Due to time limitation, I was unable to answer Tom's question adequately. Here is an outline of the proof of the Lemma:

LEMMA 0.1 (Technical Lemma). *Suppose $\mathcal{K}$ as an AEC which is also $\mathrm{PC}_{\aleph_0}$. If $I(\aleph_0, \mathcal{K}) = 1$ and $\mathcal{K}_{\aleph_1} \neq \emptyset$ then for every $M \in \mathcal{K}_{\aleph_0}$ there exits $\{M_n \mid n < \omega\} \subseteq K_{\aleph_0}$ such taht $M = \bigcap_{n<\omega} M_n$ and $M_{n+1} \precneqq M_n$ for all $n < \omega$.*

PROOF. (outline) Notice that by $\aleph_0$-categoricity it is enough to prove for *some* M instead of *every*. Fix an increasing continuous $\{M_\alpha \mid \alpha < \omega_1\} \subseteq \mathcal{K}_{\aleph_0}$. Let $F : \omega_1 \to \mathcal{K}_{\aleph_0}$ defined by $F(\alpha) = M_\alpha$.

Let $\mathcal{B} := \langle H(\chi), \in, \omega_1, \models_{L(\mathcal{K})}, \overline{\mathcal{K}}, F, \dots \rangle$, where χ is regular large enough so that $\mathcal{B}$ contains the definition of $\mathcal{K}$ and reflects all the above and $\overline{\mathcal{K}}$ is teh $\mathrm{PC}_{\aleph_0}$ definition of $\mathcal{K}$.

(*) Notice that since the theorem we proved immediately after the definition of AECs (i.e. they are closed under unions of directed systems) this also holds in $\mathcal{B}$.

By Lopez-Escobar's theorem there is a countable $\mathcal{B}^* \equiv_{\omega_1,\omega} \mathcal{B}$ such that there is $\{a_n \mid n < \omega\} \subseteq \omega_1^{\mathcal{B}^*}$ satisfying:

$$\mathcal{B}^* \models a_n \ni a_{n+1} \text{ for all } n < \omega.$$

Let

$$I := \{a \in \omega_1^{\mathcal{B}^*} \mid \mathcal{B}^* \models a \in a_n \text{ for all } n < \omega\}.$$

Notice that I is not empty. Let $M := \bigcup_{i \in I} F^{\mathcal{B}^*}(i)$, by (*) it is in $\mathcal{K}$ and let $M_n := F^{\mathcal{B}^*}(a_n)$.
Check that

$$M = \bigcap_{n<\omega} M_n.$$

An absolutness argument show that the above models are in $\mathcal{K}$, since $\mathcal{K}_{\aleph_0}^{\mathcal{B}^*} = \mathcal{K}_{\aleph_0}$.

□