\mathcal{K}}$-increasing and continuous chain of models $\langle M_i \prec_{\mathcal{K}} N \mid i < \alpha\rangle$ such that $M = M_0$ for every $i < \alpha$, we can fix $a_i \in M_{i+1} \backslash M_i$ realizing r. This construction is possible since there are λ^{++}-many realizations of r to choose from. By Theorem 4.1, $\bigcup_{i<\alpha} M_i$ realizes every type over M.

$\dashv$

We now can derive an upward categoricity transfer theorem

Corollary 4.3 (Categoricity Transfer for Tame AECs)**.** *Suppose that $\mathcal{K}$ has arbitrarily large models, satisfies the amalgamation property and is χ-tame with $\chi \geq \mathrm{LS}(\mathcal{K})$. Suppose that $\lambda \geq \max\{\chi, \mathrm{LS}(\mathcal{K})^+\}$. If $\mathcal{K}$ is categorical in both λ^+ and λ, then $\mathcal{K}$ is categorical in every μ with $\lambda \leq \mu$.*

Proof of Corollary 4.3. Let α be such that $\lambda = \aleph_\alpha$. We will prove that $\mathcal{K}$ is categorical in $\aleph_\beta$ for all $\beta \geq \alpha + 2$. The base case is Theorem 0.1. For $\beta = \gamma + 2$, Theorem 0.1 and the induction hypothesis give us that $\mathcal{K}$ is categorical in $\aleph_{\gamma+2}$.

Deriving categoricity in $\aleph_{\beta+1}$ where β is a limit ordinal $> \alpha$. Assume that $\mathcal{K}$ is categorical in every μ with $\lambda \leq \mu \leq \aleph_\beta$. We need to show that $\mathcal{K}$ is categorical in $\aleph_{\beta+1}$. Let N have cardinality $\aleph_{\beta+1}$. Fix $M \prec_{\mathcal{K}} N$ with cardinality $\aleph_\beta$. By Proposition 2.12, there is a rooted minimal type $r \in$ ga-S(M).

Let $M' \in \mathcal{K}_\kappa$ with $\lambda \leq \kappa < \aleph_\beta$ be a root of r. Suppose that there is a $(r, \aleph_\beta)$-Vaughtian pair. Then by Theorem 3.3, there is a $(r \restriction M', \kappa)$-Vaughtian pair. Our induction hypothesis tells us that $\mathcal{K}$ is categorical in κ and κ^+. Thus there are no $(r \restriction M', \kappa)$-Vaughtian pairs. And we can conclude there are no $(r, \aleph_\beta)$-Vaughtian pairs.

We now see that N realizes r $\aleph_{\beta+1}$-many times. Since there are enough realizations of r to go around, we can construct an increasing and continuous chain of models $\langle M_i \prec_{\mathcal{K}} N \mid i < \gamma\rangle$ of cardinality $\aleph_\beta$ such that for every $i < \gamma$, there is a $c_i \in M_{i+1} \backslash M_i$ realizing r and γ is a limit ordinal $< \aleph_\beta$ satisfying $\gamma = \aleph_\beta \cdot \gamma$. Now by Theorem 4.1 we see that N must realize every type over M.

Deriving categoricity in $\aleph_\beta$ for β a limit ordinal. Assume that $\mathcal{K}$ is categorical in all μ with $\lambda \leq \mu < \aleph_\beta$. For this case, it suffices to show that the every model of cardinality $\aleph_\beta$ is Galois-saturated. Given $N \in \mathcal{K}_{\aleph_\beta}$ and $M \prec_{\mathcal{K}} N$ be a model of cardinality $\aleph_\gamma$ for some $\gamma > \alpha$. Let $p \in$ ga-S(M) be given. By the Downward Löwenheim Skolem axiom of AECs, we may find $N' \in \mathcal{K}_{\aleph_{\gamma+1}}$ such that $M \prec_{\mathcal{K}} N' \prec_{\mathcal{K}} N$. By the induction hypothesis N' is Galois-saturated and realizes p. Thus N realizes p. ⊣

Combining Corollary 4.3 and Fact 0.8 yields

Corollary 4.4. *Suppose $\mathcal{K}$ is a χ-tame abstract elementary class satisfying the amalgamation and joint embedding properties. Let $\mu_0 :=$ Hanf$(\mathcal{K})$. If $\chi \leq \beth_{(2^{\mu_0})^+}$ and $\mathcal{K}$ is categorical in some $\lambda^+ > \beth_{(2^{\mu_0})^+}$, then $\mathcal{K}$ is categorical in μ for all $\mu > \beth_{(2^{\mu_0})^+}$.*

5. Implications and Open Problems

The Hart-Shelah examples [HaSh] (an alternative exposition is in chapter 19 of [Ba1]) have arbitrary large models and are categorical in several successive cardinals but fail to be categorical in some larger cardinals. By

Theorem 0.1 the AEC induced by φ_n[5] is not χ-tame for any $\chi < \aleph_n$ or fails to have the amalgamation property.

Baldwin, David Kueker, Grossberg and VanDieren have begun extending the results for categorical, tame AECs to stable, tame AECs. After the presentation of the results from this paper at the Bogotá Meeting in Model Theory 2003, Lessmann and Tapani Hytinnen have explored the implications of our arguments in more specific contexts.

Recently, by incorporating an idea of Lessmann [Le2] we were able to obtain the following strengthening of Theorem 0.1:

Theorem 5.1 ([GrVa2]). *Suppose $\mathcal{K}$ is a χ-tame AEC with the amalgamation property with arbitrarily large models. If $\mathcal{K}$ is categorical in λ^+ for some $\lambda > (\mathrm{LS}(\mathcal{K}) + \chi)$ then $\mathcal{K}$ is categorical in all $\mu > (\mathrm{LS}(\mathcal{K}) + \chi)$*

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